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# Dynamika i Obliczalność w Geometrycznej Teorii Grup

Rozprawa doktorska

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Doctoral School
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# Dynamics and computability in Geometric Group Theory

Doctoral thesis

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prof. dr hab. Aleksander Iwanow,
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## Streszczenie

Rozprawa jest podzielona na dwie części dotyczące dość niezależnych obszarów badań w geometrycznej teorii grup.

W pierwszej części pracy badamy obliczalne aspekty średniowalności. Dowodzimy obliczalną wersję twierdzenia Halla o haremach i korzystamy z niego by udowodnić obliczalną wersję twierdzenia o Alternatywie Tarskiego. Ponadto, udowadniamy nową wersję twierdzenia Halla o haremach, gdzie otrzymane skojarzenie jest realizowane jako funkcja o pewnych dodatkowych własnościach. Stosując tę wersję dla nieśredniowalnych przestrzeni zgrubnych otrzymujemy obliczalną wersję generalizacji twierdzenia Whyte'a wprowadzonej przez Schneidera.

Druga część dotyczy lokalnie eliptycznych działań grup na jednospójnych kompleksach małych skreśleń. W szczególności dowodzimy, że podgrupy torsyjne grup o $C(6)$, $C(4)$–$T(4)$ lub $C(3)$–$T(6)$ prezentacji małych skreśleń, są skończonymi grupami cyklicznymi. Jest to wniosek z ogólniejszego wyniku o istnieniu punktów stałych dla lokalnie eliptycznych działań grup na jednospójnych kompleksach małych skreśleń. Przedstawiamy zastosowanie tego wyniku dla automatycznej ciągłości. Ponadto zauważamy, że dla jednospójnych kompleksów $C(3)$–$T(6)$ można wprowadzić metrykę CAT(0). Ten fakt pozwala uzyskać mocniejsze wyniki w przypadku $C(3)$–$T(6)$. Wynika również z niego, że grupy działające na jednospójnych kompleksach $C(3)$–$T(6)$ z ograniczeniem rzędów stabilizatorów komórek spełniają Alternatywę Titsa.

# Abstract

The thesis is divided into two parts which correspond to two relatively independent areas of geometric group theory.

In the first part we study computable aspects of amenability. We prove a computable version of Hall's harem Theorem and use it to prove a computable versions of Tarski's Alternative Theorem. Moreover we also prove a new version of Hall's Harem theorem where the final matching is realized by a function with additional properties. We apply it to non-amenable computable coarse spaces to obtain a computable version of Schneider's generalization of Whyte's Theorem.

In the second part is devoted to locally elliptic actions of groups on simply connected small cancellation complexes. In particular, we prove that torsion subgroups of groups defined by $C(6)$, $C(4)$–$T(4)$, or $C(3)$–$T(6)$ small cancellation presentations are finite cyclic groups. This follows from a more general result on the existence of fixed points for locally elliptic actions of groups on simply connected small cancellation complexes. We present an application concerning automatic continuity. We also observe that simply connected $C(3)$–$T(6)$ complexes may be equipped with a CAT(0) metric. This allows us to get stronger results on locally elliptic actions in that case. It also implies that the Tits Alternative holds for groups acting on simply connected $C(3)$–$T(6)$ small cancellation complexes with a bound on the order of cell stabilisers.

## Podziękowania

Chciałbym podziękować mojemu promotorowi prof. dr. hab. Aleksandrowi Iwanowowi, za wprowadzenie w świat nauki, nieocenioną ilość czasu poświęconą mi przez lata współpracy oraz za pomoc bez której zapewne bym tutaj nie dotarł.

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*No i dyć jeszcze mej cŏłkij familie, żeście ciyngiym sōm sam kaj mie was trza. Bo skuli was żech tego ni zmaścił.*

*Mym rodzicom, boście mie richtig gryfnie wyrychtowali.*

*I mej freli, co przaja jej fest. Przeca ino bez cie żech poradził strzymać.*

# Contents

# Chapter 1

# Introduction

The topic of this dissertation is divided into two parts which correspond to two relatively independent areas of geometric group theory. However both areas arise from the topic of amenability. The fundamental idea of the latter (in a simplified form) is as follows. Let $X$ be a set and let $G$ be a group which acts on $X$ by permutations. Then the $G$-space $(G, X)$ is called *amenable* if there exists a mean $m : \ell^\infty(X) \to \mathbb{R}$ which is invariant under the action of $G$. In particular, if $X = G$, then the group $G$ is called amenable. This definition is due to to J. von Neumann [Neu29].

The theme already existed before 1929. In the 1920s and 1930s Banach and Tarski investigated the question when invariant means of $G$-spaces exist. Their study led to the Banach-Tarski Paradox [BT24] and the theory of paradoxical decompositions. In particular in 1938, Tarski proved his famous Tarski Alternative Theorem [Tar38], stating that a $G$-space $X$ is amenable if and only if $X$ does not admit a $G$-paradoxical decomposition.

The first part of this dissertation concerns possible computable version of the Tarski Alternative Theorem. It follows the trends of modern mathematical logic of analysis of classical mathematical topics (for example amenability) from the point of view of complexity of various types (see [Kec91], [BK96], [KPT05], [MU16], [CK18], [HPP08], [HKP22] and [HKP]). The idea of investigation of computable aspects of amenability is very natural and recently it appeared independently both in computability theory [Bie+12], [Mor18] and in group theory [Cav17], [Cav18].

One of versions of the Tarski Alternative Theorem considered in this part is particularly interesting since it has the same root with the second part of the dissertation.

To describe this we mention that in [Neu29] J. von Neumann also proved that amenable group cannot contain a subgroup isomorphic to the two generated free group $\mathbb{F}_2$. The question whether the converse holds i.e. whether each non-amenable group contains a subgroup isomorphic to $\mathbb{F}_2$, became popularized as the *von Neumann Conjecture* in the late 50s (see e.g. [Day57]). It was answered negatively with the construction of Tarski monster group by Olshanskii in 1980 [Ol'81]. On the other hand, in 1999 Whyte [Why99] proved so called the geometric version of this conjecture and in 2018 Schneider [Sch18] generalized Whyte's theorem to coarse spaces. The main (the most difficult) result of the first part of the dissertation is a computable version of Schneider's theorem. An additional goal of this part was to obtain a computable version of the Hall Harem Theorem [CSC10, Theorem H.4.2.]. This is motivated by the well-known fact that aforementioned classical results can be obtained by applications of this theorem.

There are some classes of groups where the von Neumann conjecture is true, for example the class of finitely generated linear groups. This is due to the theorem of Tits [Tit72], that each such group contains a free nonabelian subgroup or is virtually solvable and hence amenable. The term *Tits Alternative* usually refers to the property that all finitely generated subgroups are either virtually solvable or contain a nonabelian free group. It is believed (see e.g. [Bes00, Quest 2.8], [Bri06],[Bri07, Quest 7.1],[FHT11, Prob 12],[Cap14, §5]) that the Tits Alternative is common among 'non-positively curved' groups. However up to now it has been shown only for a few particular classes of groups. Most notable, for: Gromov-hyperbolic groups [Gro87], mapping class groups [Iva84; McC85], $Out(\mathbb{F}_n)$ [BFH00; BFH05], CAT(0) cubical groups [SW05], groups acting on a 2-dimensional CAT(0) complexes [OP22] and groups acting properly on 2-dimensional recurrent complexes [OP21].

The second part of the topic of this dissertation concerns questions related to the Tits Alternative for a class of "nonpositively curved" groups, namely for groups acting on simply connected small cancellation complexes. More precisely it concerns locally elliptic actions on small cancellation complexes.

An action of $G$ by isometries on a metric space $X$ is *elliptic* if the orbit of each $x \in X$ is bounded. An action of $G$ on $X$ is *locally elliptic* if for each $g \in G$ the action of the cyclic subgroup $\langle g \rangle$ is elliptic. Observe that actions of finite groups are always elliptic and hence actions of *torsion groups* (that is groups where every element has finite order) are locally elliptic.

In the case we consider, having a bounded orbit is equivalent to having a fixed point. Therefore 'locally elliptic' can be thought of as 'every group element fixes a point' and 'elliptic' means 'having a global fixed point'.

Recently Haettel and Osajda suggested the following:

**Meta-Conjecture.** *[HO21] Every locally elliptic action of a finitely generated group on a finite-dimensional nonpositively curved complex is elliptic. In particular, every action of a finitely generated torsion group on such a complex is elliptic.*

There are examples of finitely generated subgroups of torsion groups that are not virtually solvable. Clearly, such groups do not satisfy the Tits Alternative. Therefore, excluding infinite torsion subgroups might be seen as a first step towards establishing the alternative. We address this question in the small cancellation case showing that torsion subgroups of groups defined by small cancellation presentations are finite cyclic groups. Our approach to this is from the direction of the Meta-Conjecture above: we show that locally elliptic actions on small cancellation complexes which are free on the 1-skeleton are elliptic. This applies to an action of a group on its Cayley complex. We also provide application of the non-existence of infinite torsion subgroups to the automatic continuity.

In a specific case of C(3)-T(6) small cancellation groups we observe that the associated complexes might be equipped with a CAT(0) metric. Therefore, we are able to apply existing results on 2-dimensional CAT(0) complexes to obtain stronger results about locally elliptic actions, and we are able to conclude the Tits Alternative for groups acting properly on C(3)-T(6) small cancellation complexes.

**The structure of the dissertation.** The goal of the first part of the doctoral thesis is to obtain computable versions of Tarski's Alternative Theorem and Schneider version of Whyte's Theorem. One of the most important tools in the proofs of these theorems is Hall's Harem theorem, and consequently the major goal of this part of the thesis is investigation of computable versions of Hall's Harem Theorem.

The first part of the doctoral thesis consists of Chapters 2 - 4. In Chapter 2 we give necessary preliminaries, where we describe basic ideas of the subject. In Chapter 3 we give a brief report on the main results of the appended paper [DI22a]. Chapter 4 is devoted to versions of Hall's harem theorem and applications to Schneider-Whyte's theorem.

The goal of the second part of the dissertation is investigation of the Meta-Conjecture in the case of the groups acting on $C(6)$, $C(4)$–$T(4)$, and $C(3)$–$T(6)$ small cancellation complexes. It consists of Chapters 5 - 6. Chapter 5 gives necessary preliminaries concerning the small cancellation theory. Chapter 6 is devoted to a version of Meta-Conjecture in the case of the groups acting on $C(6)$, $C(4)$–$T(4)$, and $C(3)$–$T(6)$ small cancellation complexes with an additional assumption of the freeness of the action on 1-skeleton, and applications of that results.

# Part I

# Computable aspects of amenability

# Chapter 2

# Preliminaries for Part I

The material which is introduced in this chapter mostly appear in the appended paper [DI22a]. We give it here for the convinience of the reader, because it will be used in Chapter 4 too. In Section 2.1 we introduce Hall's Harem Theorem. Section 2.2 concerns paradoxical decompositions and Tarski's Alternative Theorem. Section 2.3 is devoted to Whyte's Theorem and its generalization to the case of coarse spaces by Schneider.

## 2.1 Hall's Harem Theorem

We mostly follow the notation of [CSC10]. A graph $\Gamma = (\mathbf{\Gamma}, E)$ is called a *bipartite graph* if the set of vertices $\mathbf{\Gamma}$ is partitioned into sets $U$ and $V$ in such a way, that the set of edges $E$ is a subset of $U \times V$. We denote such a bipartite graph by $\Gamma = (U, V, E)$.

Let $\Gamma = (U, V, E)$. We will say that an edge $(u, v)$ is *incident* to vertices $u$ and $v$. In this case we say that $u$ and $v$ are *adjacent*. When two edges $(u, v), (u', v') \in E$ have a common incident vertex we say that $(u, v), (u', v')$ are also *adjacent*. A sequence $x_1, x_2, \dots, x_n$ is called a *path* if each pair $x_i, x_{i+1}$ is adjacent, $1 \le i < n$.

Below we will denote the set of vertices $\mathbf{\Gamma}$ by the same letter with the graph as a structure, i.e. $\Gamma$. Given a vertex $x \in \Gamma$ the *neighbourhood* of $x$ is a set

$$N_\Gamma(x) = \{y \in \Gamma : (x, y) \in E\}.$$

For subsets $X \subset U$ and $Y \subset V$, we define the neighbourhood $N_\Gamma(X)$ of $X$ and the neighbourhood $N_\Gamma(Y)$ of $Y$ by

$$N_\Gamma(X) = \bigcup_{x \in X} N_\Gamma(x) \quad \text{and} \quad N_\Gamma(Y) = \bigcup_{y \in Y} N_\Gamma(y).$$

We drop the subscript $\Gamma$ if it is clear from the context.

For a given vertex $v$ a *star* of $v$ is a subgraph $S = (\{v\} \cup N_\Gamma(v), E')$ of $\Gamma$, with $E' = ((\{v\} \cup N_\Gamma(v)) \times (\{v\} \cup N_\Gamma(v))) \cap E$. A $(1, k)$-*fan* is a subset of $E$ consisting of $k$ edges incident to some vertex $u \in U$. We say that $u$ is the *root* of the fan, and when $(u, v)$ belongs to the fan we call $v$ a *leaf* of it.

**Definition 2.1.1.** An $(1, k)$-*matching* from $U$ to $V$ is a collection $M$ of pairwise disjoint $(1, k)$-fans.

The $(1,k)$-matching $M$ is called *left perfect* (resp. *right perfect*) if each vertex from $U$ is a root of a fan from $M$ (resp. each vertex from $V$ belongs to exactly one fan of $M$). The $(1,k)$-matching $M$ is called *perfect* if it is both left and right perfect.

We often view an $(1,k)$-matching as a bipartite graph $M$ where the fan of $u \in U$ is the *$M$-star* of $u$, i.e. a graph consisting of the set of all vertices adjacent to $u$ in $M$ and all edges incident to $u$ in $M$. We emphasize that a perfect $(1,k)$-matching from $U$ to $V$ is a set $M \subset E$ satisfying following conditions:

(1) each vertex $u \in U$ there exists exactly $k$ vertices $v_1, \dots v_k \in V$ such that $(u, v_1), \dots, (u, v_k) \in M$;

(2) for all $v \in V$ there is an unique vertex $u \in U$ such that $(u, v) \in M$.

Originally the Hall's Marriage Theorem (see e.g. [Die97, Theorem 2.1.2],[Bol79, Section III.3], [LP86, Theorem 2.4.2.]) provides us a condition for existence of left perfect $(1,1)$-matching in a finite bipartite graph. We are interested in so called *Hall's Harem theorem*, which is a generalization of Hall's Marriage theorem to the case of perfect $(1,k)$-matchings for the locally finite infinite graphs.

One says that graph $\Gamma = (V, E)$ is *locally finite* if for all vertices $x \in \Gamma$, the neighbourhood $N(x)$ is finite. Note that if $\Gamma$ is locally finite, then $N(X)$ is finite for any finite $X \subset V$.

**Theorem 2.1.2** (Hall's Harem theorem)**.** *[CSC10, Theorem H.4.2.] Let $\Gamma = (U, V, E)$ be a locally finite graph and let $k \in \mathbb{N}$, $k \geq 1$. The following conditions are equivalent:*

*(i) For all finite subsets $X \subset U$, $Y \subset V$ the following inequalities holds: $|N(X)| \geq k|X|$, $|N(Y)| \geq \frac{1}{k}|Y|$.*

*(ii) $\Gamma$ has a perfect $(1,k)$-matching.*

The first condition in this formulation is known as *Hall's k-harem condition*.

It is a crucial fact that the theorem holds for locally finite infinite graphs. Since infinity is necessary for applications which we are interested in, it was very surprising for the author to discover in a standard textbook for Graph Theory the following quotation: "The study of infinite graphs is an attractive, but often neglected, part of graph theory." ([Die97, p.209]). At this point we inform the reader that the statement of the Hall Marriage Theorem does not work for infinite graphs in general (see e.g. [Die97, S.2 Ex.6]). Nevertheless there are versions of this theorem for graphs of any cardinality [Aha88].

## 2.2 Tarski's Alternative Theorem

Hall's Harem theorem is useful in paradoxical decompositions. To describe it we start with the basic definition.

**Definition 2.2.1.** Let $X$ be a set and let $G$ be a group which acts on $X$ by permutations.

The $G$-space $(G, X)$ has a **paradoxical $G$-decomposition**, if there exists a finite set $K \subset G$ and families $(A_k)_{k \in K}$, $(B_k)_{k \in K}$ of subsets of $X$ such that

$$X = \Big( \bigsqcup_{k \in K} k(A_k) \Big) \bigsqcup \Big( \bigsqcup_{k \in K} k(B_k) \Big) = \Big( \bigsqcup_{k \in K} A_k \Big) = \Big( \bigsqcup_{k \in K} B_k \Big).$$

**Theorem 2.2.2.** *Let $X$ be a set and let $G$ be a group which acts on $X$ by permutations. The following are equivalent:*

- *$G$-space $(G, X)$ is amenable;*
- *$G$-space $(G, X)$ has no paradoxical decomposition.*

In the 50s Følner [Fø55] introduced the following condition and proved its equivalence to amenability.

**Definition 2.2.3.** Let $X$ be a set and let $G$ be a group which acts on $X$ by permutations. The $G$-space $(G, X)$ satisfies Følner condition if for every finite subset $K \subset G$ and every real number $\varepsilon > 0$, there exists a nonempty finite subset $F \subset X$ such that:

$$\frac{|F \setminus kF|}{|F|} < \varepsilon \text{ for all } k \in K.$$

Such a subset $F$ is called an *$\varepsilon$-Følner set with respect to $K$.*

Modern versions of the proof of the Tarski's alternative theorem use the fact that by Følner's condition of amenability, the $G$-space $(G, X)$ is not amenable if and only if there exists a finite $K \subset G$ and $n$ such that there are no $\frac{1}{n}$-Følner sets with respect to $K$. Knowing $K$ and $n$, we can construct a bipartite graph $\Gamma$ with sets of vertices being two copies of $X$. By the Hall's Harem Theorem there exists a perfect $(1, 2)$-matching in $\Gamma$ which corresponds to the paradoxical $G$-decomposition of $X$.

## 2.3 Whyte's geometric solution to the von Neumann Conjecture

As mentioned in the introduction, in its original formulation the von Neumann conjecture has been answered negatively by Olshanskii. Nevertheless, Whyte showed that geometric version of this conjecture is true. Before stating Whyte's Theorem, we recall some definitions.

**Definition 2.3.1.** A metric space $(X, d)$ is *uniformly discrete* if there exists $r > 0$ such that $d(x_1, x_2) < r \Rightarrow x_1 = x_2$.

**Definition 2.3.2.** A uniformly discrete space $X$ has *bounded geometry* if for all $r$ there is $N_r \in \mathbb{N}$ such that for any $x \in X$ $|B_r(x)| \leq N_r$.

Let us fix two metric spaces $X$ and $Y$.

**Definition 2.3.3.** A function $f : X \to Y$ is called *bilipschitz* if there exist $k \in \mathbb{R}$ such that $\frac{1}{k}d(x_1, x_2) \leq d(f(x_1), f(x_2)) \leq kd(x_1, x_2)$.

For $A \subset X$ we define:

$$\partial_r(A) = \{x \in X \setminus A \mid d(x, A) \leq r\}$$

**Definition 2.3.4.** The space $X$ is *amenable* if and only if there exists a family $S_i \subset X$, $i \in \mathbb{N}$ such that for any $r > 0$

$$\lim_{i\to\infty} \frac{|\partial_r(S_i)|}{|S_i|} = 0$$

.

We now formulate the theorem of Whyte.

**Theorem 2.3.5** (Geometric Von Neumann Conjecture)**.** *A uniformly discrete metric space of uniformly bounded geometry is non-amenable if and only if it admits a partition whose pieces are bilipschitz equivalent to the* 4*-regular tree with uniform lipschitz constant.*

Recently, Schneider [Sch18], generalized this result to the case of non-amenable coarse spaces. We recall some terminology from the coarse geometry. For the relation $E \subseteq X \times X$ on a set $X$ and $x \in X, A \subseteq X$, let

$$E[x] := \{y \in x | (x, y) \in E\},$$

and

$$E[A] := \bigcup \{E[x] | x \in A\}.$$

Furthermore, we denote by $\Gamma(E)$ the graph associated with the relation $E$, i.e. $\Gamma(E) = (X, E)$.

**Definition 2.3.6.** A *coarse space* is a pair $(X, \mathcal{E})$ consisting of a set $X$, and a collection of subsets of $X \times X$ (called *entourages*) such that:

- $\Delta_X \in \mathcal{E}$;
- if $F \subseteq E \in \mathcal{E}$, then also $F \in \mathcal{E}$;
- if $E, F \in \mathcal{E}$ then $E \cup F, E^{-1}, E \circ F \in \mathcal{E}$.

**Definition 2.3.7.** A coarse space $(X, \mathcal{E})$ is said to have *bounded geometry* if for each $E \in \mathcal{E}$ and every $x \in X$ the set $E[x]$ is finite.

Note that metric spaces are examples of coarse spaces: for a given metric space $X$, a coarse space $(X, \mathcal{E})$ is obtained by setting:

$$\mathcal{E} := \{E \subseteq X \times X | \sup\{d(x, y) | (x, y) \in E\} \leq \infty\}$$

**Definition 2.3.8.** A coarse space $(X, \mathcal{E})$ of a bounded geometry is called *amenable* if for every $\theta > 1$ and every $E \in \mathcal{E}$ there exists a non-empty finite $F \subseteq X$ such that $|E[F]| \leq \theta |F|$.

**Theorem 2.3.9.** *([Sch18]) Let $d \geq 3$. A coarse space $(X, \mathcal{E})$ of bounded geometry is not amenable if and only if there is $E \in \mathcal{E}$ such that $\Gamma(E)$ is a $d$-regular forest.*

To see why this theorem generalizes the theorem of Whyte, note that connected components of $\Gamma(E)$ can be viewed as pieces of the partition from Whyte's theorem.

In the proof of his result, Schneider takes a symmetric entourage $E$ such that $|E(F)| \geq (d-1)|F|$ for all finite $F \subset X$. Such an entourage exists by $X$ being non amenable. Then graph $\Gamma := (X, X, R)$, where $R := E \setminus \delta_X$ satisfies Hall's Harem condition for $d-1$. It follows that there exists an entourage which graph is $d$-regular. Moreover, properties of entourages allows us to refine it to remove all cycles without losing $d$-regularity.

# Chapter 3

# Description of results from the appended paper ‘Computable paradoxical decompositions’

The appended paper [DI22a] is a part of this dissertation. In this chapter we give a brief report on main results of this paper. In Section 3.1 we state a computable version of Hall's Harem Theorem. Section 3.2 is devoted to computable versions of Tarski's alternative Theorem.

## 3.1 Computable version of Hall's Harem Theorem

In the first part of [DI22a] we extend the methods of Kierstead [Kie83] to prove the computable version of Hall's Harem Theorem. In his paper, Kierstead proved the computable version of Hall's Marriage Theorem (i.e. concerning left perfect $(1,1)$-matchings in infinite graphs) for highly computable bipartite graphs. Just in case we recall the following definition.

**Definition 3.1.1.** A partial function $f : \mathbb{N} \to \mathbb{N}$ is *partially computable* if there exist a Turing program which computes it. The function $f$ is *computable* if it is partially computable and total.

**Definition 3.1.2.** Subset $X$ of $\mathbb{N}$ is called *computable* if its characteristic function is computable.

Following trends in logic we say computable instead of recursive. The facts from computability theory which we use are well-known and can be found in [Soa16].

Let $\Gamma := (V, E)$ be a graph and assume that the set of vertices $V$ is identified with $\mathbb{N}$. We call graph $\Gamma = (V, E)$ *computable* if the set $E$ is computable. A computable graph $\Gamma = (V, E)$ is *highly computable* if there is an algorithm computing sizes of neighbourhoods of each vertex.

A $(1, d)$-matching $M$ is *computable* if there is an algorithm which

- for each $i \in U$ incident to edge in $M$, finds the tuple $(i_1, i_2, \ldots, i_d)$ such that $(i, i_j) \in M$, for all $j = 1, 2, \ldots, d$;
- for $i \in V$ incident to edge in $M$, finds $i' \in U$ such that $(i', i) \in M$.

Kierstead introduced the following condition which is sufficient to construct a computable $(1,1)$-matching (which is not perfect) in a highly computable bipartite graph:

**Definition 3.1.3.** A highly computable bipartite graph $\Gamma = (U, V, E)$ satisfies the computable expanding Hall's condition ( denoted c.e.H.c.), if

- there exists computable total $h : \mathbb{N} \to \mathbb{N}$
- $h(0) = 0$
- for all finite sets $X \subset U$, the inequality $h(n) \leq |X|$ implies $n \leq |N(X)| - |X|$.

The following generalization of the Kierstead condition is introduced in [DI22a].

**Definition 3.1.4** (Theorem 2.9 in [DI22a])**.** A highly computable bipartite graph $\Gamma = (U, V, E)$ satisfies the computable expanding Hall's harem condition with respect to $d$ (denoted *c.e.H.h.c.*$(d)$), if

- there exists computable total $h : \mathbb{N} \to \mathbb{N}$;
- $h(0) = 0$
- for all finite sets $X \subset U$, the inequality $h(n) \leq |X|$ implies $n \leq |N(X)| - d|X|$
- for all finite sets $Y \subset V$, the inequality $h(n) \leq |Y|$ implies $n \leq |N(Y)| - \frac{1}{d}|Y|$.

This condition allows us to formulate a computable version of Hall's Harem Theorem.

**Theorem I.1** (Theorem 2.9 in [DI22a])**.** *If $\Gamma = (U, V, E)$ is a highly computable bipartite graph satisfying the c.e.H.h.c.$(d)$, then $\Gamma$ has a computable perfect $(1, d)$-matching.*

## 3.2 Computable version of Tarski's Alternative Theorem

In the second part of [DI22a] we apply computable version of Hall's Harem Theorem to prove two computable versions of Tarski's Alternative Theorem, one in the formulation for the very general situation of the pseudogroups of transformations, and the more precise version in the case of action of group $G$ on space $X$.

### 3.2.1 Case of pseudogroups

A *pseudogroup* $\mathcal{G}$ of transformations of a set $X$ is a set of bijections $\rho : S \to T$ between subsets $S$ and $T \subseteq X$ which satisfies some natural conditions of an action, see Definition 1.2 in [DI22a]. For $\gamma : S \to T$ in $\mathcal{G}$, we write $\alpha(\gamma)$ for the domain $S$ of $\gamma$ and $\omega(\gamma)$ for its range $T$.

Let $\mathcal{G}$ be a pseudogroup of transformations of $X$. For $R \subset \mathcal{G}$ and $A \subset X$ we define the $R$-boundary of $A$ as

$$\partial_R A = \{x \in X \setminus A : \exists \rho \in R \cup R^{-1}(x \in \alpha(\rho) \text{ and } \rho(x) \in A)\}.$$

**Definition 3.2.1.** The pseudogroup $\mathcal{G}$ satisfies the Følner condition if for any finite subset $R$ of $\mathcal{G}$ and any natural number $n$ there exists a finite non-empty subset $F = F(R,n)$ of $X$ such that $|\partial_R F| < \frac{1}{n}|F|$.

The following is a version of Tarski's theorem in case of pseudogroups, see Theorems 7 and 25 in [CSGH99].

- The pseudogroup $\mathcal{G}$ satisfies the Følner condition if and only if there is no tuple $(X_1, X_2, \gamma_1, \gamma_2)$ consisting of a non-trivial partition $X = X_1 \sqcup X_2$ and $\gamma_i \in \mathcal{G}$ with $\alpha(\gamma_i) = X_i$ and $\omega(\gamma_i) = X$ for $i = 1, 2$.

Assume that a pseudogroup $\mathcal{G}$ acts on a countable set $X$. We will identify $X$ with $\mathbb{N}$. We say that a transformation $\rho : S \to T$ from $\mathcal{G}$ is *computable* if $S$ and $T$ are computable subsets of $\mathbb{N}$ and $\rho$ is a computable function. The following definition corresponds to Definition 3.1 in [DI22a].

**Definition 3.2.2** (Definition 3.1 in [DI22a])**.** Let $\mathcal{G}$ be a pseudogroup of transformations of a set $X = \mathbb{N}$. An effective paradoxical $\mathcal{G}$-decomposition of $(\mathcal{G}, X)$ is a tuple $(X_1, X_2, \gamma_1, \gamma_2)$ consisting of a non-trivial partition $X = X_1 \sqcup X_2$ into computable sets and computable $\gamma_i \in \mathcal{G}$ with $\alpha(\gamma_i) = X_i$ and $\omega(\gamma_i) = X$ for $i = 1, 2$.

The main theorem of [DI22a] in the case of pseudogroups (Theorem 3.2) has the following form.

**Theorem I.2** (Theorem 3.2 in [DI22a])**.** *Let $(\mathcal{G}, X)$ be a pseudogroup of computable transformations defined on $\mathbb{N}$ which does not satisfy Følner's condition. Then $X$ has an effective paradoxical $\mathcal{G}$-decomposition.*

### 3.2.2 Case of groups

Let $X$ be a set identified with $\mathbb{N}$ and let $G$ be a group which acts on $X$ by computable permutations. The space $(G, X)$ has a computable paradoxical decomposition, if there exists a finite set $K \subset G$ and two families of computable sets $(A_k)_{k\in K}$, $(B_k)_{k\in K}$ such that:

$$X = \Big(\bigsqcup_{k\in K} k(A_k)\Big) \bigsqcup \Big(\bigsqcup_{k\in K} k(B_k)\Big) = \Big(\bigsqcup_{k\in K} A_k\Big) = \Big(\bigsqcup_{k\in K} B_k\Big).$$

We call $(K, (A_k)_{k\in K}, (B_k)_{k\in K})$ a computable paradoxical decomposition of $X$, see Definition 3.4 in [DI22a]. The following theorem is a counterpart of Theorem I.2 in the case of actions of groups.

**Theorem I.3** (Theorem 3.5 in [DI22a])**.** *Let $G$ be a group of computable permutations on a countable set $X$ which does not satisfy Følner's condition. Then there is a finite subset $K \subset G$ which defines a computable paradoxical decomposition.*

The remainder of [DI22a] concerns some complexity issues related to computable paradoxical decompositions.

Assume that $G$ is a computable group (i.e. a group identified with $\mathbb{N}$ as a set, whose multiplication table is computable as a subset of $\mathbb{N}^3$). The computable group $G$ has a computable paradoxical decomposition, if the left action of $G$ on $G$ has a computable paradoxical decomposition.

Theorem I.3 (and Corollary 4.2 in [DI22a]) leads to the following definition.

**Definition 3.2.3** (Definition 4.3 in [DI22a])**.** Let

$$\mathfrak{W}_{BT} = \left\{ K \subset G \text{ is finite} : \exists n \in \mathbb{N}\ (\forall \text{ finite } F \subset G)(\exists k \in K) \left( \frac{|F \setminus kF|}{|F|} \geq \frac{1}{n} \right) \right\}.$$

We call $\mathfrak{W}_{BT}$ the set of witnesses of the Banach-Tarski paradox.

In Proposition 4.4. we describe the algorithmic complexity of $\mathfrak{W}_{BT}$ and in Theorem 4.5 we give a natural example of a computable groups with computable $\mathfrak{W}_{BT}$.

**Theorem I.4** (Theorem 4.5 in [DI22a])**.** *The family $\mathfrak{W}_{BT}$ is computable for any finitely generated free group.*

On the other hand in [DI22b] we give an example of a finitely presented group with decidable world problem (hence computable) where the set $\mathfrak{W}_{BT}$ is not computable. We do not include that paper into the dissertation because of its focus on computability theory.

# Chapter 4

# Computable version of Schneider-Whyte's Theorem

Assuming that a bipartite graph is of the form $(\mathbb{N}, \mathbb{N}, E)$, where $E$ denotes the set of edges between natural numbers, an $(1, k)$-matching in such a graph clearly realizes a $k$ to 1 function, say $f : \mathbb{N} \to \mathbb{N}$. If this matching is perfect, such a function $f$ is total and surjective. We study properties which can be additionally added to such a function. The motivation follows from our work on a computable version of Schneider-Whyte's theorem. It turns out that in order to obtain a computable $d$-regular forest, some special form of Hall's harem theorem is required.

For a point $u$ such that for some $i \neq 0$ we have $f^i(u) = u$ find the smallest $i \neq 0$ with $f^i(u) = u$. Then the set $\{u, f(u), \ldots, f^{i-1}(u)\}$ is a *cycle* of $f$. The proof of our computable version of Schneider's theorem uses Hall's Harem Theorem in some version, where we are also able to recognize all the cycles in the final matching.

The results of this chapter are taken from the preprint 'A new computable version of Hall's Harem Theorem' available at arXiv [Dud21a].

In order to avoid in this thesis computability issues as much as it is possible, we omit the final result of [Dud21a]. We give the version of Hall's harem theorem which is called Main Theorem in that paper (i.e. Theorem 2.4 there). There are two reasons for this. The first one is the conviction that a classical version of Hall's harem theorem with additional restrictions on $f$ is desirable. The second reason is the fact that the task of computability of $f$ makes our proofs enormously complicated.

## 4.1 Preliminaries

### 4.1.1 Structure of the chapter

In Section 4.1.2 we define reflected graphs. In Section 4.1.3 we define functions with controlled sizes of its cycles and formulate the Main Theorem of this chapter. Section 4.1.4 provides a list of notation used in the construction. Section 4.2 shows a construction. In Section 4.3 we prove technical lemmas necessary for Section 4.4. In Section 4.4 we show that claims from Section 4.2 are true. In section 4.5 we prove Main Theorem. In Section 4.6 we explain the relation of Main Theorem with the computable version of Schneider-Whyte's Theorem.

### 4.1.2 Reflections

Throughout this chapter, $d$ is a natural number greater than 1. When $\Gamma = (U, V, E)$ is a bipartite graph, we always assume that $V \subseteq U \subseteq \mathbb{N}$, i.e. $V$ is a subset of the right copy of $U$.

The following notation substantially simplifies the presentation. For any vertex $v \in V$ there exist a vertex from $U$ which is a *copy* of $v$ (i.e. the same natural number), we denote it by $u_v$. If a vertex $u \in U$ has the copy in $V$ then we denote this copy by $v_u$.

**Definition 4.1.1.** The graph $\Gamma = (U, V, E)$ is called $U$*-reflected* if $V$ is a subset of the right copy of $U$ and for every edge $(u, v) \in E$ with $v_u \in V$ the edge $(u_v, v_u)$ is in $E$ too. If additionally $V$ is a right copy of $U$ then $\Gamma$ is called a *fully reflected* bipartite graph.

The main theorem below states that in the case of fully reflected bipartite graphs the Hall's harem condition allows us to force some additional properties at the expense of obtaining a perfect $(1,(d-1))$-matching instead of a $(1,d)$-matching. We will now give necessary details.

### 4.1.3 Controlled sizes of cycles. Main theorem

Let $f$ be a function. If for some $i \neq 0$ we have $f^i(u) = u$ then we will say that $u$ is a *periodic point*, see the introduction to this chapter. For such $u$ and the smallest $i \neq 0$ with $f^i(u) = u$, we say that $\{u, f(u), \ldots, f^{i-1}(u)\}$ is a *cycle* of $f$. Any $(1,(d-1))$ matching can be considered as a $(d-1)$ to 1 function $f : \mathbb{N} \to \mathbb{N}$. Moreover, if this matching is perfect, such a function $f$ is total and surjective. We roughly want to show that given a fully reflected bipartite graph satisfying Hall's $d$-harem condition, there is a perfect $(1,(d-1))$ matching $f : \mathbb{N} \to \mathbb{N}$, such that for each $u$ there exist $i \geq 0$ such that $f^i(u)$ is a periodic point.

**Definition 4.1.2.** Let $f : \mathbb{N} \to \mathbb{N}$ be a $(d-1)$ to 1 function. We say that $f$ has *controlled sizes of its cycles* if each of the following conditions holds:

(i) $f^2(1) = 1$;

(ii) if $n \geq 2$ and $f^i(n) = n$ then $i \leq n$;

(iii) if $n \geq 2$ and for all $i \leq n$ we have $f^i(n) \neq n$ then there exist $k \leq 2n$ and $l \leq n$ such that $f^{k+l}(n) = f^k(n)$.

The following theorem is the main result of this chapter.

**Theorem 4.1.3.** *(Main Theorem in [Dud21a]) Let $\Gamma = (U, V, E)$ be a locally finite bipartite graph such that:*

- *both $U$ and $V$ are identified with $\mathbb{N} \setminus \{0\}$,*
- *$E$ does not contain edges of the form $(u, v_u)$,*
- *$\Gamma$ is fully reflected,*
- *$\Gamma$ satisfies Hall's $d$-harem condition.*

*Then there exist a perfect* $(1, d-1)$*-matching of* $\Gamma$*, which realizes a* $(d-1)$ *to* 1 *function* $f : \mathbb{N} \to \mathbb{N}$ *with controlled sizes of its cycles.*

The proof of the theorem is based on an inductive construction of the matching.

### 4.1.4 Notation used in the construction

Since this construction is highly technical, we start with a list of the notation. We do not insist on a thorough inspection of it. In the beginning a hasty view will suffice.

- $M$ is a perfect matching that we construct.
- $M_{n-1}$ is a set of $(1, d-1)$-fans added to $M$ at the end of the $n$-th step. Thus $M = \bigcup_{n=1}^{\infty} M_{n-1}$.
- $\Gamma^{(0)} = (U^{(0)}, V^{(0)}, E^{(0)})$ is the original graph $\Gamma$.
- $U^{(n)} := U^{(n-1)} \setminus \{u \in U^{(n-1)} : \exists v \in V^{(n-1)}, (u, v) \in M_{n-1}\}$.
- $V^{(n)} := V^{(n-1)} \setminus \{v \in V^{(n-1)} : \exists u \in U^{(n-1)}, (u, v) \in M_{n-1}\}$.
- $\Gamma^{(n)} = (U^{(n)}, V^{(n)})$. We will see that $\Gamma^{(n)}$ is $U^{(n)}$-reflected.
- After the $n$-th step we obtain decompositions $U^{(n)} = U^{(n)\star} \dot\cup U^{(n)\perp}$ and $V^{(n)} = V^{(n)\star} \dot\cup V^{(n)\perp}$, where we say that $U^{(n)\perp}$ consists of elements from $U^{(n)}$ which might spoil Hall's $d$-harem condition for $\Gamma^{(n)}$.
- Put $U^{(0)\perp} = \emptyset$ and $V^{(0)\perp} = \emptyset$ (since $\Gamma^{(0)}$ is $\Gamma$, i.e. it satisfies Hall's $d$-harem condition).
- $\Gamma^{(n)\perp}$ is a graph with the sets of vertices $(U^{(n)\perp}, V^{(n)\perp})$ and the set of edges corresponding to $(1, d-1)$-fans with roots denoted by $u_i^{\perp} \in U^{(n)\perp}$.
- When $U^{(n)\perp} \setminus U^{(n-1)\perp}$ is not empty, $U^{(n)\perp} \setminus U^{(n-1)\perp} = \{u_{n-1}^{\perp}\}$ and $V^{(n)\perp} \setminus V^{(n-1)\perp}$ consists of leaves $\{v_{n-1,i}^{\perp} : 1 \leq i \leq d-1\}$.
- $\Gamma^{(n)\star} := (U^{(n)} \setminus U^{(n)\perp}, V^{(n)} \setminus V^{(n)\perp})$. We will see that $\Gamma^{(n)\star}$ satisfies Hall's harem condition.
- During the construction of $M_n$ we will define fans $M_n^j$, $j \leq n+1$. The graph $M_n$ is the union of them.
- $M_n^j$ is a fan consisting of edges denoted by $\{(u_n^j, v_{n,i}^j) : i \leq d-1\}$.
- $u_n$ is a starting vertex of the $n$-th step, it is also denoted by $u_n^0$.
- For any subgraph $\Gamma'$ of $\Gamma^{(n)}$ by $\Gamma'(-u_n^0, \ldots, -u_n^j)$ we denote the graph obtained from $\Gamma'$ by removal of $(1, d-1)$-fans of $M_n$ with roots $u_n^0, \ldots, u_n^j$.
- For any subgraph $\Gamma' = (U', V')$ of $\Gamma^{(n)}$ and any $u_j^{\perp} \in U^{(n)\perp}$ by $\Gamma'(+u_j^{\perp})$ we denote the graph induced in $\Gamma^{(n)}$ by the sets of vertices $U' \cup \{u_j^{\perp}\}$ and $V' \cup \{v_{j,i}^{\perp} : 1 \leq i \leq d-1\}$.

- For any subgraph $\Gamma' = (U', V')$ of $\Gamma^{(n)}$ and any vertex $v \in V^{(n)}$ by $\Gamma'(+v)$ (resp. $\Gamma'(-v)$) we denote the graph induced in $\Gamma^{(n)}$ by the sets of vertices $U'$ and $V' \cup \{v\}$ (resp. $V' \setminus \{v\}$).
- $\mathfrak{M}_n^1$ denotes a perfect $(1, d)$-matching in $\Gamma^{(n)\star}$ which appears in the first part of step $n + 1$.
- $\mathfrak{M}_n^2$ denotes a perfect $(1, d)$-matching in $\Gamma^{(n)}(-u_n^0, \ldots, -u_n^j) \cap \Gamma^{(n)\star}$ for some $j$, which appears in the second part of step $n + 1$.
- Elements adjacent to $u_n^{j+1}$ in the matching $\mathfrak{M}_n^2$ are denoted by $\dot{v}_{n,1}^{j+1} \ldots \dot{v}_{n,d}^{j+1}$. The element $\dot{v}_{n,1}^{j+1}$ is a candidate for $v_{n,1}^{j+1}$.
- The fan $(u_n^{\perp}, \{v_{n,j}^{\perp} \,|1 \le j \le d - 1\})$ usually appears as a part of a fan of the matching $\mathfrak{M}_n^2$. We warn the reader that it is possible that $u_n^{\perp}$ does not exist.
- We assume that all of these elements are natural numbers and are ordered according to the standard ordering of the natural numbers.

## 4.2 The construction

We assume that $\Gamma = (U, V, E)$ is a bipartite graph satisfying the Hall's $d$-harem condition, such that:

- both $U$ and $V$ are identified with $\mathbb{N} \setminus \{0\}$;
- $\Gamma$ is fully reflected;
- $E$ does not contain edges of the form $(v, u_v)$.

We now describe an inductive construction which is the heart of the proof of our main theorem. At the end of each step we formulate some claims that certain graphs satisfy Hall's $d$-harem condition. They support further steps of the construction.

### 4.2.1 Step 1, part 1

We take $u_0$, the first element of the set $U$ (it is clear that $u_0 = 1$). Using Hall's harem theorem we find a perfect $(1, d)$-matching $\mathfrak{M}_0^1$. Let $v_{0,1}^0, \ldots, v_{0,d}^0$ be elements of $V$, ordered by its numbers, such that $(u_0, v_{0,i}^0) \in \mathfrak{M}_0^1$ for all $i \le d$. Define the fan $M_0^0$ as the set of edges $(u_0, v_{0,i})$ for $i \le d - 1$. Let $\Gamma^{(0)}(-u_0) := (U \setminus \{u_0\}, V \setminus \{v_{0,1}^0, \ldots, v_{0,d-1}^0\})$.

Note that the graph $\Gamma^{(0)}(-u_0) \setminus \{v_{0,d}^0\}$ satisfies Hall's $d$-harem condition, since it obviously has a perfect $(1, d)$-matching. Furthermore, the following (more complicated) statement holds too.

*Claim* 4.2.1. The graph $\Gamma^{(0)}(-u_0)$ satisfies Hall's $d$-harem condition.

See Lemma 4.4.1 for the proof of this claim.

Before the second part of step 1 let us discuss our local goals. Let the partial function $f_0$ correspond to $M_0$ and let $\Gamma^{(1)}$ be the graph obtained after step 1, i.e. the part of $\Gamma$ after removal $M_0$. We want to force that:

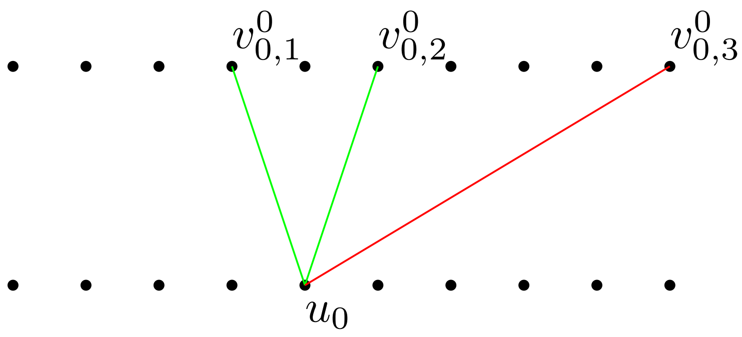



Figure 4.1: The first part of the first step, $\mathfrak{M}^1_0$-fan of $u_0$ in red and green, $M^0_0$ in green.

1. for all $n \in \mathsf{Dom}(f_0)$ there exists $i$ such that $f_0^i(n)$ is a periodic point, and
2. $\Gamma^{(1)}$ is $U^{(1)}$-reflected.

It is clear that (1)-(2) are satisfied if we add the edge $(u_{v^0_{0,1}}, v_{u_0})$ to the matching. This means that $f(v_{u_0}) = u_{v^0_{0,1}}$, i.e. $f_0^2(u_0) = f_0(f_0(v_{u_0})) = u_0$. We should organize it in a clever way.

### 4.2.2 Step 1, part 2

We denote $u^1_0 := u_{v^0_{0,1}}$ and aim to add the edge $(u^1_0, v_{u_0})$, to $M^1_0$. Note that $(u^1_0, v_{u_0}) \in \Gamma^{(0)}(-u_0)$, because $(u_0, v^0_{0,1})$ is in $\Gamma^{(0)}$ and the latter one is $U^{(0)}$-reflected. Since $\Gamma^{(0)}(-u_0)$ satisfies Hall's $d$-harem condition, it has a perfect $(1,d)$-matching $\mathfrak{M}^2_0$. We remind the reader that $\dot{v}^1_{0,1}, \dots, \dot{v}^1_{1,d}$ are elements of $V$, ordered by its numbers, such that $(u^1_0, \dot{v}^1_{0,i}) \in \mathfrak{M}^2_0$ for all $i \le d$. Since $v_{u_0}$ has the lowest number in $V$, there are two possible cases:

1. $v_{u_0} = \dot{v}^1_{0,1}$. We set $v^1_{0,i} := \dot{v}^1_{0,i}$, $1 \le i \le d-1$ (i.e. $v^1_{0,1} = v_{u_0}$).
2. $v_{u_0} \ne \dot{v}^1_{0,1}$. Then find the fan $(u, v_i) \in \mathfrak{M}^2_0$, $1 \le i \le d$, such that $v_{u_0} = v_1$ (here we assume that the ordering of $v_i$ corresponds to their indexes). We set:

   - $v^1_{0,1} := v_{u_0}$ and $v^1_{0,i} := \dot{v}^1_{0,i-1}$ for $2 \le i \le d-1$,

   and define a candidate for $\Gamma^{(1)\perp}$:

   - $\dot{u}^\perp_0 := u$;
   - $\dot{v}^\perp_{0,i-1} := v_i$ for $2 \le i \le d$.

In either case define the fan $M^1_0$ as the set of edges $(u^1_0, v^1_{0,i})$ for $1 \le i \le d-1$.

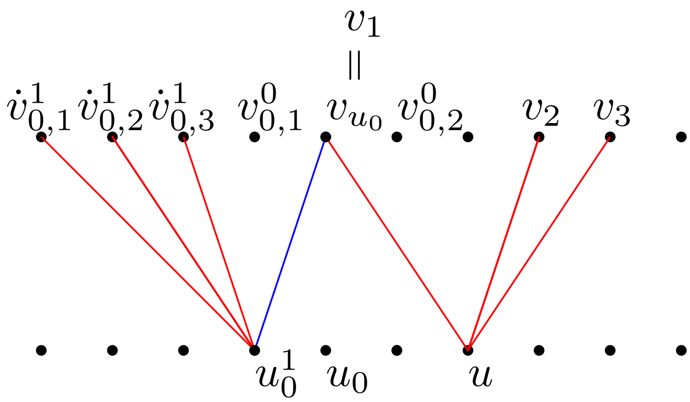



Figure 4.2: Step 1, part 2. $\mathfrak{M}_0^2$ is red. We aim to produce a cycle of length 2 in our matching by adding the edge $(u_0^1, v_{u_0})$. Since $v_{u_0}$ is matched with $u$, we force the situation from Figure 3.

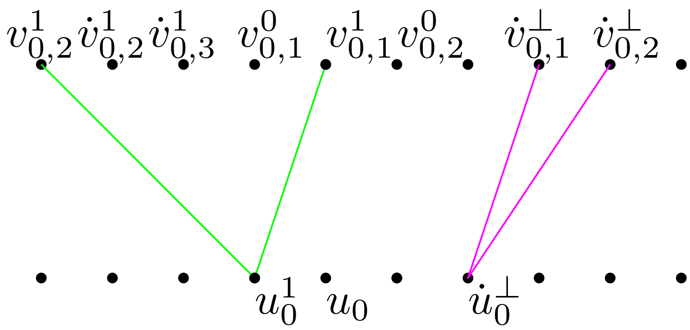



Figure 4.3: $M_0^1$ is green. It is possible that the purple fan consisting of edges $(\dot u_0^\perp, \dot v_{0,1}^\perp), (\dot u_0^\perp, \dot v_{0,2}^\perp)$ will be added to $\Gamma^{(1)\perp}$.

Put $M_0 = M_0^0 \cup M_0^1$. We obtain $\Gamma^{(1)}$ by removal of $M_0$ from $\Gamma$. Since $u_0, v_{u_0}, v_{0,1}^0$ and $u_0^1 = u_{v_{0,1}^0}$ have been removed, $\Gamma^{(1)}$ is $U^{(1)}$-reflected. It might turn out that it does not satisfy Hall's $d$-harem condition.

Let $\Gamma_0' = (U^{(1)} \setminus \{\dot u_0^\perp\}, V^{(1)} \setminus \{\dot v_{0,1}^\perp, ..., \dot v_{0,d-1}^\perp\})$. The following claim follows from Lemma 4.4.5 below.

*Claim* 4.2.2. At least one of $\Gamma_0'$ or $\Gamma^{(1)}$ satisfies Hall's $d$-harem condition.

If $\Gamma^{(1)}$ satisfies Hall's $d$-harem condition, set $\Gamma^{(1)\star} := \Gamma^{(1)}$ and $U^{(1)\perp} = \emptyset, V^{(1)\perp} = \emptyset$. If $\Gamma^{(1)}$ does not satisfy Hall's $d$-harem condition, set $\Gamma^{(1)\star} := \Gamma_0'$ and

- $u_0^\perp := \dot u_0^\perp$;
- $v_{0,i}^\perp := \dot v_{0,i}^\perp$, $1 \le i \le d-1$;
- $U^{(1)\perp} = \{u_0^\perp\}$;
- $V^{(1)\perp} = \{v_{0,i}^\perp : 1 \le i \le d-1\}$.

### 4.2.3 Step n+1, part 1

At the previous step we constructed graphs $\Gamma^{(n)}$ and $\Gamma^{(n)\star}$, where $\Gamma^{(n)}$ is $U^{(n)}$-reflected and $\Gamma^{(n)\star}$ satisfies Hall's $d$-harem condition. Let $\mathfrak{M}_n^1$ be an $(1,d)$-matching in $\Gamma^{(n)\star}$. Also note that since $|U^{(n)\perp} \setminus U^{(n-1)\perp}| \leq 1$, there are at most $n$ vertices $u_i^\perp$ in $U^{(n)\perp}$ (see Section 2.4 for the corresponding definition).

Take $u_n$, the first element of the set $U^{(n)}$. In order to define $M_n^0$ we have two possible cases:

1. There is $j$ such that $u_n = u_j^\perp \in U^{(n)\perp}$. Then we set $M_n^0$ to consist of all edges of the form $(u_j^\perp, v_{j,i}^\perp)$ and remove the fan with the root $u_j^\perp$ from $\Gamma^{(n)\perp}$. We redefine $U^{(n)\perp}$ and $V^{(n)\perp}$ accordingly (in particular $u_j^\perp$ is removed from $U^{(n)\perp}$).

2. If $u_n \notin U^{(n)\perp}$, then verify whether there is $j$ such that $(u_n, v_{n,j}^0) \in \mathfrak{M}_n^1$ and $(u_{v_{n,j}^0}, v_{u_n}) \in \Gamma^{(n)\perp}$. By the definition of $\Gamma^{(n)\perp}$ it can happen for at most one $j$. (Note here that it can also happen that $v_{u_n}$ is not even in $\Gamma^{(n)}$.) If there is such $j$, we set $M_n^0$ to consist of $(u_n, v_{n,j}^0)$ and $d-2$ of the remaining $d-1$ edges of the form $(u_n, v_{n,i}^0) \in \mathfrak{M}_n^1$ (excluding the one with the greatest index). In the case when the corresponding $j$ does not exist, we set $M_n^0$ to consist of edges $(u_n, v_{n,i}^0) \in \mathfrak{M}_n^1$ for $i \leq d-1$.

The following claim follows from Lemma 4.4.1 below.

*Claim* 4.2.3. Let $\Gamma^{(n)\star}(-u_n)$ be $\Gamma^{(n)}(-u_n) \cap \Gamma^{(n)\star}$. One of the following holds:

- $\Gamma^{(n)\star}(-u_n)$ satisfies Hall's $d$-harem condition;
- there exist some vertex $u_j^\perp \in U^{(n)\perp}$ such that the graph $\Gamma^{(n)\star}(-u_n, +u_j^\perp)$ satisfies Hall's $d$-harem condition.

If $\Gamma^{(n)\star}(-u_n)$ does not satisfy Hall's $d$-harem condition, let $u_j^\perp$ be an element from $U^{(n)\perp}$ realizing the second possibility of this claim. We remove the fan of $u_j^\perp$ with its leaves from $(U^{(n)\perp}, V^{(n)\perp})$ and then we put it into $\Gamma^{(n)\star}$. Thus the latter graph (and $U^{(n)\perp}, V^{(n)\perp}$) are updated. It is clear that now the redefined $\Gamma^{(n)\star}(-u_n)$ satisfies Hall's $d$-harem condition.

Before the second part of Step $n+1$ we describe the goals which we want to achieve after the step. We want:

1. the partial function $f_n$ corresponding to $\bigcup_{i=0}^{n} M_i$ has controlled sizes of its cycles, and

2. the graph $\Gamma^{(n+1)}$ obtained at the end of the step is $U^{(n+1)}$-reflected and the corresponding graph $\Gamma^{(n+1)\star}$ satisfies Hall's $d$-harem condition.

In order to achieve the first condition we will organize one of the the following properties:

(i) there is a sequence of vertices $u_n^0, u_n^1, u_n^2, \ldots, u_n^j$, $1 \le j \le n$ such that each edge $(u_n^i, v_{u_n^{i-1}})$ belongs to $M_n$, and for some $0 \le \ell \le j-1$ the sequence $(u_n^\ell, u_n^{\ell+1}, \ldots, u_n^j)$ is a cycle;

(ii) there is a sequence of vertices $(u_n^0, u_n^1, u_n^2, \ldots, u_n^j)$, $j \le n$ such that the edges of the form $(u_n^i, v_{u_n^{i-1}})$ belong to $M_n$, and $v_{u_n^j}$ is already adjacent to some edge from $\bigcup_{i=0}^{n-1} M_i$;

### 4.2.4 Step n+1, part 2

We begin by checking whether $v_{u_n}$ belongs to $\Gamma^{(n)\perp}$. If $v_{u_n} \in \Gamma^{(n)\perp}$ then we denote $u_n$ by $u_n^0$ and begin the following process of choosing the consecutive vertices $u_n^i$.

*First step of iteration.* Assume that for some $j_1, i$ we have $v_{u_n} = v_{j_1,i}^\perp \in V^{(n)\perp}$. Then we set
$u_n^1 = u_{j_1}^\perp$, $v_{n,k}^1 := v_{j_1,k}^\perp$, $k \le d-1$,
$M_n^1 := \{(u_n^1, v_{n,k}^1) : 1 \le k \le d-1\}$
and check whether $v_{u_n^1} \in \Gamma^{(n)\perp}(-u_n^0)$.
If it is so we repeat the iteration for $v_{u_n^1}$. Note that $v_{u_n^1} \in \Gamma^{(n)\perp}(-u_n^0, -u_n^1)$ then.

*Single step of iteration.* We verify if $v_{u_n^m} \in \Gamma^{(n)\perp}(-u_n^0, -u_n^1, \ldots, -u_n^m)$.
If it is so then for some $j_{m+1}, i$ we have $v_{u_n^m} = v_{j_{m+1},i}^\perp \in V^{(n)\perp}$. Define
$u_n^{m+1} = u_{j_{m+1}}^\perp$, $v_{n,k}^{m+1} := v_{j_{m+1},k}^\perp$, $1 \le k \le d-1$,
$M_n^{m+1} := \{(u_n^{m+1}, v_{n,k}^{m+1}) : 1 \le k \le d-1\}$
and we repeat the iteration for $v_{u_n^{m+1}}$. This ends the single iteration step.

Since $|U^{(n)\perp}| \le n$, the procedure ends after at most $n$ iterations. Therefore one of the following cases is realized for some $l \le n$:

1. $v_{u_n^l} \notin \Gamma^{(n)}$;

2. $v_{u_n^l} \in \Gamma^{(n)}$, but $v_{u_n^l} \notin \Gamma^{(n)}(-u_n^0, -u_n^1, \ldots, -u_n^l)$ (this case is impossible for $l = 0$);

3. $v_{u_n^l} \in \Gamma^{(n)\star}$.

In case (1) $v_{u_n^l}$ was already added to $M$ at preceding steps and condition (ii) described before this stage is satisfied. We finish step $n+1$, so no cycle is constructed at the step.

In case (2) we also finish step $n+1$. Note that the last iteration closes some cycle $(u_n^k, \ldots u_n^l)$ constructed at step $n+1$. The length of this cycle is not greater than $l+1$.

In case (3) we will construct a cycle $(u_n^l, u_n^{l+1})$ of length 2. This is the most complicated case. It will be considered in the next subsection. Before we start it we give an example of a cycle obtained by construction in case (2).

**Example.** The following pictures show how a cycle of length 3 can be obtained by this procedure in the graph satisfying Hall's 3-harem condition.

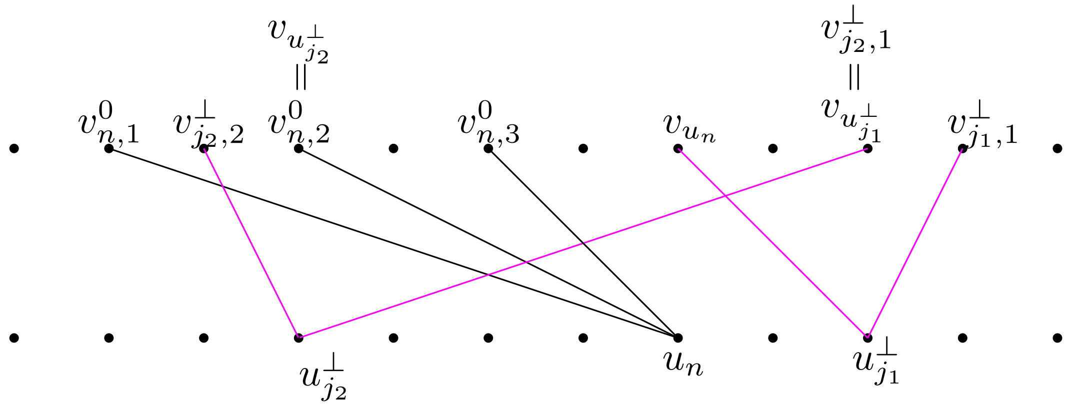



Figure 4.4: $\Gamma^{(n)\star}$ in black, $\Gamma^{(n)\perp}$ in purple.

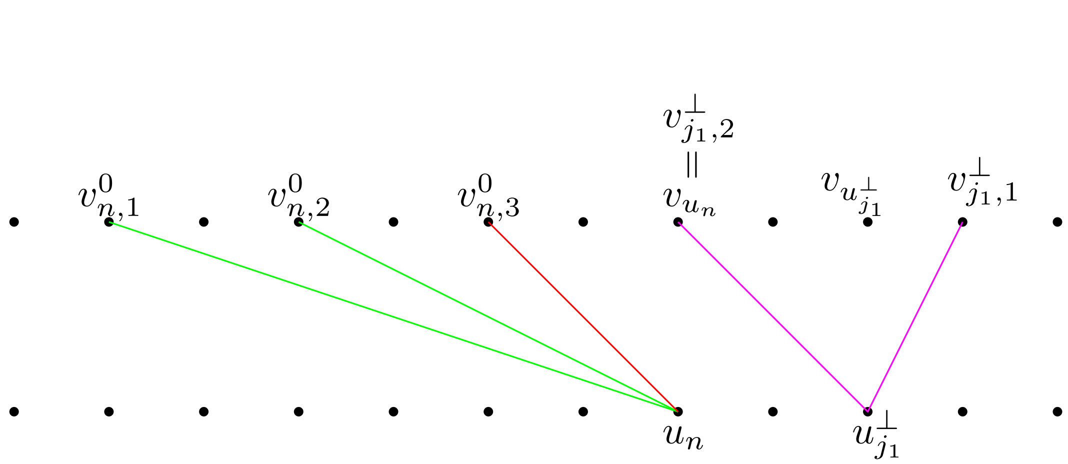



Figure 4.5: $\mathfrak{M}_1^n$ in red and green, $M_n^0$ in green. We have $v_{u_n} = v^\perp_{j_1,2}$.

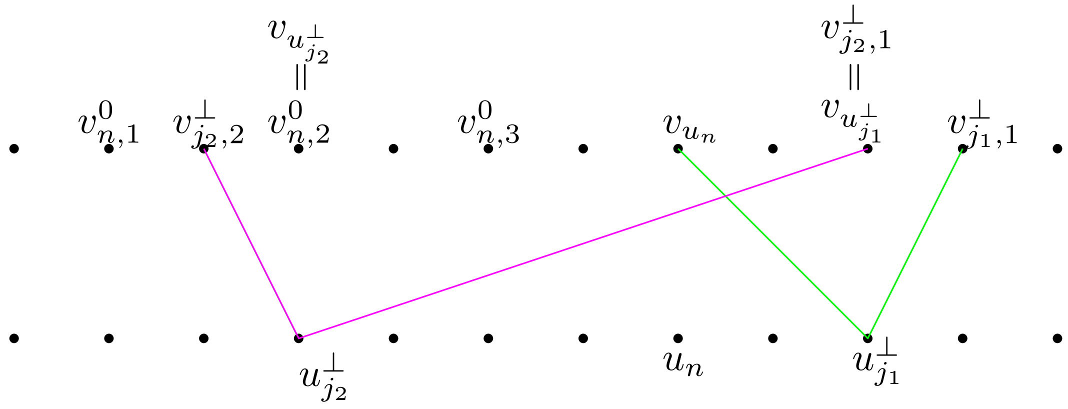



Figure 4.6: $M_n^1$ in green. We have $v_{u^\perp_{j_1}} = v^\perp_{j_2,2}$. Moreover $v_{u^\perp_{j_2}} = v^0_{n,2}$.

To show cycle on one picture, we take 4 copies of the set $\mathbb{N}$. First and second correspond to sets $V$ and $U$ at the part 1 of the step. Second and third correspond to the sets $V$ and $U$ at the first iteration step and third and fourth correspond to the sets $U$ and $V$ at the second step of iteration.

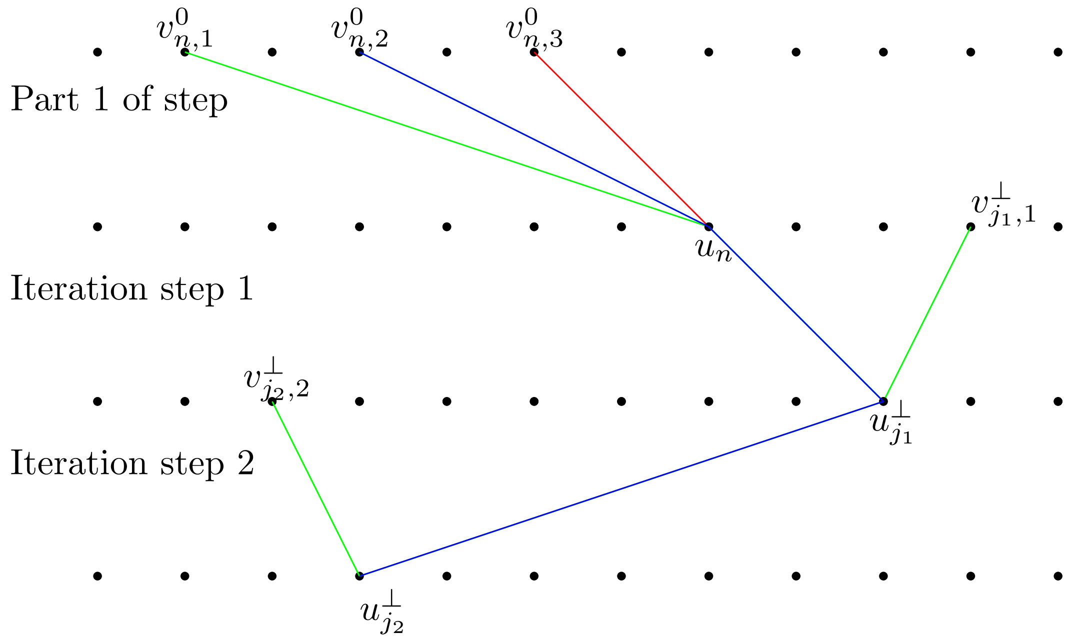



Figure 4.7: Resulting $M_n$ with the obtained cycle of length 3 marked in blue. Red edge is the edge from $\mathfrak{M}_1^n$ that was not added to $M_n^0$.

### 4.2.5 Case (3) and the end

In case (3) we will construct a cycle $(u_n^l, u_n^{l+1})$ of length 2 as follows.

We start with a new term $u_n^{l+1} := u_{v^l_{n,1}}$.

Since the edge $(u_n^l, v_{n,1}^l)$ is in $\Gamma^{(n)}$ and $v_{u_n^l}$ is not in $\Gamma^{(n)}(-u_n, -u_n^1, \ldots, -u_n^l)$, applying $U^{(n)}$-reflectedness of $\Gamma^{(n)}$ we see $(u_n^{l+1}, v_{u_n^l}) \in \Gamma^{(n)}(-u_n, -u_n^1, \ldots, -u_n^l)$.

Observe that since $u_{j_1}^{\perp}, \dots, u_{j_l}^{\perp}$ are in $U^{(n)\perp}$,

$$\Gamma^{(n)}(-u_n, -u_n^1, \dots, -u_n^l) \cap \Gamma^{(n)\star} = \Gamma^{(n)}(-u_n) \cap \Gamma^{(n)\star} = \Gamma^{(n)\star}(-u_n).$$

We also remind the reader that by the first part of this step the graph $\Gamma^{(n)\star}(-u_n)$ satisfies Hall's $d$-harem condition. Thus we can find an $(1, d)$-matching $\mathfrak{M}_n^2$ in $\Gamma^{(n)\star}(-u_n)$. Let us fix it.

We now check whether there is $j$ with $u_n^{l+1} = u_j^{\perp}$.
If it is so, we set $\dot{v}_{n,i}^{l+1} := v_{j,i}^{\perp}$ for $1 \le i \le d-1$.
If there is no $j$ such that $u_n^{l+1} = u_j^{\perp}$, then $\dot{v}_{n,1}^{l+1} \dots \dot{v}_{n,d}^{l+1}$ will denote the elements adjacent to $u_n^{l+1}$ under $\mathfrak{M}_n^2$.
There are two cases:

A) $(u_n^{l+1}, v_{u_n^l}) \in \mathfrak{M}_n^2$, i.e. $v_{u_n^l} = \dot{v}_{n,k}^{l+1}$ for some $1 \le k \le d$. In this case $u_n^{l+1}$ cannot be $u_j^{\perp}$ for any $j$.

B) $(u_n^{l+1}, v_{u_n^l}) \notin \mathfrak{M}_n^2$, i.e. there exists some $u \in \Gamma^{(n)\star}(-u_n)$, such that $v_{u_n^l} = v_k$ for some $1 \le k \le d$, where $v_1 \dots v_d$ denote the elements adjacent to $u$ under $\mathfrak{M}_n^2$. In this case it is possible that $u_n^{l+1}$ coincides with some $u_j^{\perp}$.

In case A) we produce a cycle of length 2 by including the pair $(u_n^{l+1}, v_{u_n^l})$ into $M_n^{l+1}$. In fact we include it into $M_n^{l+1}$ together with a fan with the root $u_n^{l+1}$ and $(d-2)$ leaves taken among $\dot{v}_{n,i}^{l+1}$. To be precise we organize it as follows. If $k = d$, we set $v_{n,i-1}^{l+1} := \dot{v}_{n,i}^{l+1}$ for $2 \le i \le d$. If $k \ne d$ we set $v_{n,i}^{l+1} := \dot{v}_{n,i}^{l+1}$ for $1 \le i \le d-1$. The set $M_n^{l+1}$ consists of edges $(u_n^{l+1}, v_{n,i}^{l+1})$ for $1 \le i \le d-1$. The procedure is finished.

In case B) we produce a cycle of length 2 by including the pair $(u_n^{l+1}, v_{u_n^l})$ into $M_n^{l+1}$. In fact, as in the previous case we include it into $M_n^{l+1}$ together with a fan with the root $u_n^{l+1}$ and $(d-2)$ leaves taken among $\dot{v}_{n,i}^{l+1}$, which will be denoted $v_{n,1}^{l+1} = v_{u_n^l}$ and $v_{n,i+1}^{l+1} = \dot{v}_{n,i}^{l+1}$, $1 \le i \le d-2$. We define $\dot{u}_n^{\perp} := u$ and rename the remaining $d-1$ vertices $v_j$ to $\dot{v}_{n,i}^{\perp}$. The procedure is finished. It is worth noting here that if $u_n^{l+1}$ coincides with some $u_j^{\perp}$, then the vertex $\dot{v}_{n,d}^{l+1}$ does not exist, i.e the only vertex $\dot{v}_{n,d-1}^{l+1}$ from the fan of $u_n^{l+1}$ would be outside of $M_n^{l+1}$.

Let $M_n = \bigcup_{k=0}^{l+1} M_n^k$. We obtain the graph $\Gamma^{(n+1)}$ from $\Gamma^{(n)}$ by removal of $M_n$-fans. Since for each $u \in U^{(n)} \setminus U^{(n+1)}$ the element $v_u$ is also removed, then the graph is $U^{(n+1)}$-reflected.

In cases (1), (2) and (3)A) we set $\mathfrak{U} := U^{(n+1)} \setminus U^{(n)\perp}, \mathfrak{V} := V^{(n+1)} \setminus V^{(n)\perp}$. Let $\mathfrak{T} = (\mathfrak{U}, \mathfrak{V})$ and $\dot{\Gamma}^{(n)\perp} := \Gamma^{(n)\perp} \cap \Gamma^{(n+1)}$. We note here that in cases (1) and (2) only elements of $\Gamma^{(n)\perp}$ were added to $M_n$ at this part of the step. Therefore in these cases $\mathfrak{T}$ coincides with $\Gamma^{(n)\star}(-u_n)$ and satisfies Hall's $d$-harem condition.

In case (3)B) we set $\mathfrak{U} := U^{(n+1)} \setminus (U^{(n)\perp} \cup \{\dot{u}_n^\perp\}), \mathfrak{V} := V^{(n+1)} \setminus (V^{(n)\perp} \cup \{\dot{v}_{n,i}^\perp : 1 \leq i \leq d-1\})$. Let $\mathfrak{T} = (\mathfrak{U}, \mathfrak{V})$ and $\dot{U}^{(n)\perp} := (U^{(n)\perp} \cup \{\dot{u}_n^\perp\}) \cap U^{(n+1)}$. We define $\dot{V}^{(n)\perp}$ accordingly: $\dot{V}^{(n)\perp} := (V^{(n)\perp} \cup \{\dot{v}_{n,k}^\perp : 1 \leq k \leq d-1\}) \cap V^{(n+1)}$, and set

$$\dot{\Gamma}^{(n)\perp} := (\dot{U}^{(n)\perp}, \dot{V}^{(n)\perp}).$$

The following claim follows from Lemma 4.4.5 below.

*Claim* 4.2.4. At least one of the following holds:

- $\mathfrak{T}$ satisfies Hall's $d$-harem condition;
- there exists some vertex $u_j^\perp \in \dot{U}^{(n)\perp}$ such that the graph $\mathfrak{T}(+u_j^\perp)$ satisfies Hall's $d$-harem condition.
- there exist vertices $u_i^\perp, u_j^\perp \in \dot{U}^{(n)\perp}$ such that the graph $\mathfrak{T}(+u_i^\perp, +u_j^\perp)$ satisfies Hall's $d$-harem condition.

Depending on the output of this claim we define the final output of step $n+1$. In the case (3)B) (i.e. $\dot{u}_n^\perp$ exists) if $u_i^\perp \neq \dot{u}_n^\perp \neq u_j^\perp$ (for any output of the claim), we set $u_n^\perp := \dot{u}_n^\perp$ and $v_{n,k}^\perp := \dot{v}_{n,k}^\perp$ for $1 \leq k \leq d-1$. Otherwise, or in the remaining cases $u_n^\perp, v_{n,k}^\perp$ are not defined.

In the first case of the claim we set $\Gamma^{(n+1)\star} := \mathfrak{T}$ and

$$U^{(n+1)\perp} := (U^{(n)\perp} \cup \{u_n^\perp\}) \cap U^{(n+1)},$$

$$V^{(n+1)\perp} := (V^{(n)\perp} \cup \{v_{n,k}^\perp : 1 \leq k \leq d-1\}) \cap V^{(n+1)}.$$

In the second case we set $\Gamma^{(n+1)\star} := \mathfrak{T}(+u_j^\perp)$and

$$U^{(n+1)\perp} := ((U^{(n)\perp} \setminus \{u_j^\perp\}) \cup \{u_n^\perp\}) \cap U^{(n+1)},$$

$$V^{(n+1)\perp} := ((V^{(n)\perp} \setminus \{v_{j,k}^\perp : 1 \leq k \leq d-1\}) \cup \{v_{n,k}^\perp : 1 \leq k \leq d-1\}) \cap V^{(n+1)}.$$

In the third case we set $\Gamma^{(n+1)\star} := \mathfrak{T}(+u_i^\perp, +u_j^\perp)$ and

$$U^{(n+1)\perp} := ((U^{(n)\perp} \setminus \{u_i^\perp, u_j^\perp\}) \cup \{u_n^\perp\}) \cap U^{(n+1)},$$

$$V^{(n+1)\perp} := ((V^{(n)\perp} \setminus \{v_{i,k}^\perp, v_{j,k}^\perp : 1 \leq k \leq d-1\}) \cup \{v_{n,k}^\perp : 1 \leq k \leq d-1\}) \cap V^{(n+1)}.$$

## 4.3 Technical Lemmas

The notation used in this section does not correspond the construction above.

Throughout this section $\Gamma = (U, V, E)$ denotes a bipartite graph and $\Gamma^\perp = (U^\perp, V^\perp, E^\perp)$ denotes its subgraph.
The graph $\Gamma^\star = (U^\star, V^\star, E^\star)$ is an induced subgraph of $\Gamma$ such that

$$U^\star \cap U^\perp = \emptyset = V^\star \cap V^\perp.$$

Below we always assume that $d$ is a natural number greater than 1.

The following situation will arise several times in our arguments in Section 4.4. Let $X$ be a subset of $V$ such that

$$|N_\Gamma(X)| \geq \frac{1}{d}|X|,$$

but

$$|N_\Gamma(X) \setminus U^\perp| < \frac{1}{d}|X|.$$

Thus we can conclude that $N_\Gamma(X) \cap U^\perp \neq \emptyset$.

The following lemma describes typical circumstances which lead to this situation.

**Lemma 4.3.1.** *Let $\Gamma = (U, V, E)$ be $U$-reflected and let $\Gamma^\star$ be a subgraph of $\Gamma$ induced by the sets of vertices $U^\star = U \setminus U^\perp$, $V^\star = V \setminus V^\perp$. Assume that $\Gamma^\star$ satisfies Hall's $d$-harem condition and assume that for each $Y \subset U^\perp$ we have $|N_{\Gamma^\perp}(Y)| \geq (d-1)|Y|$.*

*Then for any $X \subseteq V$ we have $|N_\Gamma(X)| \geq (d-1)|X|$. In particular $|N_\Gamma(X)| \geq \frac{1}{d}|X|$*

*Proof.* Let $U_X := \{u \in U : v_u \in X\}$. By $U$-reflectedness of $\Gamma$ we have $|U_X| = |X|$. Consider the sets

$$U_X^\star := U_X \setminus U^\perp$$

and

$$U_X^\perp := U_X \cap U^\perp.$$

Using $U$-reflectedness of $\Gamma$ again, we see

$$|N_\Gamma(X)| \geq |N_{\Gamma^\star}(U_X^\star)| + |N_\Gamma(U_X^\perp) \cap V^\perp|.$$

Since Hall's $d$-harem condition is satisfied for any subset of $U^\star$,

$$|N_{\Gamma^\star}(U_X^\star)| \geq d|U_X^\star|.$$

Moreover we have $|N_\Gamma(U_X^\perp) \cap V^\perp| \geq (d-1)|U_X^\perp|$. Therefore

$$|N_{\Gamma^\star}(U_X^\star)| + |N_\Gamma(U_X^\perp) \cap V^\perp| \geq (d-1)|U_X|.$$

Since $d \geq 2$ it follows that

$$|N_\Gamma(X)| \geq (d-1)|X| \geq \frac{1}{d}|X|.$$

$\square$

This lemma cannot be applied in the situation of Part 1 of a step of the main construction, since some vertices from a $U$-reflected graph are removed. In that case, we have a graph $\Gamma$ which is not $U$-reflected, but for some $v \in V$ its subgraph $\Gamma(-v)$ is $U$-reflected. Because of this change, in the following lemmas both $\Gamma^\perp$ and $\Gamma^\star$ are subgraphs of $\Gamma(-v)$.

**Lemma 4.3.2.** *Let $v$ be a vertex from $V$ and $\Gamma(-v) = (U, V(-v))$. Assume that $\Gamma(-v)$ is $U$-reflected. Let $\Gamma^\perp = (U^\perp, V^\perp, E^\perp)$ be a subgraph of $\Gamma(-v)$ and let $\Gamma^\star$ be defined as the subgraph of $\Gamma(-v)$ induced by the sets of vertices $U^\star := U \setminus U^\perp$, $V^\star := V(-v) \setminus V^\perp$.*

*Assume that $\Gamma^\star$ satisfies Hall's $d$-harem condition and assume that for each $Y \subset U^\perp$ we have $|N_\Gamma(Y) \cap V^\perp| \geq (d-1)|Y|$.*

*Then for any $X \subseteq V \setminus \{v\}$ the inequality $|N_\Gamma(X)| \geq (d-1)|X| - 1$ holds. Moreover, the equality $|N_\Gamma(X)| = (d-1)|X| - 1$ can happen only if $v \in N_\Gamma(U_X)$ and $N_{\Gamma^\star}(U_X) = \emptyset$, where $U_X := \{u \in U : v_u \in X\}$.*

*Proof.* By $U$-reflectedness of $\Gamma(-v)$ we have $|U_X| = |X|$ and

$$|N_\Gamma(X)| + 1 \geq |N_{\Gamma(-v)}(U_X)|.$$

Furthermore, if $v \notin N_\Gamma(U_X)$, then

$$|N_\Gamma(X)| \geq |N_{\Gamma(-v)}(U_X)|.$$

Consider the sets

$$U_X^\star := U_X \setminus U^\perp \text{ and } U_X^\perp := U_X \cap U^\perp.$$

Applying the argument of Lemma 4.3.1 to $\Gamma(-v)$ we obtain

$$|N_{\Gamma(-v)}(U_X)| \geq |N_{\Gamma^\star}(U_X^\star)| + |N_\Gamma(U_X^\perp) \cap V^\perp| \geq (d-1)|U_X|.$$

Thus $|N_\Gamma(X)| \geq |N_{\Gamma(-v)}(U_X)| - 1 \geq (d-1)|X| - 1$.

Observe that $|N_{\Gamma(-v)}(U_X)| = (d-1)|U_X|$ only if $N_{\Gamma(-v)}(U_X) \subseteq V^\perp$, i.e. $N_{\Gamma^\star}(U_X) = \emptyset$. If $v \notin N_\Gamma(U_X)$ then

$$|N_\Gamma(X)| \geq |N_{\Gamma(-v)}(U_X)| \geq (d-1)|X|.$$

Therefore $|N_\Gamma(X)| = (d-1)|X| - 1$ only if $v \in N_\Gamma(U_X)$ and $N_{\Gamma^\star}(U_X) = \emptyset$.

□

The following definition will be useful in the proofs of the lemmas of the next section.

**Definition 4.3.3.** Let $\Gamma := (U, V, E)$ be a bipartite graph, and $\Gamma^\star = (U^\star, V^\star, E^\star)$ be a subgraph in $\Gamma$ satisfying Hall's $d$-harem condition. Assume that $M$ is a corresponding perfect $(1, d)$-matching in $\Gamma^\star$.

Let $X$ be a subset of $V$ and let $x \in X$. We say that $x$ *is accessible from $y \in N_\Gamma(X)$ through $X$ by matching $M$*, (denoted by $y \xhookrightarrow{M,X} x$) if there exist two sequences of vertices $\{v_0', \ldots, v_n'\} \subset X$ and $\{u_0', \ldots, u_{n-1}'\} \subset N_\Gamma(X)$ such that

- $v_n' = x$;
- $(u_i', v_i') \in M$ and $(u_i', v_{i+1}') \in E \setminus M$, where $i < n$;
- $(y, v_0') \in E$.

In the technical lemma below we use the following notion of Kierstead, [Kie83]. The subset $X$ of $U$ (resp. of $V$) is called *connected* if for all $x, x' \in X$ there exist a path $x = p_0, p_1, \ldots, p_k = x'$ in $\Gamma$ such that $p_i \in X \cup N_\Gamma(X)$ for all $i \leq k$. Note here that this definition concerns only bipartite graphs.

**Lemma 4.3.4.** *Let $\Gamma = (U, V, E)$ be a bipartite graph. Let $\Gamma^\star = (U^\star, V^\star, E^\star)$ be a subgraph of $\Gamma$ satisfying Hall's d-harem condition and $M$ be a corresponding perfect $(1, d)$-matching.*

*Assume that*

- $\widehat{v} \in N_\Gamma(U^\star) \setminus V^\star$;
- *$X$ is a minimal connected subset in $V^\star \cup \{\widehat{v}\}$ with $|N_\Gamma(X)| < \frac{1}{d}|X|$;*
- $\widehat{u} \in N_\Gamma(X) \setminus U^\star$;

*Then $\widehat{u} \xhookrightarrow{M,X} \widehat{v}$.*

*Proof.* It is clear that $\widehat{v} \in X$. Let $X'$ denote the subset of all elements of $X$ that are accessible from $\widehat{u}$ through $X$ by the matching $M$. Assume that $\widehat{v} \notin X'$. Since $\widehat{u} \in N_\Gamma(X)$, we then see that there exist $u_0', v_0'$ such that $\widehat{u}, u_0' \in N_\Gamma(v_0')$ and $(u_0', v_0') \in M$. Therefore $X' \neq \emptyset$. We will show that $|N_\Gamma(X \setminus X')| < \frac{1}{d}|X \setminus X'|$.

Let $|X'| = l$ and let $U_M(X') = \{u \in U \mid (\exists v \in X')(u, v) \in M\}$. Since elements of $X \setminus X'$ are not accessible from $\widehat{u}$ through $X$ by the matching $M$, then $N_\Gamma(X \setminus X') \cap U_M(X') = \emptyset$.

Using this we see that

$$|N_\Gamma(X \setminus X')| \leq |N_\Gamma(X)| - |U_M(X')|.$$

Since $M$ is a $(1, d)$-matching, each element of $U_M(X')$ can be matched with at most $d$ elements from $X'$. Therefore $|U_M(X')| \geq \lceil \frac{l}{d} \rceil$ and

$$|N_\Gamma(X \setminus X')| \leq |N_\Gamma(X)| - |U_M(X')| \leq |N_\Gamma(X)| - \lceil \frac{l}{d} \rceil < \frac{1}{d}(|X| - l) = \frac{1}{d}|X \setminus X'|.$$

So $|N_\Gamma(X \setminus X')| < \frac{1}{d}|X \setminus X'|$ and $X \setminus X'$ is smaller than $X$, a contradiction with the choice of the latter. As a result $\widehat{v} \in X'$, and consequently $\widehat{u} \xhookrightarrow{M,X} \widehat{v}$. □

## 4.4 Graphs constructed in parts 1 and 2 of each step satisfy Hall's $d$-harem condition

In this section the notation is taken from the construction of Section 3.

### 4.4.1 Part 1

**Lemma 4.4.1.** *For any $n$ one of the following statements holds:*

- *$\Gamma^{(n)\star}(-u_n)$ satisfies Hall's d-harem condition;*
- *there exist some vertex $u_j^\perp \in U^{(n)\perp}$ such that the graph $\Gamma^{(n)\star}(-u_n, +u_j^\perp)$ satisfies Hall's d-harem condition.*

*Remark* 4.4.2. If $U^{(n)\perp} = \emptyset$, then by Lemma 4.4.1 the graph $\Gamma^{(n)\star}(-u_n)$ satisfies Hall's $d$-harem condition. In particular, this lemma proves Claim 4.2.1.

*Proof.* We know that the graph $\Gamma^{(n)\star}$ satisfies Hall's $d$-harem condition. Let $\mathfrak{v}$ denote the only vertex from the set $\{v^0_{n,1}, \dots, v^0_{n,d}\}$ that belongs to $\Gamma^{(n)\star}(-u_n)$. The choice of $u_n, v^0_{n,1}, \dots, v^0_{n,d}$ ensures that the graph $\Gamma^{(n)\star}(-u_n, -\mathfrak{v})$ satisfies Hall's $d$-harem condition as well. Since $U^{(n)\star}(-u_n) = U^{(n)\star}(-u_n, -\mathfrak{v})$, for any $X \subset U^{(n)\star}(-u_n)$ we have

$$|N_{\Gamma^{(n)\star}(-u_n)}(X)| \geq |N_{\Gamma^{(n)\star}(-u_n,-\mathfrak{v})}(X)| \geq d|X|.$$

The corresponding property also holds for all subsets of $V^{(n)\star}(-u_n)$ that do not contain $\mathfrak{v}$. Therefore if $\Gamma^{(n)\star}(-u_n)$ does not satisfy Hall's $d$-harem condition, then a witness of this is a finite $X \subset V^{(n)\star}(-u_n)$ which contains $\mathfrak{v}$.

If such $X$ is not connected, then the neighbourhood of $X$ is a disjoint union of the neighbourhoods of the connected subsets of $X$. Therefore there exists a minimal connected set $X$ such that $\mathfrak{v} \in X \subset V^{(n)\star}(-u_n)$ and $|N_{\Gamma^{(n)\star}(-u_n)}(X)| < \frac{1}{d}|X|$. Choose $X$ with these properties. We now want to prove that there is some $u_j^\perp \in N_{\Gamma^{(n)}(-u_n)}(X)$. By the inequality of the previous line it suffices to show that

$$|N_{\Gamma^{(n)}(-u_n)}(X)| \geq \frac{1}{d}|X|. \tag{4.1}$$

First, we consider the case when $X = \{\mathfrak{v}\}$. The vertex $u_\mathfrak{v}$ either belongs to $U^{(n)\star}(-u_n)$ or to $U^{(n)\perp}$. Since $\Gamma^{(n)\perp}$ consists of $(1, d-1)$-fans, in each case we have $|N_{\Gamma^{(n)}(-u_n)}(u_\mathfrak{v})| \geq d - 1$. Moreover, the equality

$$|N_{\Gamma^{(n)}(-u_n)}(u_\mathfrak{v})| = d - 1 \tag{4.2}$$

implies that $v_{u_n} \notin \Gamma^{(n)}$. Indeed, if $v_{u_n} \in \Gamma^{(n)}$ then $v_{u_n} \in N_{\Gamma^{(n)}(-u_n)}(u_\mathfrak{v})$ by reflectedness. On the other hand by the induction assumption, equality (4.2) implies that $(u_\mathfrak{v}, v_{u_n}) \in \Gamma^{(n)\perp}$. Thus $(u_n, \mathfrak{v})$ would be added to the matching $M$ at the first part of $n+1$-st step of the construction, i.e. $\mathfrak{v} \notin \Gamma^{(n)\star}(-u_n)$, a contradiction.

Our next observation is

$$|N_{\Gamma^{(n)}(-u_n)}(\mathfrak{v})| \geq |N_{\Gamma^{(n)}(-u_n)}(u_\mathfrak{v})| - 1.$$

This follows by $U^{(n)}$-reflectedness of $\Gamma^{(n)}$: $v_{u_n}$ is the only possible element adjacent to $u_\mathfrak{v}$ that does not have the left copy in $U^{(n)}(-u_n)$. Additionally note that the equality

$$|N_{\Gamma^{(n)}(-u_n)}(\mathfrak{v})| = |N_{\Gamma^{(n)}(-u_n)}(u_\mathfrak{v})| - 1 \tag{4.3}$$

holds only if $v_{u_n} \in \Gamma^{(n)}(-u_n)$.

Therefore equalities (4.2), (4.3) are not consistent, i.e.:

$$|N_{\Gamma^{(n)}(-u_n)}(\mathfrak{v})| > (d-1) - 1 \text{ and } |N_{\Gamma^{(n)}(-u_n)}(\mathfrak{v})| \geq \frac{1}{d}.$$

Since $X$ is a singleton,

$$|N_{\Gamma^{(n)}(-u_n)}(X)| \geq \frac{1}{d}|X|.$$

As we mentioned above this means that there is some $u_j^\perp \in N_{\Gamma^{(n)}(-u_n)}(X)$.

Assume $X \neq \{\mathfrak{v}\}$. Then $|N_{\Gamma^{(n)\star}(-u_n)}(X)| \geq 1$, and we have $|X| > d$ by the assumption $|N_{\Gamma^{(n)\star}(-u_n)}(X)| < \frac{1}{d}|X|$. Consider this case. Let $U_X := \{u \in U : v_u \in X\}$ in $\Gamma^{(n)}(-u_n)$. Applying the fact that $V^{(n)}$ is a subset of the copy of $U^{(n)}$ we arrive at two possibilities:

(i) $v_{u_n} \notin X$ and $|X| = |U_X|$;

(ii) $v_{u_n} \in X$ and $|X| = |U_X| + 1$.

We will show that in either case the inequality (4.1) follows from Lemma 4.3.2.

Let $\mathfrak{S}$ denote the fan from the matching $\mathfrak{M}_n^1$ containing $v_{u_n}$ and $\mathfrak{S}'$ denote $\mathfrak{S}$ with $v_{u_n}$ removed. The conditions of Lemma 4.3.2 are satisfied if we consider $\Gamma^{(n)}(-u_n)$ as $\Gamma$ in that lemma, $v_{u_n}$ as $v$, and $\Gamma^{(n)\perp}(+\mathfrak{v}) \cup \mathfrak{S}'$ as $\Gamma^{\perp}$. Indeed, $\Gamma^{(n)}(-u_n, -v_{u_n})$ is $U^{(n)}(-u_n, -v_{u_n})$-reflected. Moreover, the corresponding graph $\Gamma^{\star}$ from the lemma is obtained by removal of $\Gamma^{(n)\perp}(+\mathfrak{v}) \cup \mathfrak{S}'$ from $\Gamma^{(n)}(-u_n, -v_{u_n})$, therefore it is the same as $\Gamma^{(n)\star}(-u_n, -\mathfrak{v})$ with $\mathfrak{S}$ removed. Since $\mathfrak{S}$ is a fan from a perfect $(1, d)$-matching in $\Gamma^{(n)\star}(-u_n, -\mathfrak{v})$, we know that $\Gamma^{\star}$ satisfies Hall's $d$-harem condition.

Therefore in case (i) by Lemma 4.3.2 we have

$$|N_{\Gamma^{(n)}(-u_n)}(X)| \geq (d-1)|X| - 1.$$

This fact combined with inequalities $d \geq 2$ and $|X| > d$ implies

$$|N_{\Gamma^{(n)}(-u_n)}(X)| \geq \frac{1}{d}|X|.$$

In case (ii) by Lemma 4.3.2 we have:

$$|N_{\Gamma^{(n)}(-u_n)}(X \setminus \{v_{u_n}\})| \geq (d-1)(|X|-1) - 1,$$
$$\text{i.e. } |N_{\Gamma^{(n)}(-u_n)}(X)| \geq (d-1)(|X|-1) - 1.$$

Let us show that the inequality is strict. Indeed, by Lemma 4.3.2 the equality

$$|N_{\Gamma^{(n)}(-u_n)}(X \setminus \{v_{u_n}\})| = (d-1)(|X|-1) - 1$$

implies that $v_{u_n} \in N_{\Gamma^{(n)}(-u_n)}(U_{X\setminus\{v_{u_n}\}})$ and $N_{\Gamma^{(n)\star}(-u_n,-\mathfrak{v})}(U_{X\setminus\{v_{u_n}\}}) = \emptyset$. Therefore $v_{u_n} \in V^{(n)\perp}$. On the other hand, $v_{u_n} \in X \subset \Gamma^{(n)\star}$, a contradiction with the choice of $X$.

Again, inequalities $d \geq 2$ and $|X| > d$ imply

$$|N_{\Gamma^{(n)}(-u_n)}(X)| \geq (d-1)(|X|-1) \geq \frac{1}{d}|X|.$$

Therefore, the assumption $|N_{\Gamma^{(n)\star}(-u_n)}(X)| < \frac{1}{d}|X|$ implies

$$N_{\Gamma^{(n)}(-u_n)}(X) \cap U^{(n)\perp}(-u_n) \neq \emptyset,$$

i.e. there exists some $u_j^{\perp} \in N_{\Gamma^{(n)}(-u_n)}(X)$.

There are $d-1$ vertices adjacent to $u_j^{\perp}$ in $\Gamma^{(n)\perp}$. We denote them by $v_{j,1}^{\perp}, \ldots, v_{j,d-1}^{\perp}$.

Since $\mathfrak{M}_n^1$ is a perfect $(1, d)$-matching in the graph $\Gamma^{(n)\star}(-u_n, -\mathfrak{v})$, which in turn is a subgraph of the bipartite graph $\Gamma^{(n)}(-u_n)$, we can use Lemma 4.3.4 for $\mathfrak{v}$, $X$, $u_j^{\perp}$ and arrive at $u_j^{\perp} \xrightarrow{\mathfrak{M}_n^1, X} \mathfrak{v}$. This gives us a sequences of vertices $\{v_0', \ldots, v_n'\}, \{u_0', \ldots, u_{n-1}'\}$ as in Definition 4.3.3.

In order to prove that the graph $\Gamma^{(n)\star}(-u_n, +u_j^{\perp})$ satisfies Hall's $d$-harem condition, we construct a perfect $(1, d)$-matching in it. We set

$$M' := (\mathfrak{M}_n^1 \setminus \{(u_n, v_{n,1}^0), \ldots, (u_n, v_{n,d}^0), (u_0', v_0'), \ldots (u_{n-1}', v_{n-1}')\}) \cup$$

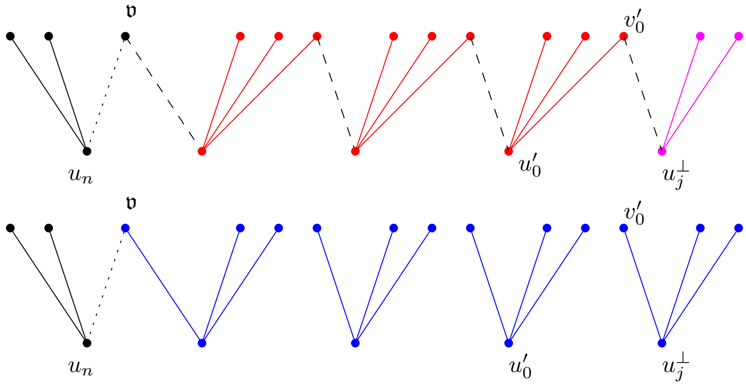



Figure 4.8: We replace the red fans in the matching $\mathfrak{M}_n^1$ by the blue fans to obtain the matching $M'$ in $\Gamma^{(n)\star}(-u_n, +u_j^\perp)$.

$$\{(u_j^\perp, v_{j,1}^\perp), \dots, (u_j^\perp, v_{j,d-1}^\perp), (u_j^\perp, v'_0), (u'_0, v'_1), \dots (u'_{n-1}, v'_n)\},$$

where $v'_n = \mathfrak{v} \in \{v_{n,1}^0, \dots, v_{n,d}^0\}$.

We remind the reader that $\mathfrak{M}_n^1$ is a perfect $(1, d)$-matching in the graph $\Gamma^{(n)\star}$. We have obtained $M'$ by removing $d$ edges adjacent to $u_n$, adding $d$ edges incident to $u_j^\perp$, and the following replacement: for each of $u'_i$ we replace one edge incident to it by another incident edge (then $\mathfrak{v}$ becomes adjacent to one edge in $M'$). It follows that the matching $M'$ is a perfect $(1, d)$-matching in the graph $\Gamma^{(n)\star}(-u_n, +u_j^\perp)$. Therefore that graph satisfies Hall's $d$-harem condition. $\square$

### 4.4.2 Notation used in proof of Lemma 4.4.5

Before stating the second lemma, we remind the reader the notation used in it.

- In Case 3A
  - $\mathfrak{U} := U^{(n+1)} \setminus U^{(n)\perp}$
  - $\mathfrak{V} := V^{(n+1)} \setminus V^{(n)\perp}$
  - $\mathfrak{T} := (\mathfrak{U}, \mathfrak{V}, \mathfrak{E})$, where $\mathfrak{E}$ is induced in $\Gamma$ by the sets of vertices $\mathfrak{U}, \mathfrak{V}$.
  - $\dot{U}^{(n)\perp} = U^{(n)\perp} \cap U^{(n+1)}$;
  - $\dot{V}^{(n)\perp} = V^{(n)\perp} \cap V^{(n+1)}$
  - $\dot{\Gamma}^{(n)\perp} := (\dot{U}^{(n)\perp}, \dot{V}^{(n)\perp}, \dot{E}^{(n)\perp})$, where $\dot{E}^{(n)\perp}$ is defined according to the set of fans of elements from $\dot{U}^{(n)\perp}$ putted into $\dot{\Gamma}^{(n)\perp}$.
- In Case 3B
  - $\mathfrak{U} := U^{(n+1)} \setminus (U^{(n)\perp} \cup \{\dot{u}_n^\perp\})$
  - $\mathfrak{V} := V^{(n+1)} \setminus (V^{(n)\perp} \cup \{\dot{v}_{n,i}^\perp : 1 \leq i \leq d-1\})$.
  - $\mathfrak{T} := (\mathfrak{U}, \mathfrak{V}, \mathfrak{E})$, where $\mathfrak{E}$ is induced in $\Gamma$ by the sets of vertices $\mathfrak{U}, \mathfrak{V}$.
  - $\dot{U}^{(n)\perp} := (U^{(n)\perp} \cup \{\dot{u}_n^\perp\}) \cap U^{(n+1)}$ with $\dot{V}^{(n)\perp}$ defined accordingly: $\dot{V}^{(n)\perp} := (V^{(n)\perp} \cup \{\dot{v}_{n,k}^\perp : 1 \leq k \leq d-1\}) \cap V^{(n+1)}$;

– $\dot{\Gamma}^{(n)\perp} := (\dot{U}^{(n)\perp}, \dot{V}^{(n)\perp}, \dot{E}^{(n)\perp})$, where $\dot{E}^{(n)\perp}$ is defined according to the set of fans of elements from $\dot{U}^{(n)\perp}$ putted into $\dot{\Gamma}^{(n)\perp}$.

- $\mathfrak{M}_n^2$ is a perfect $(1,d)$-matching in the graph $\Gamma^{(n)\star}(-u_n)$;
- $\dot{v}_{n,i}^{l+1}$, $1 \le i \le d$ (or $d-1$), are vertices adjacent to $u_n^{l+1}$ in $\mathfrak{M}_n^2$ (or in $\Gamma^{(n)\perp}$);
- $\dot{v}_1, \dot{v}_2$ are the elements among $\dot{v}_{n,i}^{l+1}$, $1 \le i \le d$ (or $d-1$), which are not added to $M_n^{l+1}$; there are at most two of them.

Before stating the lemma corresponding to the claims 4.2.2 and 4.2.4 we state two lemmas that will be useful in its proof.

**Lemma 4.4.3.** *Depending on the existence of $\dot{v}_2$ either the graph $\mathfrak{T}(-\dot{v}_1)$ or $\mathfrak{T}(-\dot{v}_1, -\dot{v}_2)$ satisfies Hall's $d$-harem condition.*

*Proof.* We only consider the case when $\dot{v}_2$ exists, the other one is analogous.

It follows from the procedure that:

$$\mathfrak{U}(-\dot{v}_1, -\dot{v}_2) = U^{(n)\star}(-u_n) \setminus \{u_n^{l+1}, \dot{u}_n^{\perp}\}$$

and

$$\mathfrak{V}(-\dot{v}_1, -\dot{v}_2) = V^{(n)\star}(-u_n) \setminus \{\dot{v}_{n,1}^{l+1}, \ldots, \dot{v}_{n,d}^{l+1}, \dot{v}_{n,1}^{\perp}, \ldots, \dot{v}_{n,d}^{\perp}\}.$$

Since $\mathfrak{M}_n^2$ is a perfect $(1,d)$-matching in $\Gamma^{(n)\star}(-u_n)$ and $\mathfrak{T}(-\dot{v}_1, -\dot{v}_2)$ is obtained from $\Gamma^{(n)\star}(-u_n)$ by removal of two fans from $\mathfrak{M}_n^2$, we know that $\mathfrak{T}(-\dot{v}_1, -\dot{v}_2)$ satisfies Hall's $d$-harem condition. □

**Lemma 4.4.4.** *For any $X \subset V^{(n+1)}$, we have*

$$|N_{\Gamma^{(n+1)}}(X)| \ge \frac{1}{d}|X|.$$

*Proof.* The inequality follows from Lemma 4.3.1. Indeed, consider $\Gamma^{(n+1)}$ to be $\Gamma$ from that lemma. Note that the construction guarantees that $\Gamma^{(n+1)}$ is $U^{(n+1)}$-reflected. Depending on existence of $\dot{v}_2$ consider either $\dot{\Gamma}^{(n)\perp}(+\dot{v}_1)$ or $\dot{\Gamma}^{(n)\perp}(+\dot{v}_1, +\dot{v}_2)$ to be $\Gamma^{\perp}$ from that lemma. Then the corresponding graph $\Gamma^{\star}$ from the lemma is equal to either $\mathfrak{T}(-\dot{v}_1)$ or $\mathfrak{T}(-\dot{v}_1, -\dot{v}_2)$ and by Lemma 4.4.3 it satisfies Hall's $d$-harem condition. Therefore conditions of Lemma 4.3.1 are satisfied. □

### 4.4.3 Part 2

**Lemma 4.4.5.** *For any $n$ one of the following holds:*

- *$\mathfrak{T}$ satisfies Hall's $d$-harem condition;*
- *there exist some vertex $u_j^{\perp} \in \dot{U}^{(n)\perp}$ such that the graph $\mathfrak{T}(+u_j^{\perp})$ satisfies Hall's $d$-harem condition.*
- *there exists vertices $u_i^{\perp}, u_j^{\perp} \in \dot{U}^{(n)\perp}$ such that the graph $\mathfrak{T}(+u_i^{\perp}, +u_j^{\perp})$ satisfies Hall's $d$-harem condition.*

*Remark* 4.4.6. If $|\dot{U}^{(n)\perp}| \le 1$, then Lemma 4.4.5 can be restated as follows. One of the following holds:

- $\mathfrak{T}$ satisfies Hall's $d$-harem condition;
- $\Gamma^{(n+1)}$ satisfies Hall's $d$-harem condition.

Therefore this lemma proves Claim 4.2.2.

*Proof of Lemma 4.4.5.* Assume that $\mathfrak{T}$ does not satisfy Hall's $d$-harem condition. Let $u_n^{l+1}$ be the root of the last fan added to the matching $M_n$ in the second part of the $n$-th step; it belongs to the produced cycle of length 2. Recall that $\dot{v}_1, \dot{v}_2$ denote the vertices of the form $\dot{v}_{n,i}^{l+1}$ that were not added to $M_n^{l+1}$. In particular in Case 3A there is only one such a vertex and in Case 3B there is either one or two such vertices. From now on we consider the case of two additional vertices: $\dot{v}_1$ and $\dot{v}_2$. The case of only one of them, $\dot{v}_1$, is similar.

It is clear that the inequality $|N_{\mathfrak{T}}(X)| \geq |N_{\mathfrak{T}(-\dot{v}_1,-\dot{v}_2)}(X)|$ holds for all $X \subset \mathfrak{U}$. This inequality also holds for subsets of $\mathfrak{V}$ which do not intersect $\{\dot{v}_1, \dot{v}_2\}$. By Lemma 4.4.3 $\mathfrak{T}(-\dot{v}_1, -\dot{v}_2)$ satisfies Hall's $d$-harem condition, therefore if $\mathfrak{T}$ does not satisfy Hall's $d$-harem condition then a witness of this is some finite subset of $\mathfrak{V}$ containing at least one of $\dot{v}_1, \dot{v}_2$.

The rest of the proof is divided into two parts.

*Part 1.* We check whether the graph $\mathfrak{T}(-\dot{v}_2)$ satisfies Hall's $d$-harem condition. If it does, we set

- $M'$ is a perfect $(1, d)$-matching in $\mathfrak{T}(-\dot{v}_2)$;
- $\dot{\Gamma}^{(n+1)\perp} := \dot{\Gamma}^{(n)\perp}$;
- $\Gamma' = \mathfrak{T}$, denoting $\Gamma' = (U', V')$,

and finish Part 1 of the proof.

If it does not, then there exists a minimal connected set $X$ such that $\dot{v}_1 \in X \subset \mathfrak{V}(-\dot{v}_2)$ and $|N_{\mathfrak{T}(-\dot{v}_2)}(X)| < \frac{1}{d}|X|$. Observe that $|N_{\Gamma^{(n+1)}}(X)| \geq \frac{1}{d}|X|$ by Lemma 4.4.4.

The inequalities

$$|N_{\Gamma^{(n+1)}}(X)| \geq \frac{1}{d}|X| \text{ and } |N_{\mathfrak{T}(-\dot{v}_2)}(X)| < \frac{1}{d}|X|$$

imply

$$N_{\Gamma^{(n+1)}}(X) \cap \dot{U}^{(n)\perp} \neq \emptyset,$$

i.e. there exists some $u_j^{\perp} \in N_{\Gamma^{(n+1)}}(X)$. Similarly as in Lemma 4.4.1 we denote by $v_{j,1}^{\perp}, \ldots, v_{j,d-1}^{\perp}$ the remaining vertices of the fan from $\dot{\Gamma}^{(n)\perp}$ containing $u_j^{\perp}$.

Since $\mathfrak{M}_n^2$ is a perfect $(1, d)$-matching in the graph $\mathfrak{T}(-\dot{v}_1, -\dot{v}_2)$, which in turn is a subgraph of the bipartite graph $\Gamma^{(n+1)}$, we can use Lemma 4.3.4 for $u_j^{\perp}, X, \dot{v}_1$ and arrive at $u_j^{\perp} \xhookrightarrow{\mathfrak{M}_n^2, X} \dot{v}_1$. This gives us sequences $\{v'_0, \ldots, v'_n\}, \{u'_0, \ldots, u'_{n-1}\}$ as in Definition 4.3.3. We now apply an argument similar to one from the proof of Lemma 4.4.1. We set

$$\begin{aligned} M' := (\mathfrak{M}_n^2 \setminus \{(u_n^{l+1}, \dot{v}_{n,1}^{l+1}), \ldots, (u_n^{l+1}, \dot{v}_{n,d}^{l+1}), (u'_0, v'_0), \ldots, (u'_{n-1}, v'_{n-1})\}) \cup \\ \{(u_j^{\perp}, v_{j,1}^{\perp}), \ldots, (u_j^{\perp}, v_{j,d-1}^{\perp}), (u_j^{\perp}, v'_0), (u'_0, v'_1), \ldots, (u'_{n-1}, v'_n)\}, \end{aligned}$$

where $v'_n = \dot{v}_1$. Observe that

$$(\mathfrak{U} \cup \{u_j^\perp\}, (\mathfrak{V} \cup \{v_{j,1}^\perp, \dots, v_{j,d-1}^\perp\}) \setminus \{\dot{v}_2\}) = \mathfrak{T}(+u_j^\perp) \setminus \{\dot{v}_2\} = \mathfrak{T}(+u_j^\perp, -\dot{v}_2).$$

We remind the reader that $\mathfrak{M}_n^2$ is a perfect $(1,d)$-matching in the graph $\Gamma^{(n)\star}(-u_n)$. We have obtained $M'$ from $\mathfrak{M}_n^2$ by removing $d$ edges incident to $u_n^{l+1}$, adding $d$ edges incident to $u_j^\perp$, and the following replacement: for each $u'_i$ we replace one edge incident to it by another incident edge. It follows that the matching $M'$ is a perfect $(1,d)$-matching in the graph $\mathfrak{T}(+u_j^\perp, -\dot{v}_2)$. Therefore that graph satisfies Hall's $d$-harem condition.

We define $\Gamma' := \mathfrak{T}(+u_j^\perp)$, denote $\Gamma' = (U', V')$ and put

$$\dot{\Gamma}^{(n+1)\perp} = (\dot{U}^{(n)\perp} \setminus \{u_j^\perp\}, \dot{V}^{(n)\perp} \setminus \{v_{j,k}^\perp : 1 \le k \le d-1\}).$$

This ends the first part of the proof.

*Part 2.* We check whether the graph $\Gamma'$ satisfies Hall's $d$-harem condition. If it does, then by Part 1 $\Gamma'$ has to be equal to $\mathfrak{T}(+u_j^\perp)$ and the proof is finished by the second option of the formulation. If it does not then repeating the reasoning of Part 1 we see that there exists a minimal connected set $X$ such that $\dot{v}_2 \in X \subset V'$ and $|N_{\Gamma'}(X)| < \frac{1}{d}|X|$. Again, using Lemma 4.4.4 we obtain

$$N_{\Gamma^{(n+1)}}(X) \cap \dot{U}^{(n)\perp} \neq \emptyset,$$

i.e. there exists some $u_i^\perp \in N_{\Gamma^{(n+1)}}(X)$. We denote by $v_{i,1}^\perp, \dots, v_{i,d-1}^\perp$ the vertices adjacent to $u_j^\perp$ in $\dot{\Gamma}^{(n+1)\perp}$.

The matching $M'$ obtained in the first part of the proof is a perfect $(1,d)$-matching for either the graph $\mathfrak{T}(-\dot{v}_2)$, or the graph $\mathfrak{T}(+u_j^\perp, -\dot{v}_2)$. Each of them is a subgraph of $\Gamma^{(n+1)}$. Therefore we can use Lemma 4.3.4 for $u_i^\perp, X, \dot{v}_2$. We have $u_i^\perp \xhookrightarrow{M',X}\!\!\!\rightarrow \dot{v}_2$. Again we can apply the argument of Lemma 4.4.1 to obtain an appropriate matching:

$$M'' := (M' \setminus \{(u'_0, v'_0), \dots, (u'_{n-1}, v'_{n-1})\}) \cup$$

$$\{(u_i^\perp, v_{i,1}^\perp), \dots, (u_i^\perp, v_{i,d-1}^\perp), (u_i^\perp, v'_0), (u'_0, v'_1), \dots, (u'_{n-1}, v'_n)\}.$$

We have obtained $M''$ from $M'$ by adding $d$ edges incident to $u_i^\perp$, and replacing one edge in matching for each of $u'_k$ in such a way, that $\dot{v}_2$ is adjacent to one edge in $M''$.

The final argument depends on two possible outputs of Part 1. If the graph $\mathfrak{T}(-\dot{v}_2)$ does not satisfy Hall's $d$-harem condition (i.e. $u_j^\perp$ is involved), then $M''$ is a perfect $(1,d)$-matching in the graph $\mathfrak{T}(+u_i^\perp, +u_j^\perp)$. If it does, we redefine $u_j^\perp := u_i^\perp$ and then $M''$ becomes a perfect $(1,d)$-matching in the graph $\mathfrak{T}(+u_j^\perp)$. Therefore if $\mathfrak{T}$ does not satisfy Hall's $d$-harem condition, then either $\mathfrak{T}(+u_j^\perp)$ or $\mathfrak{T}(+u_i^\perp, +u_j^\perp)$ satisfies this condition. □

## 4.5 Proof of the Main Theorem

*Proof of Theorem 4.1.3.* Let us apply the construction of Section 4.2. This construction works modulo Claims 4.2.1, 4.2.2, 4.2.3, 4.2.4. Claims 4.2.1 and 4.2.3 follow from

Lemma 4.4.1. Claims 4.2.2 and 4.2.4 follow from Lemma 4.4.5. Since for every $n$ the union $\bigcup_{i=1}^{n} M_i$ consist of $(1, d-1)$-fans, the final set of edges $M$ is an $(1, d-1)$-matching.

For every $u \in U$ there is a step where an edge incident to $u$ is added to $M$. Then if the copy $v_u$ was not added to $M$ earlier, in the second part of this step this copy is added to $M$. It follows that $M$ is a perfect $(1, d-1)$-matching of the graph $\Gamma$.

It remains to verify that $M$ realizes a function, say $f$, with controlled sizes of its cycles. Since the edges $(u_0, v^0_{0,1})$ and $(v_{u_0}, u_{v^0_{0,1}})$ are added to $M$ at step 1 we see $f^2(u_0) = f(u_{v^0_{0,1}}) = u_0$. Since $u_0 = 1$, condition $(i)$ of Definition 4.1.2 is satisfied.

Till the end of the proof a natural number $n$ will be used both for vertices and numbers of steps. Note that as a vertex it appears in $M$ at the $n$-th step at latest. This follows from minimality of $u_n$ in $U^{(n-1)}$. The length of a cycle created at the $n$-th step cannot be greater than $\max\{2, n\}$ so if $n$ is in a cycle then $f^i(n) = n$ for some $i \leq n$ and condition $(ii)$ of Definition 4.1.2 is satisfied.

It remains to show that condition $(iii)$ is satisfied. Let $\mathfrak{f}_i$ be the partial function (living in $\mathbb{N}$) realized by $M_i$ and $f_n$ be the partial function realized by $\bigcup_{i=0}^{n} M_i$. Let $F_n = \{(f_n(m), m) : m \in \mathsf{Dom}(f_n)\}$ be the graph of $f_n$ on $\mathsf{Dom}(f_n) \cup \mathsf{Rng}(f_n)$. Then $M_n = \{(\mathfrak{f}_n(m), m) : m \in \mathsf{Dom}(\mathfrak{f}_n)\}$ is the graph of $\mathfrak{f}_n$ and is a subgraph of $F_n$ too.

Observe that $F_n$ has at most $n+1$ connected components. Indeed, each subgraph $M_i$ has exactly one connected component. Furthermore, one of the following properties holds:

- $M_n$ is a connected component of $F_n$;
- $M_n$ is a fan with a root which already appears in $F_{n-1}$ as a vertex of degree 1 (see the way how fans of $U$-elements are added to $M$).

The construction guarantees that $F_n$ consists only of vertices of degree 1 and $d$. When a vertex has degree 1 its $F_n$-neighbourhood is only its $f_n$-image. When a vertex has degree $d$ its $f_n$-image and $d-1$ preimages are in $F_n$. In particular each connected component of $F_n$ contains a cycle. The length of the cycle is not greater than $\max\{n, 2\}$.

Since the value $f_{n-1}(n)$ is defined, $n$ belongs to some connected component of $f_{n-1}$. Thus there exist $k$ and $l$ such that $f^{k+l}_{n-1}(n) = f^k_{n-1}(n)$. These $k$ and $l$ work for the equality $f^{k+l}(n) = f^k(n)$. It remains to show that $k$ can be bounded by $2n$ and $l$ by $n$.

The latter estimate is easy: the biggest cycle that can be constructed before the $n$-th step does not have more than $\max\{n-1, 2\}$ elements. Below we will use the bound $n$ for $l$ for simplicity.

In order to show that $k$ is bounded by $2n$ let us estimate the size of a subset of $U$ without a cycle that can be added to the matching $M$ in the process of the $n$-th step of the construction. It must consist of elements of $U^{(n-1)\perp}$ added to the matching at the iteration of part 2 of the $n$-th step together with $u_n$. Therefore we can bound it by the maximal possible number of elements in $U^{(n-1)\perp}$ increased by 1.

Let us denote the number of elements from $U^{(s)\perp}$ added to $M$ at the $s$-th step by $\ell_s$. Clearly, $f^{\ell_s+1}(n) \in M_j$ for some $s \leq n$ and $j \leq n-1$. If no cycle is constructed in the $j$-th step then $f^{\ell_j+1}(f^{\ell_s+1}(n)) \in M_i$ for some $i \leq j-1$. Iterating this argument we arrive at

$$k \leq \sum_{s=1}^{n} (\ell_s + 1).$$

Since at the $n$-th step the size of $\bigcup_{s=1}^{n} U^{(s)\perp}$ does not exceed $n-1$,

$$\sum_{s=1}^{n} \ell_s \leq n-1.$$

We see that $k \leq 2n-1$. Thus condition $(iii)$ of Definition 4.1.2 is satisfied. □

## 4.6 Computable entourages of coarse spaces

To simplify the matter, in the case of bipartite graphs, from now on we will always identify $U$ and $V$ with computable subsets of $\mathbb{N}$. Further, admitting that $U \cap V \neq \emptyset$ we distinguish these sets saying that $U$ is taken from the left copy of $\mathbb{N}$ but $V$ is taken from the right one. In particular we often consider graphs where $U = V = \mathbb{N}$.

The following proposition explains our original motivation which led us to the final results of [Dud21a].

**Proposition 4.6.1.** *Let $(\mathbb{N}, \mathcal{E})$ be a coarse space of bounded geometry and let $E \in \mathcal{E}$ be a symmetric entourage. Let $f$ be a computable function realizing a perfect $(1, d-1)$-matching for $d \geq 3$ in the graph $\Gamma = (\mathbb{N}, \mathbb{N}, R)$, where $R = E \setminus \Delta_{\mathbb{N}} \subseteq \mathbb{N} \times \mathbb{N}$.*

*If $f$ has controlled sizes of its cycles, then there exists a computable $E' \in \mathcal{E}$ such that $\Gamma(E')$ is a $d$-regular forest. Moreover there exist an algorithm which for any $m, n \in \mathbb{N}$ recognizes if $m$ and $n$ are in the same connected component of $\Gamma(E')$.*

*Proof.* We adapt the proof of Theorem 2.2 of [Sch18].

Since $f$ is a total, computable, surjective, $(d-1)$ to 1 function, the graph of $f$ (denoted by $\Gamma(f)$) is computable and $d$-regular. We remind the reader that $f$ satisfies the following properties:

(i) $f^2(1) = 1$;

(ii) if $n \geq 2$ and $f^i(n) = n$ then $i \leq n$;

(iii) if $n \geq 2$ and for all $i \leq n$ we have $f^i(n) \neq n$ then there exist $k \leq 2n$ and $l \leq n$ such that $f^{k+l}(n) = f^k(n)$;

(iv) for each $n$ the pair $(n, f(n))$ belongs to $R$.

Since $R = E \setminus \Delta_X$ the last property implies that $f$ does not have fixed points. Let

$$P(f) = \{n \in \mathbb{N} \mid \exists m \geq 1 (f^m(n) = n)\} \text{ and } P_0(f) = \{n \in P(f) \mid \forall m \geq 1 (f^m(n) \geq n)\},$$

i.e. the union of all cycles and the set of minimal representatives of cycles. By property (ii) there is an algorithm which for every $n \in \mathbb{N}$ verifies whether $n \in P(f)$, i.e. $P(f)$ is computable. Observe that $P_0(f)$ is computable too. Indeed, 1 obviously belongs to $P_0(f)$. When $n \geq 2$ and $n \in P(f)$, then verifying if $f^i(n) \geq n$ for all $i \leq n$ we can check whether $n \in P_0(f)$ (apply (ii) again).

Since each component does not have two disjoint cycles we see that whenever $n, m \in P_0(f)$ and $n \neq m$, then $n$ and $m$ do not belong to the same connected component of the graph $\Gamma(f)$. Thus

$$P(f) = \dot{\bigsqcup}_{n \in P_0(f)} \{f^m(n) | m \in \mathbb{N}\}.$$

There is an algorithm which for every $n \in \mathbb{N}$ finds the $P_0(f)$-representative of the connected component of $n$. Indeed, if for example $n \notin P(f)$ then applying (iii) we can find $i \leq 2n$ such that $f^i(n) \in P(f)$ and later $j \leq n$ such that $f^{i+j}(n) \in P_0(f)$.

Based on this we want to construct a new computable function $f_\star$ such that its graph (denoted by $\Gamma(f_\star)$) is a computable $d$-regular forest. Let us start with two auxiliary functions $g, h : P_0(f) \times \mathbb{N} \to \mathbb{N}$ such that, for all $n \in P_0(f)$ and $m \geq 1$ the following properties hold:

- $g(n, 0) = n$ and $h(n, 0) = f(n)$;
- $\{g(n, m), h(n, m)\} \cap P(f) = \emptyset$;
- $f(g(n, m)) = g(n, m - 1)$, and $f(h(n, m)) = h(n, m - 1)$.

We want $g, h$ to be computable functions. Since $P_0(f)$ and $P(f)$ are computable and the graph $\Gamma(f)$ is computable and $d$-regular the following rule gives required algorithm. Given $n \in P_0(f)$ and $m \geq 1$ and having defined $g(n, m - 1)$ find the minimal $x$ such that $x \notin P(f)$ and $f(x) = g(n, m - 1)$. Then let $g(n, m) = x$. The definition of $h(n, m)$ is similar. Clearly $g$ and $h$ are injective and have disjoint ranges.

Now we define $f_\star : \mathbb{N} \to \mathbb{N}$ for $x \in \mathbb{N}$ in the following way:

$$f_\star(x) = \begin{cases} g(y, m + 2) & \text{if } x = g(y, m) \text{ for } y \in P_0(f) \text{ and even } m \geq 0, \\ g(y, m - 2) & \text{if } x = g(y, m) \text{ for } y \in P_0(f) \text{ and odd } m \geq 3, \\ f^2(x) & \text{if } x = h(y, m) \text{ for } y \in P_0(f) \text{ and } m \geq 2, \\ f(x) & \text{otherwise.} \end{cases}$$

We remind the reader that for each $n$ there is $i \leq 3n$ such that $f^i(n) \in P_0(f)$. Thus for any $m' > 3n$ the number $n$ does not belong to $\{g(y, m'), h(y, m')\}$. This guarantees that $f_\star$ is computable. In the proof of Theorem 2.2 in [Sch18] it is proved that $\Gamma(f_\star) \in \mathcal{E}$, the function $f_\star$ does not have cycles and for each $x \in \mathbb{N}$ the size $|f_\star^{-1}(x)|$ is $d - 1$. Therefore the graph $\Gamma(f_\star)$ is a computable $d$-regular forest.

To see the last statement of the proposition note that if $C \subset \mathbb{N}$ is a connected components of $\Gamma(f)$ then it is also the set of vertices of a connected component of $\Gamma(f_\star)$. Since for each $n$ one can compute $n' \in P_0(f)$ such that $n$ and $n'$ are in same connected component of $\Gamma(f)$ we have an algorithm which for any $n$ and $m$ verifies whether $n$ and $m$ are in the same tree in $\Gamma(f_\star)$. □

We now formulate a final theorem of [Dud21a].

**Theorem 4.6.2.** *Let $d \geq 3$. Let a coarse space $(\mathbb{N}, \mathcal{E})$ of a bounded geometry be non-amenable. Furthermore, assume that there exists a highly computable symmetric $E \in \mathcal{E}$ such that for any finite $F \subseteq \mathbb{N}$ we have $|E[F]| \geq (d+2)|F|$. Then there exists a computable $E' \in \mathcal{E}$ such that $\Gamma(E')$ is a $d$-regular forest. Moreover, there exist an algorithm which for any $m, n \in \mathbb{N}$ recognizes if $m$ and $n$ are in the same connected component of $\Gamma(E')$.*

To see that this statement is a computable version of Schneider's result we remind the reader that when $(X, \mathcal{E})$ is not amenable, then for every finite $d$ there is a symmetric entourage such that $|E(F)| \geq d|F|$ for every finite $F \subseteq X$ (see discussion before Proposition 2.1 in [Sch18]).

This theorem is proved in [Dud21a]. Here we only explain some difficulties arising in this task. In order to apply Proposition 4.6.1 to Theorem 4.6.2 one may use Theorem 4.1.3. It gives a matching which satisfies almost all properties of Proposition 4.6.1. The only property which is lost is computability. To remedy this one can imagine that the computable version of Hall's Harem theorem from [DI22a] should work.

Indeed, this theorem can be applied under circumstances resembling the situation of Theorem 4.6.2. Let $E$ be a highly computable symmetric entourage as in Theorem 4.6.2 and consider the graph $\Gamma(R)$ defined for the symmetric relation $R := E \setminus \Delta_{\mathbb{N}} \subseteq \mathbb{N} \times \mathbb{N}$. Since the coarse space $(\mathbb{N}, \mathcal{E})$ is of bounded geometry, the neighbourhood of each vertex is finite. By high computability of $E$ it follows that there is an algorithm which computes the sizes of neighbourhoods of vertexes. Clearly $\Gamma(R)$ is a highly computable graph.

It remains to show that $\Gamma(R)$ satisfies *c.e.H.h.c.*$(d)$. The following inequality holds for $R$:

$$|R[F]| \geq |E[F] \setminus F| \geq |E[F]| - |F| \geq (d+1)|F|.$$

Thus, for all finite sets $X \subset \mathbb{N}$, the following holds

$$|N_{\Gamma(R)}(X)| - d|X| \geq (d+1)|X| - d|X| = |X|.$$

Hence $n \leq |X|$ implies that $n \leq |N(X)| - d|X| \leq |N(X)| - \frac{1}{d}|X|$. Since the identity map on $\mathbb{N}$ is a total computable function, it follows that $\Gamma(R)$ satisfies *c.e.H.h.c.*$(d)$. As a result applying [DI22a, Theorem 2.9] we get a computable function $f$ realizing a perfect $(1, d-1)$ matching in the graph $\Gamma(R)$.

We now meet another obstacle: applying [DI22a] we cannot guarantee that this computable $f$ satisfies any condition (e.g. having controlled sizes of its cycles) that will allow us to recognize periodic points in a computable way. The latter task is necessary for computability of $f_{\star}(x)$ in the natural adaptation of the proof of Theorem 2.2 of [Sch18] (see Theorem 4.6.1). This explains why we need a computable version of the Main Theorem.

# Part II

# Locally elliptic actions on small cancellation complexes

# Chapter 5

# Preliminaries for Part II

The main results of this part are available at arXiv as a preprint 'Torsion subgroups of small cancellation groups' [Dud21b]. In comparison to the preprint, this part is extended by some preliminary material.

In Section 5.1 we give a brief introduction to the small cancellation theory. In Section 5.2 we define the quadrization of a complex and show properties of quadrizations of $C(4)$–$T(4)$ complexes. In Section 5.3 we define the Wise complex and demonstrate some of its properties in the case of $C(6)$ complexes.

## 5.1 Small Cancellation Theory

In 1911 Dehn [Deh11] provided algorithms which solve word and conjugacy problems for the fundamental groups of closed orientable two-dimensional manifolds. A crucial feature of these groups is that they are defined by a single relator $r$, with the property that there is a little cancellation in the product of $r$ and its cyclic conjugates.

Dehn's algorithms have been extended to a large class of groups with similar '*Small Cancellation*' properties of its defining relations.

The theory was sporadically developed before being fully treated in the book by Lyndon and Schupp [LS01], which is a standard reference on the subject.

The small cancellation theory is a classical powerful tool for constructing examples of groups with interesting features, as well as for exploring properties of well-known groups. It might be seen as a bridge between the combinatorial and the geometric group theories, and as one of the first appearances of nonpositive curvature techniques in studying groups. Constituting a classical part of mathematics, small cancellation techniques are still being developed, having numbers of variations, and have also been used in proving new important results nowadays.

Traditionally, small cancellation theory is defined using group presentations. We follow the generalization of [MW02], with small cancellation conditions defined for combinatorial 2-complexes. This approach agrees with the definition of small cancellation conditions for group presentations, when we consider the Cayley complex of a group presentation.

We recall some fundamental notions of algebraic topology following Hatcher's textbook [Hat00].

A *homotopy* is a family of maps $f_t : X \to Y, t \in [0,1]$ such that the associated map $F : X \times [0,1] \to Y$ where $F(x,t) = f_t(x)$ is continuous. One says that two maps $f_0, f_1 : X \to Y$ are *homotopic* if there exist a homotopy $f_t$ connecting them. A map is *nullhomotopic* if it is homotopic to a constant map.

**Definition 5.1.1.** A *CW complex* is a space $X$ constructed in the following way:

1. Start with a discrete set $X^0$ called 0-*skeleton*. Elements of $X^0$ are called 0-cells.
2. Inductively, form the $n$-*skeleton* $X^n$ from $X^{n-1}$ by attaching $n$-cells $e^n_\alpha$ (where each $e^n_\alpha$ is a copy of $\mathbb{D}^n$) via maps $\varphi : \mathbb{S}^{n-1} \to X^{n-1}$. As a set $X^n = X^{n-1} \bigsqcup_\alpha e^n_\alpha$.

This process can either stop at some finite stage setting $X = X^n$ or can continue indifnitely setting $X = \bigcup_n X^n$. In this thesis we only consider the first case.

Note that a graph is a 1-dimensional CW-complex with 0-cells called *vertices* and 1-cells called *edges*. Distances between vertices in a connected graph are always measured by the *standard graph metric* which is defined for a pair of vertices $u$ and $v$ as the number of edges in the shortest path connecting $u$ and $v$.

A graph $\Gamma$ is *simplicial* if there is no edge in $\Gamma$ with both endpoints attached to one vertex and no two edges of $\Gamma$ having their endpoints attached to the same unordered pair of vertices.

Let $X$ and $Y$ be CW-complexes. A map from $X$ to $Y$ is *combinatorial* if it is a continuous map whose restriction to every open cell $e$ of $X$ is a homeomorphism from $e$ to an open cell of $Y$. A complex is *combinatorial* if the attaching map of each of its $n$-cells is combinatorial for a suitable subdivision of the sphere $\mathbb{S}^{n-1}$.

A *polygon* is a 2-disc with a cell structure that consists of $n$ 0-cells, $n$ 1-cells and a single 2-cell. For any 2-cell $C$ of a 2-complex $X$ there exists a map $R \to X$ where $R$ is a polygon and the attaching map for $C$ factors as $\mathbb{S}^1 \to \partial R \to X$. Because of that, by a *cell*, we will mean a map $R \to X$ where $R$ is a polygon.

A *path* in $X$ is a combinatorial map $P \to X$ where $P$ is either a subdivision of the interval or a single 0-cell. For given paths $P_1 \to X$ and $P_2 \to X$ such that the terminal point of $P_1$ is equal to the initial point of $P_2$, their *concatenation* is the natural path $P_1P_2 \to X$. Similarly, a *cycle* is a map $C \to X$ where $C$ is a subdivision of a circle $\mathbb{S}^1$. We will often identify paths and cycles with their images in the complex $X$.

Two paths in $X$ are homotopic if they are joined by homotopy and have the same endpoints. A *homotopy class* of a path $f$ is the equivalence class of a path $f$ under the equivalence relation of homotopy.

A space $X$ is *simply connected* if there is a unique homotopy class of paths connecting any two points in $X$.

From now on we only consider 2-dimensional CW-complexes and we will refer to them as “2-complexes”.

The following definition is crucial in the small cancellation theory.

**Definition 5.1.2.** Let $X$ be a combinatorial 2-complex. A non-trivial path $P \to X$ is a *piece* of $X$ if there are 2-cells $R_1$ and $R_2$ such that $P \to X$ factors as $P \to R_1 \to X$ and as $P \to R_2 \to X$ but there does not exist a homeomorphism $\partial R_1 \to \partial R_2$ such that there is a commutative diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \partial R_2 \arrow[d] \\ \partial R_1 \arrow[r] \arrow[ur] & X \end{tikzcd}$$

Intuitively, a piece of $X$ is a path which is contained in boundaries of 2-cells of $X$ in at least two distinct ways.

A *disc diagram* $D$ is a compact contractible subspace of a 2-sphere $\mathbb{S}^2$ with the structure of a combinatorial 2-complex. The *area* of diagram $D$ is the number of its 2-cells. If $D$ is a disc diagram, then the diagram $D$ *in* $X$ is $D$ along with a combinatorial map from $D$ to $X$ denoted by $D \to X$.

**Definition 5.1.3.** Let $X$ be a 2-complex. A disc diagram $D \to X$ is *reduced* if for every piece $P \to D$ the composition $P \to D \to X$ is a piece in $X$.

Let us regard $\mathbb{S}^2$ as $\mathbb{R}^2 \cup \{\infty\}$ and assume that $D$ is a disc diagram that does not contain the point $\infty$. The *boundary cycle* $\partial D$ of $D$ is the attaching map of the 2-cell that contains the point $\infty$.

The following theorem is known as the Lyndon-van Kampen lemma [MW02, Lemma 2.17].

**Theorem 5.1.4.** *If $X$ is a 2-complex and $P \to X$ is a nullhomotopic closed path, then there exists a reduced disc diagram $D \to X$ such that $P \to D$ is the boundary cycle of D, and $P \to X$ is the composition $P \to D \to X$.*

We will now define small cancellation conditions $C(p)$ and $T(q)$.

**Definition 5.1.5.** Let $p$ be a natural number. We say that a 2-complex $X$ satisfies the $C(p)$ *small cancellation condition* provided that for each 2-cell $R \to X$ its attaching map $\partial R \to X$ is not a concatenation of fewer than $p$ pieces in $X$.

In the following definition of $T(q)$ condition we will use the notion of *valence* of a 0-cell $v \in X$ in the complex $X$, i.e. the number of ends of 1-cells incident to it. We denote it by $\delta_X(v)$ and drop the subscript if it is clear from the context.

**Definition 5.1.6.** Let $q$ be a natural number. We say that a 2-complex $X$ satisfies the $T(q)$ *small cancellation condition* if there does not exist a reduced map $D \to X$ where $D$ is a disc diagram containing an internal 0-cell $v$ such that $2 < \delta(v) < q$.

If a complex satisfies both $C(p)$ and $T(q)$ conditions, then we call it a $C(p)$–$T(q)$ complex. It is known that $C(3)$–$T(6)$, $C(4)$–$T(4)$ and $C(6)$ complexes are nonpositively curved.

The following proposition is a known property of simply connected $C(6)$ complexes [OP18].

**Lemma 5.1.7.** *Let $\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n$ be a pairwise intersecting 2-cells of a $C(6)$ complex $X$. Then $\widehat{x}_1 \cap \widehat{x}_2 \cap \ldots \cap \widehat{x}_n$ is either a piece or a vertex.*

The following propositions are known properties of simply connected $C(4)$–$T(4)$ complexes [Hod20, Proposition 3.5, 3.8].

**Proposition 5.1.8.** *Let $F_1$ and $F_2$ be a pair of intersecting 2-cells of a simply connected $C(4)$–$T(4)$ complex. Then the intersection $F_1 \cap F_2$ is connected.*

**Proposition 5.1.9** (Strong Helly Property)**.** *Let $F_1$, $F_2$ and $F_3$ be pairwise intersecting 2-cells of a simply connected $C(4)$–$T(4)$ complex. Then the intersection of some pair of these 2-cells is contained in the remaining 2-cell, i.e. for some permutation $\sigma$ of the indices, we have*

$$F_{\sigma(1)} \cap F_{\sigma(2)} \subset F_{\sigma(3)}.$$

In the case of $C(3)$–$T(6)$ complexes, we will use the fact that all pieces are short, which was first observed by Pride [Pri88] in the following Lemma.

**Lemma 5.1.10.** *If $X$ is a $T(q)$ complex for $q \geq 5$, then every piece in $X$ has length 1.*

## 5.2 Quadric complexes and quadrization of a complex

In this section we define the quadrization of a complex and state some results concerning quadrizations of $C(4)$–$T(4)$ complexes. We begin with some necessary notions concerning CAT(0) square complexes.

An important for us type of complexes are *square* complexes. Square complexes are 2-complexes whose 2-cells are 4-gons. In this case, instead of the usual terms, 0-cell, 1-cell and 2-cell, we will use vertex, edge and square, respectively.

Let $X$ be a square complex and $L$ be a subcomplex of $X$ with vertices $\{u_1, \dots, u_n, v_1, \dots, v_n\}$, and the set of edges consisting of the edges of the form $(u_i, u_{i+1}), (v_i, v_{i+1})$ and $(v_i, u_i)$. We call such a complex $L$ a *ladder of length $n$* and denote it by $\{u_1, \dots, u_n | v_1, \dots, v_n\}$.

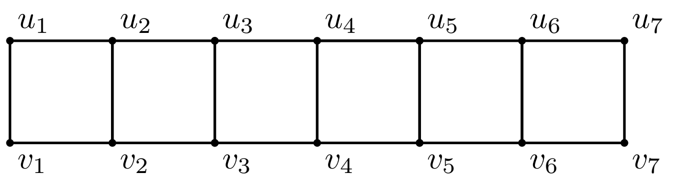



Figure 5.1: Ladder of length 7.

A *CAT(0) square complex* is a square complex for which the metric obtained by making each square isometric to the regular Euclidean square of side length 1 satisfies the CAT(0) condition, which is a metric nonpositive curvature condition concerning thinness of geodesic triangles. We use a combinatorial characterization of the CAT(0) condition for the square complexes which follows from Gromov's link condition, and take it as the definition. In the following definition, a *link* of a 0-cell $v$ of a 2-complex $X$ (denoted $X_v$) is the graph whose vertices correspond to the ends of 1-cells of $X$ incident to $v$, and an edge joins vertices corresponding to the ends of 1-cells $e_1, e_2$ iff there is a 2-cell $F$ such that $e_1, e_2 \in \partial F$.

**Definition 5.2.1.** A square complex $X$ is CAT(0) if it is simply connected and for any 0-cell $v \in X$ the shortest embedded cycle in the link of $v$ has length at least 4.

If $X$ is a disc diagram, the CAT(0) condition means that each internal vertex is incident to at least four squares.

Let $X$ be a 2-complex with embedded 2-cells. Let $X_0, X_2$ be the sets of 0-cells and 2-cells of $X$. Let $\Gamma_X$ be a bipartite graph on the vertex set $X_0 \cup X_2$ where an edge joins $v \in X_0$ with $F \in X_2$ whenever $v \in \partial F$. The following notion was introduced by N.Hoda.

**Definition 5.2.2.** [Hod20, Section 3.2] The *quadrization* $Y$ of a complex $X$ is the 4-flag completion $Y = \overline{\Gamma_X}$ i.e. a complex obtained from $\Gamma_X$ by spanning a 2-cell on each of its nontrivial 4-cycles.

If we additionally assume that every 1-cell of $X$ is contained in the boundary of a 2-cell then simply connectedness of $X$ implies simply connectedness of its quadrization [Hod20, Lemma 3.9].

All 2-cells are squares, therefore the quadrization of a complex is a square complex. We will sometimes consider cells in both the complex $X$ and its quadrization $Y$ at the same time. In such cases we will denote by $\widehat{x}$ the 0- or 2-cell corresponding to the vertex $x$ from $Y$.

A path $P \to Y$ will be denoted by $(x_1, \ldots, x_n)$, where $x_i$ are the vertices in $Y$ that are the images of vertices of $P$, in particular, $x_1, x_n$ are the images of ends. If a path is a piece, then we will denote it by $(x, x')$ where $x$ and $x'$ are its endpoints. A cycle $C \to Y$ will be denoted by $\langle x_1, \ldots, x_n \rangle$.

Each square $F$ in $Y$ has four vertices: two from $X_0$ and two from $X_2$. Boundary of $F$ is a cycle $\partial F = \langle x_1, x_2, x_3, x_4 \rangle$. We will use notation $F = [x_1, x_2, x_3, x_4]$. Observe that either $x_1, x_3 \in X_0$ and $x_2, x_4 \in X_2$, or $x_1, x_3 \in X_2$ and $x_2, x_4 \in X_0$; wherever possible we will use the first of these options. Let $F = [x_1, x_2, x_3, x_4]$ be a square in $Y$. Since $X$ is $C(4)$–$T(4)$, the intersection of $\widehat{x}_2, \widehat{x}_4$ is connected and contains $\widehat{x}_1, \widehat{x}_3$. Therefore $(\widehat{x}_1, \widehat{x}_3)$ is a piece in $X$. On the other hand, if $(\widehat{x}'_1, \widehat{x}'_3)$ is a piece in $X$ contained in $\widehat{x}'_2 \cap \widehat{x}'_4$, then $F' = [x'_1, x'_2, x'_3, x'_4]$ is a square in $Y$.

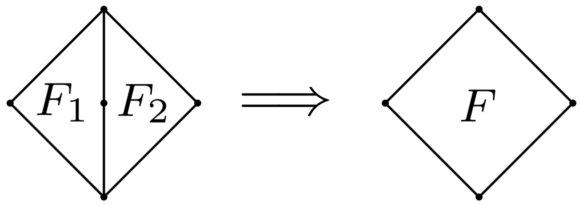
In [Hod20] N.Hoda proved that the 1-skeleton of the quadrization of a simply connected $C(4)$–$T(4)$ complex is a hereditary modular graph. In his paper Hoda called such complexes *quadric*. We only need a part of his result. Hoda defines quadric complexes as simply connected square complexes satisfying, among others, the following conditions, called *rules of replacement*.

**Proposition 5.2.3.** *[Hod20, Definition 1.1.] Let $Y$ be the quadrization of the $C(4)$–$T(4)$ complex $X$.*

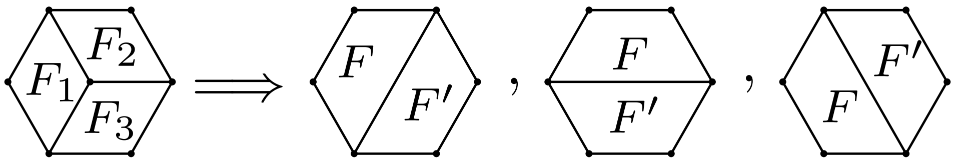
*(A) If there are two squares $F_1, F_2 \in Y$ such that $\partial(F_1 \cup F_2)$ is a cycle of length 4, then there is $F \in Y$ such that $\partial F = \partial(F_1 \cup F_2)$.*

*(B) If there are three squares $F_1, F_2, F_3 \in Y$ such that $\partial(F_1 \cup F_2 \cup F_3)$ is a cycle of length 6, then there exist $F, F' \in Y$ such that $\partial(F_1 \cup F_2 \cup F_3) = \partial(F \cup F')$, i.e. this cycle has a diagonal that divides it into two 4-cycles.*

**Proposition 5.2.4.** *Let $Y$ be the quadrization of a $C(4)$–$T(4)$ complex $X$ and $\alpha$ be some cycle in $Y$. If $D$ is a minimal area disc diagram for $\alpha$ in $Y$, then $D$ is a CAT(0) square complex.*




(A)




(B)

Figure 5.2: Replacement rules for quadric complexes

*Proof.* Suppose that there is an internal vertex $v$ in $D$ incident to $k < 4$ squares. Since $v$ is internal, $k$ is either 2 or 3. Let $D'$ be the union of squares incident to $v$.

Suppose $k = 2$. Let $F_1 = [v, x_1, x_2, x_3], F_2 = [v, x'_1, x'_2, x'_3]$. Since $v$ is an internal vertex, without loss of generality $x_1 = x'_1, x_3 = x'_3$. The vertex $x_2$ cannot be the same vertex as $x'_2$ as then both $F_1$ and $F_2$ would be spanned by the same 4-cycle what contradicts the definition of quadrization. Therefore $\langle x_2, x_1, x'_2, x_3\rangle = \partial D'$. By the first rule of replacement, it spans a square that contradicts the minimality of the area of $D$.

Suppose that $k = 3$. Without loss of generality we may assume that every internal vertex of $D$ is incident to at least three squares. Then $D'$ consists of 3 squares bounded by a 6-cycle. By the second rule of replacement there exists diagram $D''$ such that $\partial D'' = \partial D'$ and $D''$ consists of two squares, a contradiction to minimality of the area of $D$. $\square$

Let $Y$ be the quadrization of a $C(4)$–$T(4)$ complex. Let $L$ be the subcomplex of $Y$ with the set of vertices

$$\{u_1, \dots, u_n, v_1, \dots, v_n, w_1, \dots, w_n, c\},$$

the set of edges consisting of the edges of the form $(u_i, u_{i+1})$, $(v_i, v_{i+1})$, $(w_i, w_{i+1})$, $(w_i, v_i)$, $(v_i, u_i)$ and $(c, w_n), (c, u_n)$ and the set of squares consisting of squares $[u_i, u_{i+1}, v_i, v_{i+1}]$, $[v_i, v_{i+1}, w_i, w_{i+1}]$ and $[c, u_n, v_n, w_n]$. We call such a complex a *double ladder of length $n$ with a cap* and denote it by $\{u_1, \dots, u_n|v_1, \dots, v_n|w_1, \dots, w_n|c\}$. The square $[c, u_n, v_n, w_n]$ is called a *cap* (see Fig. 5.3).

**Proposition 5.2.5.** *If $\{u_1, \dots, u_n|v_1, \dots, v_n|w_1, \dots, w_n|c\}$ is a double ladder with a cap, then at least one of the following conditions holds:*

*1) $(c, v_{n-1}) \in Y$;*

*2) for some $i \in \{3, \dots, n\}$ $(u_i, v_{i-2}) \in Y$;*

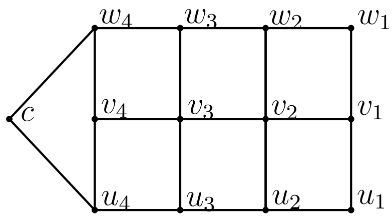



Figure 5.3: Double ladder $\{u_1, \ldots, u_4 | v_1, \ldots, v_4 | w_1, \ldots, w_4 | c\}$ of length 4 with a cap.

*3) for some* $i \in \{3, \ldots, n\}$ $(w_i, v_{i-2}) \in Y$*;*

*4)* $(u_2, w_1) \in Y$*;*

*5)* $(u_1, w_2) \in Y$*.*

*Proof.* Assume that none of these holds. By definition of a double ladder with a cap, $\langle c, w_n, w_{n-1}, v_{n-1}, u_{n-1}, u_n \rangle$ is a 6-cycle in $Y$, that bounds three squares. By the second rule of replacement there exists a diagonal splitting into two 4-cycles. Therefore one of $(c, v_{n-1}), (u_n, w_{n-1}), (u_{n-1}, w_n)$ belongs to $Y$; see Fig. 5.4

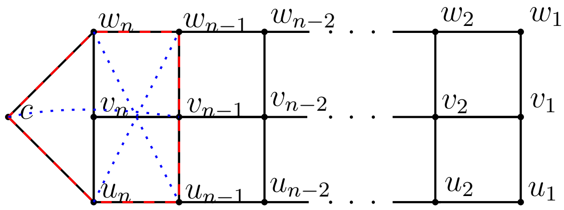



Figure 5.4: The 6-cycle $\langle c, w_n, w_{n-1}, v_{n-1}, u_{n-1}, u_n \rangle$ bounds three squares in red. One of blue edges belongs to $Y$.

Since 1) does not hold, we obtain a shorter double ladder with the cap being $[w_n, u_{n-1}, v_{n-1}, w_{n-1}]$ or $[u_n, u_{n-1}, v_{n-1}, w_{n-1}]$.

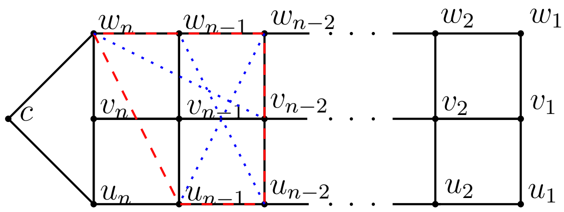



Figure 5.5: One of the possible options for the second 6-cycle that bounds three squares in red. One of the blue edges belongs to $Y$.

Since there is no $i$ such that 2) or 3) holds, by induction we obtain that $(u_2, w_1) \in Y$ or $(u_1, w_2) \in Y$, a contradiction. □

## 5.3 Systolic complexes and Wise complex

In this section we define systolic complexes and state some results concerning the Wise complex of a $C(6)$ complex.

We begin by introducing an important notion of *simplicial* complexes. These complexes are $n$-dimensional complexes whose $n$-cells are $n$-simplices. As already mentioned, we only consider 2-complexes. In the case of simplicial complexes, instead of the usual terms, 0-cell, 1-cell and 2-cell, we will use vertex, edge and triangle, respectively.

Let $X$ be a simplicial complex and $L$ be a subcomplex of $X$ with vertices $\{u_1, \ldots, u_n, v_1, \ldots, v_{n+1}\}$, and the set of edges consisting of the edges of the form $(u_i, u_{i+1}), (v_i, v_{i+1}),$ $(u_i, v_i)$ and $(u_i, v_{i+1})$. We call such a complex $L$ a *ladder of length* $n$ and denote it by $\{u_1, \ldots, u_n | v_1, \ldots, v_{n+1}\}$.

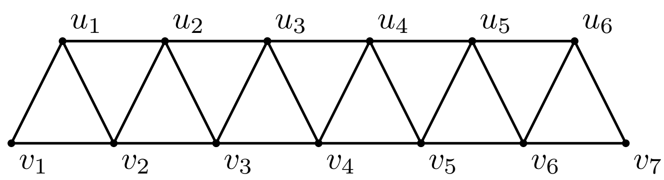



Figure 5.6: Ladder of length 6.

A *CAT(0) simplicial complex* is a simplicial complex for which the metric obtained by making each triangle isometric to the equilateral Euclidean triangle satisfies the CAT(0) condition. As in the square complex case we use a combinatorial characterization, following from Gromov's link condition, as the definition.

**Definition 5.3.1.** A simplicial complex $X$ is CAT(0) if it is simply connected and for any 0-cell $v \in X$ the shortest embedded cycle in the link of $v$ has length at least 6.

If $X$ is a disc diagram, the CAT(0) condition means that each internal vertex is incident to at least six triangles.

Let $X$ be a 2-complex with embedded 2-cells, such that every 1-cell is contained in the boundary of a 2-cell.

Let $\mathcal{U}$ be a cover of $X$. The *nerve* of the cover $\mathcal{U}$ is a simplicial complex whose vertex set is $\mathcal{U}$, and vertices $U_0, \ldots, U_n$ span an $n$-simplex if and only if $\bigcap_{0 \leq i \leq n} U_i \neq \emptyset$.

**Definition 5.3.2.** The *Wise complex* $Y$ of a complex $X$ is the nerve of the covering of $X$ by closed 2-cells.

By definition Wise complex is a simplicial complex. Similarly as in the case of quadrization, we will sometimes consider some cells in both the complex $X$ and its Wise complex $Y$ at the same time. In such cases by $\widehat{x}$ we will denote the 2-cell corresponding to the vertex $x$ from $Y$.

It is known that the Wise complex of a simply connected $C(6)$ complex is systolic [Wis03, Theorem 10.6], see also [OP18].

**Definition 5.3.3.** Let $X$ be a simplicial complex. $X$ is *$k$-large* if $X$ is flag and every cycle in $X$ of length less than $k$ has a diagonal, i.e. in $X$ there is an edge connecting nonconsecutive vertices of the cycle.

**Lemma 5.3.4.** *[JS06, Lemma 1.3] Suppose that $X$ is $k$-large and $\mathbb{S}^1_m$ denotes the triangulation of $\mathbb{S}^1$ with $m$ 1-cells. If $m < k$ then any simplicial map $f : \mathbb{S}^1_m \to X$ extends to a simplicial map from disc $\mathbb{D}^2$, triangulated so that the triangulation on the boundary is $\mathbb{S}^1_m$ and so that there are no interior vertices in $\mathbb{D}^2$.*

**Definition 5.3.5.** A simplicial complex $X$ is *systolic* if $X$ is simply connected, and links of all vertices of $X$ are 6-large.

Proposition below follows from [Che00, Claim 1, Theorem 8.1], we note here that *bridged complexes* in that paper are exactly systolic complexes.

**Proposition 5.3.6.** *Let $\alpha$ be a cycle in a systolic complex $X$. If $D$ is a minimal area disc diagram for $\alpha$ in $X$, then $D$ is a CAT(0) simplicial complex.*

Let $L$ be the flag subcomplex of $Y$ spanned by the set of vertices

$$\{u_1, \dots, u_n, v_1, \dots, v_n, w_1, \dots, w_n\}$$

and the set of edges:

$$\{(u_i, u_{i+1}), (v_i, v_{i+1}), (w_i, w_{i+1}), (w_i, v_{i+1}), (v_{i+1}, u_i) : 1 \le i \le n-1\} \cup$$

$$\{(w_i, v_i), (v_i, u_i) : 1 \le i \le n\} \cup \{(u_n, w_n)\}.$$

We call such a complex a *double ladder of length $n$ with a cap* and denote it by

$$\{u_1, \dots, u_n | v_1, \dots, v_n | w_1, \dots, w_n\}.$$

The triangle $[u_n, v_n, w_n]$ is called a *cap*, see Fig. 5.7.

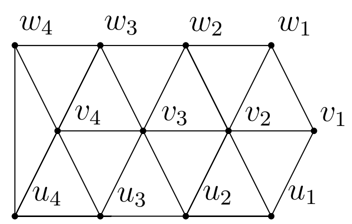



Figure 5.7: Double ladder $\{u_1, \dots, u_4 | v_1, \dots, v_4 | w_1, \dots, w_4\}$ of length 4 with a cap.

**Proposition 5.3.7.** *If $\{u_1, \dots, u_n | v_1, \dots, v_n | w_1, \dots, w_n\}$ is a double ladder with a cap, then at least one of the following conditions holds:*

*1) for some $i \in \{3, \dots n\}$ $(u_i, v_{i-1}) \in Y$;*

*2) for some $i \in \{3, \dots n\}$ $(w_i, v_{i-1}) \in Y$;*

*3) $(u_1, w_1) \in Y$.*

*Proof.* Assume that none of these holds. By definition of a double ladder with a cap, $\langle w_n, w_{n-1}, v_{n-1}, u_{n-1}, u_n\rangle$ is a 5-cycle in $Y$. By systolicity, one of $(w_n, v_{n-1}), (u_n, v_{n-1}), (u_{n-1}, w_{n-1})$ belongs to $Y$.

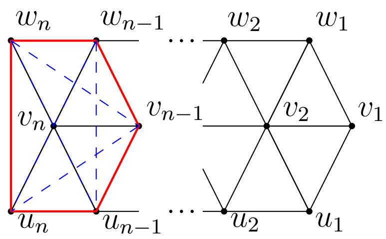



Figure 5.8: The 5-cycle $\langle w_n, w_{n-1}, v_{n-1}, u_{n-1}, u_n\rangle$ in red. One noncrossing pair of blue edges belongs to $Y$.

Since neither 1) nor 2) holds, we obtain a shorter double ladder with the cap $[u_{n-1}, v_{n-1}, w_{n-1}]$. Since there is no $i$ such that 1) or 2) holds, by induction we obtain that $(u_1, w_1) \in Y$. Contradiction. $\square$

# Chapter 6

# Torsion subgroups of small cancellation groups

Most of that chapter concerns Theorem II.2, stating that the Meta-Conjecture from the introduction is true for small cancellation complexes, if we additionally assume that the action is free on the 1-skeleton. In Sections 6.2 - 6.6 we prove technical results, necessary for the proof of Theorem II.2. Moreover, in Section 6.6 we prove existence of CAT(0) structure for $C(3)$–$T(6)$ complexes and conclude that the Tits Alternative holds for groups acting almost freely on $C(3)$–$T(6)$ complexes. The remaining sections finish the proof of Theorem II.2 and apply it to prove automatic continuity and that torsion subgroups of groups defined by small cancellation presentations are finite.

## 6.1 Description of the results

The main result is the following theorem.

**Theorem II.1.** *Torsion subgroups of groups defined by $C(6)$, $C(4)$–$T(4)$, and $C(3)$–$T(6)$ small cancellation presentations are finite cyclic groups.*

The theorem establishes a particular case of Meta-Conjecture from the introduction. It may be seen as a small cancellation counterpart of a (still) conjectural feature of CAT(0) groups (see [Swe99]). Note however that in our result we allow the presentations to be infinite. Theorem II.1 is an immediate consequence of the following result.

**Theorem II.2.** *Let $X$ be a simply connected $C(6)$, $C(4)$–$T(4)$, or $C(3)$–$T(6)$ small cancellation complex. Let $G$ be a group acting on $X$ by automorphisms such that the action induces a free action on the 1-skeleton $X^1$ of $X$. If the action is locally elliptic, then $G$ is a finite cyclic group. In particular, $G$ fixes a 2-cell of $X$.*

Although we believe that some versions of Theorem II.2 hold without the assumptions on the freeness of the action, in the current statement these cannot be omitted. For example, it has been shown by Serre [Ser03, Theorem 15, Chapter I.6.1] that any countable infinitely generated group acts without fixed points on a tree, which is a 1-dimensional small cancellation complex (the conjectures in [HO21] concern mostly finitely generated groups). Furthermore, our assumptions are tailored for the application to Theorem II.1.

In general, there is no known way of equipping a small cancellation complex with a "reasonable" CAT(0) structure (and some experts doubt it can be done at all). Nevertheless, we observe that $C(3)$–$T(6)$ complexes admit such a structure. This is a relatively simple observation following a remark by Pride [Pri88, p.165]. Although the next result implies a number of significant features of $C(3)$–$T(6)$ groups, it seems it has not been observed before.

**Theorem II.3.** *Let $X$ be a simply connected $C(3)$–$T(6)$ small cancellation complex. Then there exists a metric on $X$ turning it into a CAT$(0)$ triangle complex $\mathfrak{X}$ such that every automorphism of $X$ induces an automorphism of $\mathfrak{X}$.*

The existence of a CAT(0) structure allows us to extend Theorem II.2 in the $C(3)$–$T(6)$ case, using results of [NOP22].

**Corollary II.4.** *A finitely generated group acting locally elliptically on a simply connected $C(3)$–$T(6)$ small cancellation complex fixes a point.*

From the CAT(0) property of $C(3)$–$T(6)$ complexes we also conclude the following result closely related to the non-existence of infinite torsion subgroups. Recall that a group satisfies the *Tits Alternative* if each of its finitely generated subgroups either contains a free nonabelian subgroup or is virtually solvable. It is believed that "nonpositively curved" groups satisfy the Tits Alternative, but this has been proved only in a limited number of cases. See [SW05; OP21; OP22] for more details. The following theorem states that the Tits Alternative holds for groups acting *almost freely* (there is a bound on the order of cell stabilisers) on $C(3)$–$T(6)$ complexes.

**Corollary II.5.** *Let $G$ be a group acting almost freely on a simply connected $C(3)$–$T(6)$ small cancellation complex. Then $G$ is virtually cyclic, or virtually $\mathbb{Z}^2$, or contains a nonabelian free group.*

Finally, let us present an application of Theorem II.1 to *automatic continuity.* This property has its origins in the descriptive set theory and, roughly, says that every group homomorphism between topological groups is continuous. Recently automatic continuity has been established for a large class of homomorphisms into "nonpositively curved" groups, see e.g. [KMV22] – we extend some results of that paper in the following, where $G$ is equipped with a discrete topology.

**Theorem II.6.** *Let $G$ be a group acting geometrically on a locally finite, simply connected $C(6)$, $C(4)$–$T(4)$, or $C(3)$–$T(6)$ small cancellation complex $X$ such that the action induces a free action on the 1-skeleton of $X$. If $H$ is a subgroup of $G$ then any group homomorphism $\varphi : L \to H$ from a locally compact group $L$ is continuous or there exists a normal open subgroup $N \subseteq L$ such that $\varphi(N)$ is a finite group.*

## Related results

For finitely generated groups acting on uniformly locally finite simply connected $C(4)$–$T(4)$ complexes a version of Theorem II.2 has been proved recently in [HO21, Corollary B(3)]. The proof there uses the fact (established in [Cha+ a]) that groups acting geometrically on simply connected $C(4)$–$T(4)$ small cancellation complexes are Helly. In particular, in the case of finitely presented groups the $C(4)$–$T(4)$ part of Theorem II.1 follows from [HO21, Corollary B(3)]. We do not need such finiteness assumptions for our proof. As for further results establishing the Conjecture and, more generally, the (Meta-)Conjecture from [HO21] in specific cases of small cancellation let us mention the case of CAT(0) square complexes (an example of $C(4)$–$T(4)$) following from [Sag95; LV20], the case of 2-dimensional systolic complexes (an example of $C(3)$–$T(6)$) and other 2-dimensional CAT(0) complexes from [NOP22], the case of (graphical) $C(18)$ complexes from [HO21, Theorem F], the case of $C'(1/4)$–$T(4)$ complexes from [Gen21], and the case of some two-dimensional buildings from [SST20].

Whether the Tits Alternative holds for small cancellation groups is an open problem. In [Col73; AJ77; EH88] the Weak Tits Alternative is shown, see [SW05] for a discussion.

The $C(4)$–$T(4)$ case of Theorem II.6 has been established in [HO21, Corollary H].

### Idea of the proof of Theorem II.2

We restrict here to $C(6)$, and $C(4)$–$T(4)$ cases. First, we show that each element of the group $G$ fixes the center of exactly one 2-cell. Therefore $G$ does not have elements of infinite order, as such elements would not act freely on the 1-skeleton of $X$. Then we show that if $G$ acts on $X$ without a global fixed point, then $G$ has an element of infinite order, contradiction.

To prove that latter implication, consider two elements $f, g \in G$ with $\mathrm{Fix}(f) \neq \mathrm{Fix}(g)$. We consider a "dual" complex $Y$ of $X$: it is the quadrization in the $C(4)$–$T(4)$ case, and the Wise complex in the $C(6)$ case. These complexes are: quadric and systolic, respectively. In particular, $Y$ is simply connected and $G$ acts by automorphisms on $Y$.

Both $\mathrm{Fix}_Y(f)$ and $\mathrm{Fix}_Y(g)$ consist of one vertex each, we denote them by $x$ and $y$. Since $Y$ is connected, we can find a geodesic $\gamma := (x_0 := y, x_1, \ldots, x_n := x)$ in $Y$. We find $k, l$ such that $x_1$ (resp. $x_{n-1}$) is at distance at least 3 from $g^l x_1$ (resp. $f^k x_{n-1}$) in the link of $y$ (resp. $x$). For such $k, l$ and any $i$, we show that the path $\alpha_i := \bigcup_{0 \leq j \leq i} (f^k g^l)^j (\gamma \cup f^k \gamma)$ is a geodesic. Therefore $f^k g^l$ has infinite order.

## 6.2 Curvature

Let $D$ be a disc diagram that considered as a CW-complex is a square complex. In this section we will call such diagrams *square disc diagrams*.

The *square curvature* of a vertex $v \in D$ is defined as

$$\kappa_D^{\square}(v) = 4 - 2\delta_D(v) + \rho_D^{\square}(v),$$

where $\delta_D(v)$, as previously, denotes the valence of vertex $v$ and $\rho_D^{\square}(v)$ denotes the number of squares incident to $v$, cf. Fig.6.1.

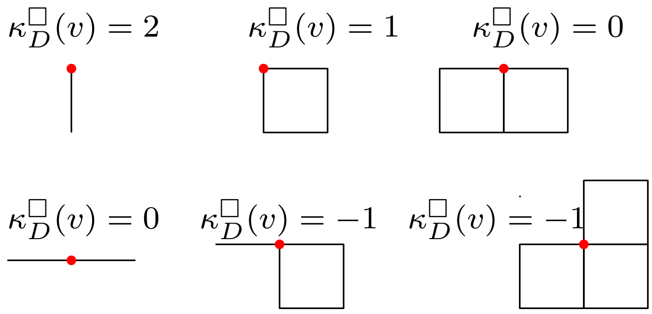



Figure 6.1: All the possible neighborhoods of a boundary vertex $v$ (red) of a square disc diagram $D$ such that $-1 \leq \kappa_D^{\square}(v) \leq 2$.

Analogously we define the curvature for simplicial complexes. Let $D$ be a disc diagram that considered as a CW-complex is a simplicial complex. In this section, we will call such diagrams *simplicial disc diagrams.*

The *simplicial curvature* of a vertex $v \in D$ is defined as follows

$$\kappa_D^{\triangle}(v) = 6 - 3\delta_D(v) + 2\rho_D^{\triangle}(v),$$

where $\rho^{\triangle}(v)$ denotes the number of triangles incident to $v$, cf. Fig.6.2.

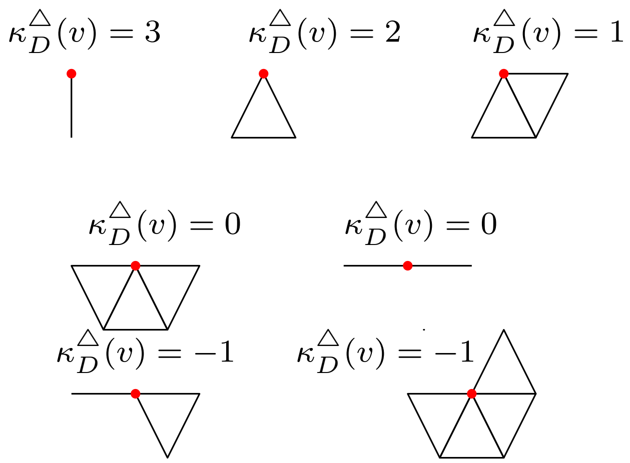



Figure 6.2: All the possible neighborhoods of a boundary vertex $v$ (red) of a simplicial disc diagram $D$ such that $-1 \leq \kappa_D^{\triangle}(v) \leq 3$.

We now state a version of the combinatorial Gauss-Bonnet theorem for CAT(0) square and simplicial disc diagrams. Generalization of the Gauss-Bonnet theorem to arbitrary 2-complexes was stated and proven by McCammond and Wise in [MW02, Theorem 4.6]. The version for CAT(0) square disc diagrams was already stated by Hoda in [Hod20, Proposition 1.8].

**Proposition 6.2.1** (Gauss-Bonnet Theorem for a CAT(0) Disc Diagrams)**.** *Let $D$ be a CAT(0) square disc diagram and let $E$ be a CAT(0) simplicial disc diagram.*

*Then:*

$$\sum_{v\in D}\kappa_D^{\square}(v)=4 \ \textit{and} \ \sum_{v\in E}\kappa_E^{\triangle}(v)=6,$$

*moreover:*

$$\sum_{v\in\partial D}\kappa_D^{\square}(v)\geq 4 \ \textit{and} \ \sum_{v\in\partial E}\kappa_E^{\triangle}(v)\geq 6.$$

*Proof.* We only show the proof in the case of simplicial disc diagrams, case of square disc diagrams is analogous, and was already proved by Hoda.

For a disc diagram the Euler characteristic $\chi(E)$ is equal to 1. It can be computed by subtracting the number of edges from the number of vertices and triangles. That is, each edge contributes $-1$ to the Euler characteristic and each vertex or 2-cell contributes $+1$. Distributing $\frac{-1}{2}$ to each of the endpoints of the edge, and $+\frac{1}{3}$ to each of vertices at the boundary of each triangle we obtain:

$$\chi(E)=\sum_{v\in E}\frac{\kappa_E^{\triangle}(v)}{6}.$$

In the case of internal vertices, we have $\delta_E(v)=\rho_E^{\triangle}(v)$ and therefore by the CAT(0) property $\kappa_E^{\triangle}(v)$ is nonpositive for the internal vertices. □

The square and the simplicial curvatures have analogous properties, from now on we will often not differentiate between them and just denote it by $\kappa_D(v)$ with the type of the curvature known from context.

**Proposition 6.2.2.** *Let $D$ be either a square or simplicial disc diagram and let $\gamma\subset\partial D$ be a geodesic in $D$. Then none of the internal vertices of $\gamma$ has the curvature greater than 1. Moreover, if $u,v$ are internal vertices of $\gamma$ with $\kappa_D(u)=\kappa_D(v)=1$, then there is a vertex $w\in\gamma$ between $v$ and $u$ with $\kappa_D(u)\leq -1$.*

*Proof.* Let $\gamma:=(v_0,v_1,\ldots,v_n)$. If $\kappa_D^{\square}(v_i)>1$, then, by Figure 6.1, we have $v_{i-1}=v_{i+1}$. If $\kappa_D^{\triangle}(v_i)>1$, then, by Figure 6.2, either $v_{i-1}=v_{i+1}$ or there exists an edge between $v_{i-1}$ and $v_{i+1}$. Either case contradicts the fact that $\gamma$ is a geodesic.

Let $u=v_i$ and $v=v_j$. Assume, without loss of generality, that $\kappa_D(v_s)=0$ for $i<s<j$. If $D$ is a square disc diagram, the distance between $v_{i-1}$ and $v_{j+1}$ is equal to at most the distance between $v_i$ and $v_j$. If $D$ is a simplicial disc diagram, the distance between $v_{i-1}$ and $v_{j+1}$ is equal to at most the distance between $v_i$ and $v_j$ increased by one. Either case contradicts the fact that $\gamma$ is a geodesic. □

In fact, these properties of geodesics stated above characterize geodesics in systolic and quadric complexes.

**Lemma 6.2.3.** *Let $\gamma:=\{g_0,g_1,\ldots,g_n\}$ be a path in an either systolic or quadric complex $X$ and let $\gamma':=\{g'_0=g_0,g'_1,\ldots,g'_m=g_n\}$ be a geodesic between the endpoints of $\gamma$ such that a minimal area disc diagram $D$ for $\gamma\cup\gamma'$ in $X$ has the smallest area. If for all $0<i\leq j<n$ we have*

$$\sum_{i\leq k\leq j}\kappa_D(g_k)\leq 1,$$

*then $\gamma$ is a geodesic.*

*Proof.* By Proposition 6.2.2 all vertices along $\gamma'$ have nonpositive curvature in $D$. Indeed, if for some $0 < i < m$ we have $\kappa_D(g'_i) = 1$, then there exists a vertex $g''_i \in D$ incident to $g'_{i-1}$ and $g'_{i+1}$ (see Figures 6.1 and 6.2). Therefore, there exists a smaller area disc diagram $D'$ for $\gamma \cup \gamma''$, where $\gamma'' := \{g'_0, \dots, g'_{i-1}, g''_i, g'_{i+1} \dots, g'_n\}$. Therefore, the sum of the curvatures along $\gamma'$ is bounded by 0.

On the other hand, the curvature along $\gamma$ is bounded by 1. Then by nonpositiveness of the curvature of the internal vertices and Proposition 6.2.1, it follows that the sum of the curvature of vertices $g_0, g_n$ is at least 5 in the case of the simplicial disc diagrams and at least 3 in the case of square disc diagrams. In either case at least one of $g_0, g_n$ has to be a spur, i.e. a vertex of valence 1.

We will show that $D$ is a tree. Assume that $D$ is not a tree and without loss of generality assume that $g_0$ is a spur. We take the smallest $i$ such that $g_i$ is on the boundary of a 2-cell. Then the curvature of $g_i = g'_i$ has to be negative (see Figures 6.1, 6.2), and the curvature of vertices $g_j = g'_j$ for $0 < j < i$ is 0. Therefore, the sum of the curvatures along $\gamma'$ is negative. The sum of the curvatures of vertices $g_j$ for $i < j < n$ is bounded by 1. Therefore the sum of the curvature along $\gamma$ is 0. Thus the sum of the curvatures of vertices $g_0, g_n$ is at least 7 in the case of the simplicial disc diagrams, or at least 5 in the case of square disc diagrams. This is possible only if one of these vertices is not connected to the rest of the diagram, a contradiction.

If $D$ is a tree and $\gamma$ is not a geodesic, then there exists vertex $g_j$ for $0 < j < n$ that is a spur, but spurs have curvature greater than 1, a contradiction. $\square$

## 6.3 Combining geodesics

In this section, in Lemma 6.3.2 and Lemma 6.3.4, we show that some particular paths obtained via gluing two geodesics at their endpoints are geodesics themselves. This is used in Section 6.5 to construct automorphisms of infinite order.

### 6.3.1 $C(4)$–$T(4)$ case.

Here we assume that $X$ is a simply connected $C(4)$–$T(4)$ small cancellation complex such that every 1-cell of $X$ is contained in the boundary of a 2-cell. Let $Y$ be the quadrization of $X$. Let $G$ be a finitely generated group acting on $X$ by automorphisms and assume that this action induces a free action on the 1-skeleton $X^1$ of $X$. It is clear that $G$ acts on $Y$ by automorphisms and this action induces a free action on the sets of edges and vertices from $X_0$. Moreover, since by Proposition 5.1.8 any non-empty intersection of two 2-cells from $X$ is connected, the action of $G$ on $Y$ induces a free action on the set of squares as well. From now on, if $D$ is a diagram in $Y$ and $v$ is a vertex in $D$ then $v$ is mapped to a vertex in $Y$ denoted by $v^Y$. In some cases, we denote by $u^D$ a vertex in $D$ that is mapped to a vertex $u \in Y$. Note that there is some ambiguity, as more than one vertex can be mapped to $u$, therefore we only use this notation when this ambiguity does not matter.

The aim of this section is to prove a technical Lemma 6.3.2, which is necessary for the proof of the $C(4)$–$T(4)$ part of Lemma 6.5.3, the main lemma of Section 6.5. But first, we need to prove the following.

**Lemma 6.3.1.** *Let $D$ be a minimal area disc diagram in $Y$ and $x \in D$ be a vertex such that $x^Y$ is fixed by some $h \in G$. Let $(u_n, u_{n-1}, \dots, u_1, u_0 = x =$*

*$v_0, v_1, \dots, v_n, v_{n+1}, \overline{u}_n, \overline{u}_{n-1}, \dots, \overline{u}_1, \overline{u}_0 = \overline{x} = \overline{v}_0, \overline{v}_1, \dots, \overline{v}_n, \overline{v}_{n+1})$ be a tuple of vertices such that (see Figure 6.3):*

*1) $hu_i^Y = v_i^Y$ for $0 \le i \le n$;*

*2) $h\overline{u}_i^Y \neq \overline{v}_i^Y$ for $0 \le i \le n$;*

*3) $\{u_n, u_{n-1}, \dots, u_1, x, v_1, \dots, v_n, v_{n+1} | \overline{u}_n, \overline{u}_{n-1}, \dots, \overline{u}_1, \overline{x}, \overline{v}_1, \dots, \overline{v}_n, \overline{v}_{n+1}\}$ is a ladder in $D$;*

*4) $\{h\overline{x}^Y, h\overline{u}_1^Y \dots, h\overline{u}_n^Y | x^Y, v_1^Y, \dots, v_n^Y | \overline{x}^Y, \overline{v}_1^Y, \dots, \overline{v}_n^Y | \overline{v}_{n+1}^Y\}$ is a double ladder with a cap in $Y$.*

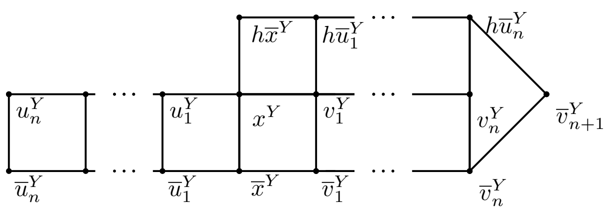



Figure 6.3:

*Then at least one of the following holds:*

*(i) $(\overline{v}_s^Y, v_{s-2}^Y) \in Y$ for some $2 \le s \le n+1$;*

*(ii) $(\overline{u}_s^Y, u_{s-2}^Y) \in Y$ for some $2 \le s \le n$ ;*

*(iii) $d_{Y_x}(h\overline{x}^Y, \overline{x}^Y) < 2$.*

*Proof.* By Proposition 5.2.5 at least one of the following holds:

(a) $(\overline{v}_{n+1}^Y, v_{n-1}^Y) \in Y$;

(b) $(h\overline{u}_s^Y, v_{s-2}^Y) \in Y$ for some $2 \le s \le n$;

(c) $(\overline{v}_s^Y, v_{s-2}^Y) \in Y$ for some $2 \le s \le n$;

(d) $(h\overline{u}_1^Y, \overline{x}^Y) \in Y$;

(e) $(\overline{v}_1^Y, h\overline{x}^Y) \in Y$.

The case (i) is satisfied if (a) or (c) holds.

If (b) holds, then case (ii) is satisfied. Indeed, if for some $2 \le s \le n$ we have $(h\overline{u}_s^Y, v_{s-2}^Y = hu_{s-2}^Y) \in Y$, then $(\overline{u}_s^Y, u_{s-2}^Y) \in Y$.

The case (iii) is satisfied if either (d) or (e) holds. Indeed, if one of these cases holds, then $R_1 := [\overline{x}^Y, x^Y, h\overline{x}^Y, h\overline{u}_1^Y]$ or $R_2 := [\overline{x}^Y, x^Y, h\overline{x}^Y, \overline{v}_1^Y]$ is a square in $Y$, therefore $d_{Y_x}(h\overline{x}^Y, \overline{x}^Y) < 2$. □

**Lemma 6.3.2.** *Let $x \in X_2$ and $g \in G$ be such that $x \in \mathrm{Fix}_Y(g)$. Assume that for every square $P \in Y$ containing $x$ in its boundary, we have $P \cap gP = \{x\}$. Let $\gamma_1 := (x = x_0, x_1, \dots, x_n)$ and $\gamma_2 := (x = y_0, y_1, \dots, y_m)$ be a geodesics in $Y$, such that $n \le m$ and for all $i \le n$ we have $gx_i = y_i$. Then $\alpha := \gamma_1 \cup \gamma_2$ is a geodesic.*

*Proof.* Assume that $\alpha$ is not a geodesic. Then there exists a geodesic $\beta$ between $x_n$ and $y_m$. From the set of geodesics between $x_n$ and $y_m$ choose $\beta$ such that the minimal area disc diagram $D$ for the path $\alpha \cup \beta$ has the smallest area.

We want to show that for any $u, v \in \alpha$ such that $x_n < u^Y < v^Y < y_m$, if we have $\kappa_D(u) = \kappa_D(v) = 1$, then there is a vertex $w \in \alpha$, such that $u^Y < w^Y < v^Y$ and $\kappa_D(w) \leq -1$. It is clear that in such case the sum of the curvature along any subpath of $\alpha$ is bounded by 1, and the assumptions of Lemma 6.2.3 are satisfied.

First, observe that between $u$ and $v$ there is no vertex with curvature 2. Indeed, a boundary vertex has curvature 2 iff it is an end of a spur (see Figure 6.1). By Proposition 6.2.2 such a vertex cannot be an internal vertex of a geodesic, so it has to be mapped to $x$. But if $x^D$ is a spur then $g$ fixes a vertex from $X_0$, a contradiction. Therefore each vertex between $u$ and $v$ is incident to at least one square (see Figure 6.1).

Assume that there are no vertices of curvature at most $-1$ between $u$ and $v$. Without loss of generality, we can assume that all vertices between $u$ and $v$ have curvature 0 and by Figure 6.1 each of them is incident to two squares. Observe, that $u^Y \neq x \neq v^Y$. Indeed, if $\kappa_D(x^D) = 1$ then $x^D$ is incident to exactly one square $P \in D$. Therefore there is $P^Y \in Y$ corresponding to $P$ such that $P^Y \cap gP^Y \neq \{x\}$, a contradiction.

By Proposition 6.2.2, $u$ and $v$ cannot be internal vertices of the same geodesic. Therefore we can assume that $u^Y \in \gamma_1$, $v^Y \in \gamma_2$.

Now we will find a tuple satisfying the assumptions 1)-4) of Lemma 6.3.1. We begin by taking the sequence $u_N = u, u_{N-1}, \ldots, u_1, u_0 = x^D = v_0, v_1, \ldots, v_{M-1}, v_M = v$ of consecutive vertices in $\partial D$ between $u$ and $v$. We have $gu_p^Y = v_p^Y$ for $p \leq N$.

We assumed that $u_N, v_M$ have curvature 1 and all of vertices between them have curvature 0. Because of that the neighborhood of $u_N, \ldots, u_1, x^D s, v_1, \ldots, v_M$ has the following form. The vertex $x^D$ is adjacent to three vertices, two of them being $u_1$, $v_1$, we denote the remaining adjacent vertex by $\overline{x}$. The vertex $u_N$ (resp. $v_M$) is adjacent to two vertices, one of them $u_{N-1}$ (resp. $v_{M-1}$), we denote the remaining adjacent vertex by $\overline{u_N}$ (resp. $\overline{v_M}$). For $0 < s < N$ (resp. $0 < p < M$) the vertex $u_s$ (resp. $v_p$) is adjacent to three vertices, two of them being $u_{s-1}$, $u_{s+1}$ (resp. $v_{p-1}$, $v_{p+1}$). We denote the remaining adjacent vertex by $\overline{u}_s$ (resp. $\overline{v}_p$), see Figure 6.4.

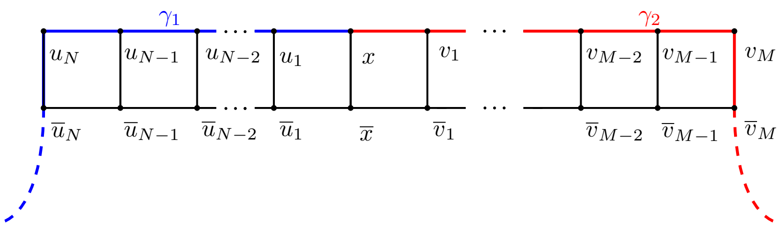



Figure 6.4: Disc diagram $D$ in the neighborhood of $u_N, \ldots, u_1, x, v_1, \ldots, v_M$.

Let the sequence $\overline{u}_N^Y, \overline{u}_{N-1}^Y, \ldots, \overline{u}_1^Y, \overline{u}_0^Y = \overline{x}^Y = \overline{v}_0^Y, \overline{v}_1^Y, \ldots, \overline{v}_{M-1}^Y, \overline{v}_M^Y$, be the corresponding sequence of vertices in $Y$. We have $\overline{u_N}^Y \in \gamma_1$ and $\overline{v_M}^Y \in \gamma_2$.

For any $0 < p \leq M$, $0 < s \leq N$ we define squares $V_p = [v_{p-1}^Y, v_p^Y, \overline{v}_p^Y, \overline{v}_{p-1}^Y]$ and $U_s = [u_{s-1}^Y, u_s^Y, \overline{u}_s^Y, \overline{u}_{s-1}^Y]$ (we remind that $x = u_0^Y = v_0^Y$ and $\overline{x}^Y = \overline{u}_0^Y = \overline{v}_0^Y$). We consider the subcomplex $S \subset Y$ consisting of $gU_1, \ldots gU_N, V_1, \ldots V_M$. Let us remind that for $s \leq N$ we have $gu_s^Y = v_s^Y$. We note here that it is possible that $g\overline{u}_s^Y = \overline{v}_s^Y$ as shown in Figure 6.5.

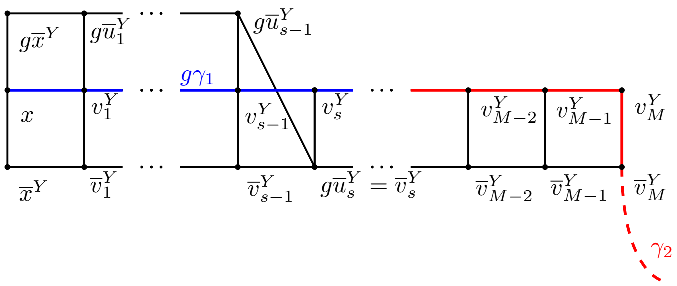



Figure 6.5: Example of the subcomplex $S$. In this case $g\overline{u}_s^Y = \overline{v}_s^Y$ and $g\overline{u}_{s-1}^Y \neq \overline{v}_{s-1}^Y$.

It is clear that one of the following cases holds.

1) $g\overline{x}^Y = \overline{x}^Y$.

2) $g\overline{x}^Y \neq \overline{x}^Y$, and there is some $p \leq N$, such that $g\overline{u}_p^Y = \overline{v}_p^Y$ and $g\overline{u}_s^Y \neq \overline{v}_s^Y$ for $s < p$. If $p = N$ then we have $N = M$ as $\overline{v}_N^Y = g\overline{u}_N^Y \in \alpha$.

3) $g\overline{x}^Y \neq \overline{x}^Y$, $gu^Y \neq v^Y$ and for $s \leq N$ we have $g\overline{u}_s^Y \neq \overline{v}_s^Y$. In this case we have $g\overline{u}_N^Y = v_{N+1}^Y$.

We will show that each of these cases results in a contradiction.

In case 1) $g$ is not an identity and fixes $\overline{x}^Y$. But $\widehat{\overline{x}^Y}$ is a 0-cell in $X$, therefore we have a contradiction to the freeness of the action of $G$ on $X^1$.

In case 2) we have $g\overline{u}_p^Y = \overline{v}_p^Y$ and $gu_{p-1}^Y \neq v_{p-1}^Y$. It follows that $\partial(gU_p \cup V_p) = \langle \overline{v}_p^Y, \overline{v}_{p-1}^Y, v_{p-1}^Y, g\overline{u}_{p-1}^Y \rangle$ is a 4-cycle in $Y$. Since $Y$ is the quadrization of the complex $X$, each 4-cycle spans a square and $[\overline{v}_p^Y, \overline{v}_{p-1}^Y, v_{p-1}^Y, g\overline{u}_{p-1}^Y] \in Y$. Figure 6.6 show the complex $S$.

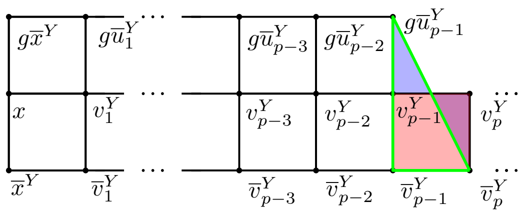



Figure 6.6: The boundary of the union of $gU_p$ (blue+purple) and $V_p$ (red+purple) is a 4-cycle (green).

Thus the cycle $\langle \overline{v}_p^Y, \overline{v}_{p-1}^Y, \ldots, \overline{v}_1^Y, \overline{x}^Y, x, g\overline{x}^Y, g\overline{u}_1^Y, \ldots, g\overline{u}_{p-1}^Y \rangle$ bounds a double ladder with a cap, where the square $[\overline{v}_p^Y, \overline{v}_{p-1}^Y, v_{p-1}^Y, g\overline{u}_{p-1}^Y]$ is a cap. Clearly, the tuple

$$(u_{p-1}, u_{p-2}, \ldots, u_1, x, v_1, \ldots, v_{p-1}, v_p, \overline{u}_{p-1}, \overline{u}_{p-2}, \ldots, \overline{u}_1, \overline{x}, \overline{v}_1, \ldots, \overline{v}_{p-1}, \overline{v}_p)$$

satisfies the conditions of Lemma 6.3.1. Therefore one of the following holds:

(i) for some $2 \le s \le p$, $(\overline{v}_s^Y, v_{s-2}^Y) \in Y$;

(ii) for some $2 \le s \le p-1$, $(\overline{u}_s^Y, u_{s-2}^Y) \in Y$;

(iii) there exists a square $P$ in $Y$ such that $x \in P$ and $P \cap gP \supsetneq \{x\}$.

In case (i) both $\overline{v}_M$ and $v_{s-2}$ are vertices belonging to the geodesic $\gamma_2$. Therefore $(v_{s-2}, \dots, v_M, \overline{v}_M)$ is a geodesic, but $(v_{s-2}, \overline{v}_s, \dots, \overline{v}_M)$ is a shorter path between the same pair of vertices, a contradiction.

In case (ii) both $\overline{u}_N$ and $u_{s-2}$ are vertices belonging to the geodesic $\gamma_1$. Therefore $(u_{s-2}, \dots, u_N, \overline{u}_N)$ is a geodesic, but $(u_{s-2}, \overline{u}_s, \dots, \overline{u}_N)$ is a shorter path between the same pair of vertices, a contradiction.

If (iii) holds, then we have a square $P \ni x$ such that $P \cap gP \neq \{x\}$. A contradiction.

In case 3) we have a double ladder with a cap in $Y$ bounded by the cycle

$$\langle \overline{v}_{N+1}^Y, \overline{v}_N^Y, \dots, \overline{v}_1^Y, \overline{x}^Y, x, g\overline{x}^Y, g\overline{u}_1^Y, \dots, g\overline{u}_N^Y \rangle$$

(see Figure 6.7).

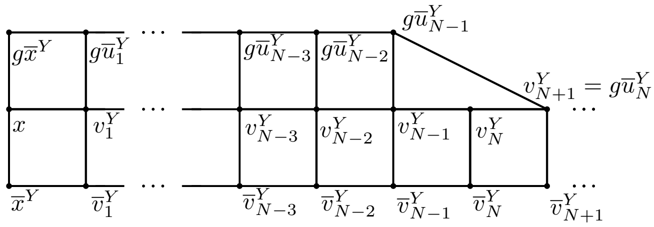



Figure 6.7: Double ladder with a cap in $Y$. The square $[v_{N+1}^Y, \overline{v}_{N+1}^Y, \overline{v}_N^Y, v_N^Y]$ is a cap.

Clearly, the set

$$\{u_N, u_{N-1}, \dots, u_1, x, v_1, \dots, v_N, v_{N+1}, \overline{u}_N, \overline{u}_{N-1}, \dots, \overline{u}_1, \overline{x}, \overline{v}_1, \dots, \overline{v}_N, \overline{v}_{N+1}\}$$

satisfies the conditions of Lemma 6.3.1. Again each case following from this lemma results in a contradiction.

Therefore between any two vertices $x_n < u < v < y_m$ such that $\kappa_D(u) = \kappa_D(v) = 1$, there is a vertex $w$ such that $u^Y < w^Y < v^Y$ and the curvature of $w$ is at most $-1$. As already mentioned, in such a case $\alpha$ satisfies the conditions of Lemma 6.2.3, thus $\alpha$ is a geodesic. $\square$

### 6.3.2 $C(6)$ case.

This subsection is quite similar to the previous one, with lemmas and proofs in this section being analogous to the ones from the previous section. We begin this section with analogous assumptions.

We assume that $X$ is a simply connected $C(6)$ small cancellation complex such that every 1-cell of $X$ is contained in the boundary of a 2-cell. Let $Y$ be the Wise complex of $X$. Let $G$ be a finitely generated group acting on $X$ by automorphisms and assume that this action induces a free action on the 1-skeleton $X^1$ of $X$. It is clear that $G$ acts on $Y$ by automorphisms. From now on, if $D$ is a diagram in $Y$ and $v$ is a vertex in $D$ then $v$ is mapped to a vertex in $Y$ denoted by $v^Y$.

The aim of this section is to prove a technical Lemma 6.3.4, which is a $C(6)$ analogue of Lemma 6.3.2 and is necessary for the proof of the $C(6)$ part of Lemma 6.5.3, the main lemma of the Section 6.5. We begin with a $C(6)$ analogue of Lemma 6.3.1.

**Lemma 6.3.3.** *Let $D$ be a minimal area disc diagram in $Y$ and $x \in D$ be a vertex such that $x^Y$ is fixed by some $h \in G$.*
*Let $(u_n, u_{n-1}, \ldots, u_1, u_0 = x = v_0, v_1, \ldots, v_n, \overline{u}_n, \overline{u}_{n-1}, \ldots, \overline{u}_1, \overline{u}_0, \overline{v}_0, \overline{v}_1, \ldots, \overline{v}_{n-1}, \overline{v}_n)$ be a tuple of vertices such that (see Figure 6.8):*

1) *$hu_i^Y = v_i^Y$ for $0 \le i \le n$;*
2) *$h\overline{u}_i^Y \neq \overline{v}_i^Y$ for $0 \le i \le n$;*
3) *$\{u_n, u_{n-1}, \ldots, u_1, x, v_1, \ldots, v_n | \overline{u}_{n+1}, \overline{u}_n, \ldots, \overline{u}_1, \overline{u}_0, \overline{v}_0, \overline{v}_1, \ldots, \overline{v}_n, \overline{v}_{n+1}\}$ is a ladder in $D$;*
4) *$\{h\overline{u}_0^Y, h\overline{u}_1^Y \ldots, h\overline{u}_n^Y | x^Y, v_1^Y \ldots, v_n^Y | \overline{v}_0^Y, \ldots, \overline{v}_n^Y\}$ is a double ladder with a cap in $Y$.*

*Then at least one of the following holds:*

*(i) for some $1 \le s \le n$, $(\overline{v}_s^Y, v_{s-1}^Y) \in Y$;*

*(ii) for some $1 \le s \le n$, $(\overline{u}_s^Y, u_{s-1}^Y) \in Y$;*

*(iii) $d_{Y_x}(h\overline{u}_0^Y, \overline{u}_0^Y) < 3$.*

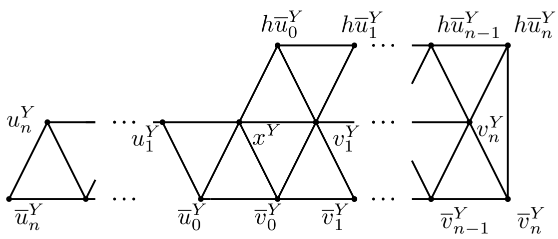



Figure 6.8:

*Proof.* By Proposition 5.3.7 at least one of the following holds:

(a) for some $1 \le s \le n$ $(\overline{v}_s^Y, v_{s-1}^Y) \in Y$;

(b) for some $1 \le s \le n$ $(h\overline{u}_s^Y, v_{s-1}^Y) \in Y$;

(c) $(h\overline{u}_0^Y, \overline{v}_0^Y) \in Y$;

The case (i) is satisfied if (a) holds.

The case (ii) is satisfied if (b) holds. Indeed, if for some $1 \le s \le n+1$ we have $(h\overline{u}_s^Y, v_{s-1}^Y = hu_{s-1}^Y) \in Y$, then $(\overline{u}_s^Y, u_{s-1}^Y) \in Y$.

The case (iii) is satisfied if (c) holds. Indeed, $d_{Y_x}(h\overline{u}_0^Y, \overline{u}_0^Y) \le 2$, follows from $\overline{u}_0^Y$ being incident to $\overline{v}_0^Y$. $\square$

**Lemma 6.3.4.** *Let $x \in Y^0$ and $g \in G$, such that $x \in \mathrm{Fix}_Y(g)$. Assume that for every vertex $y \in Y$ belonging to the link of $x$ in $Y$, we have $d_{Y_x}(gy, y) \geq 3$. Let $\gamma_1 := (x_0 = x, x_1, \ldots, x_n)$ and $\gamma_2 := (y_0 = x, y_1, \ldots, y_m)$ be a geodesics in $Y$, such that $n \leq m$ and for all $i \leq n$ we have $gx_i = y_i$. Then $\alpha := \gamma_1 \cup \gamma_2$ is a geodesic.*

*Proof.* The proof is analogous to the proof of Lemma 6.3.2 and has exactly the same structure. We take a geodesic $\beta$ between $x_n$ and $y_m$ such that minimal area disc diagram $D$ for the path $\alpha \cup \beta$ has the smallest area.

As previously, we want to use Lemma 6.2.3. Therefore we need to show that assumptions of that Lemma are satisfied, and again we do it by showing that for any pair of vertices $u, v \in \alpha$, such that $x_n < u^Y < v^Y < y_m$, if we have $\kappa_D(u) = \kappa_D(v) = 1$, then there is a vertex $w \in \alpha$, such that $u^Y < w^Y < v^Y$ and $\kappa_D(w) \leq -1$.

First, observe that between $u$ and $v$ there is no vertex with curvature greater than 1. A boundary vertex has curvature 3 iff it is an end of the spur and 2 iff it belongs to the boundary of exactly one triangle (see Figure 6.2). By Proposition 6.2.2 such a vertex cannot be an internal vertex of a geodesic, so it has to be mapped to $x$. But for every vertex $y \in Y$ belonging to the link of $x$ in $Y$, we have $d_{Y_x}(gy, y) \geq 3$. Therefore, the curvature of $x$ is at most 0. Assume that there are no vertices of the curvature at most $-1$ between $u$ and $v$. Then each vertex between $u$ and $v$ is incident to at most three triangles (see Figure 6.2). We can assume that all vertices between $u$ and $v$ have curvature 0 and by Figure 6.2 each of them is incident to exactly three triangles.

By Proposition 6.2.2, $u$ and $v$ cannot be internal vertices of the same geodesic. Therefore, we can assume that $u \in \gamma_1$, $v \in \gamma_2$.

Analogously to the $C(4)$–$T(4)$ version of this Lemma, we now want to find a tuple satisfying the assumptions 1)-4) of Lemma 6.3.3. Therefore we take the sequence $u_N = u, u_{N-1}, \ldots, u_1, u_0 = x^D = v_0, v_1, \ldots, v_{M-1}, v_M = v$ of consecutive vertices in $\partial D$ between $u$ and $v$. Since $N < n$ we have $gu_p^Y = v_p^Y$ for $p \leq N$.

We assumed that $u_N, v_M$ have curvature 1 and all of vertices between them have curvature 0. Because of that the neighborhood of $u_N, \ldots, u_1, x, v_1, \ldots, v_M$ has the following form. Vertex $x$ is adjacent to four vertices, two of them being $u_1$, $v_1$, we denote the remaining adjacent vertices by $\overline{u}_0, \overline{v}_0$ so that $(u_1, \overline{u}_0), (v_1, \overline{v}_0) \in D$. For $0 < s < N$ (resp. $0 < p < M$) the vertex $u_s$ (resp. $v_p$) is adjacent to four vertices, two of them being $u_{s-1}$, $u_{s+1}$ (resp. $v_{p-1}$, $v_{p+1}$). One of the remaining two vertices is connected to $u_{s-1}$ (resp. $v_{p-1}$), we denote it by $\overline{u}_{s-1}$ (resp. $\overline{v}_{p-1}$), the last vertex is denoted by $\overline{u}_s$ (resp. $\overline{v}_p$). Vertex $u_N$ (resp. $v_M$) is adjacent to three vertices, two of them being $u_{N-1}$, $\widehat{u}_{N-1}$ (resp. $v_{M-1}$, $\widehat{v}_{M-1}$), we denote the remaining adjacent vertex by $\overline{u}_N$ (resp. $\overline{v}_M$), see Figure 6.9.

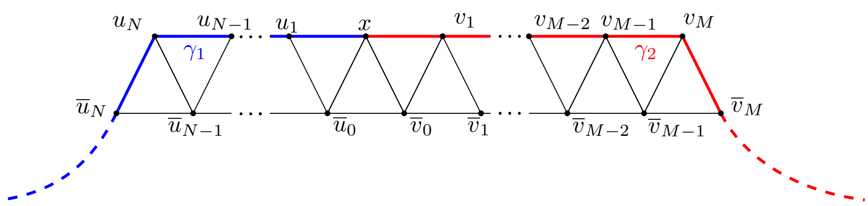



Figure 6.9: The disc diagram $D$ in the neighborhood of $u_N, \ldots, u_1, x, v_1, \ldots, v_M$.

Let the sequence $\overline{u}_N^Y, \overline{u}_{N-1}^Y, \ldots, \overline{u}_1^Y, \overline{u}_0^Y, \overline{v}_0^Y, \overline{v}_1^Y, \ldots, \overline{v}_{M-1}^Y, \overline{v}_M^Y$, be the corresponding sequence of vertices in $Y$. We have $\overline{u}_N^Y \in \gamma_1$ and $\overline{v}_M^Y \in \gamma_2$.

For any $0 < p \leq M$, $0 < s \leq N$ we define the triangles $V_p = [v_{p-1}^Y, v_p^Y, \overline{v}_{p-1}^Y]$, $\overline{V}_p = [v_p^Y, \overline{v}_p^Y, \overline{v}_{p-1}^Y]$, $U_s = [u_{s-1}^Y, u_s^Y, \overline{u}_{s-1}^Y]$, and $\overline{U}_s = [u_s^Y, \overline{u}_s^Y, \overline{u}_{s-1}^Y]$ (we remind that $x = u_0^Y = v_0^Y$). We consider the subcomplex $S \subset Y$ consisting of $gU_1, \ldots, gU_N, g\overline{U}_1, \ldots, g\overline{U}_N, V_1, \ldots, V_M, \overline{V}_1, \ldots, \overline{V}_M$. Let us remind that for $s \leq N$ we have $gu_s^Y = v_s^Y$. Similarly as in the $C(4)$–$T(4)$ case, it is possible that $g\overline{u}_s^Y = \overline{v}_s^Y$, as shown in Figure 6.10.

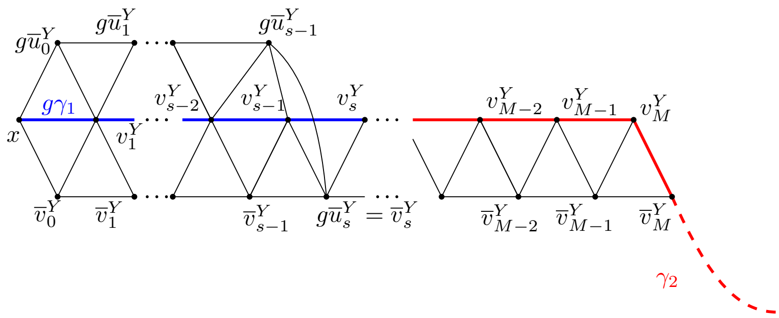



Figure 6.10: Example of the subcomplex $S$. In this case $g\overline{u}_s^Y = \overline{v}_s^Y$ and $g\overline{u}_{s-1}^Y \neq \overline{v}_{s-1}^Y$.

It is clear that one of the following cases holds.

1) $g\overline{u}_0^Y = \overline{v}_0^Y$.

2) $g\overline{u}_0^Y \neq \overline{v}_0^Y$, and there is some $p \leq N$, such that $g\overline{u}_p^Y = \overline{v}_p^Y$ and $\overline{v}_{s+1}^Y \neq g\overline{u}_s^Y \neq \overline{v}_s^Y$ for $s < p$. If $p = N$ then we have $N = M$ as $\overline{v}_N^Y = g\overline{u}_N^Y \in \alpha$.

3) $g\overline{u}_0^Y \neq \overline{v}_0^Y$, and there is some $p \leq N$, such that $g\overline{u}_p^Y = \overline{v}_{p+1}^Y$ and $\overline{v}_{s+1}^Y \neq g\overline{u}_s^Y \neq \overline{v}_s^Y$ for $s < p$.

4) $g\overline{u}_0^Y \neq \overline{v}_0^Y$, $gu^Y \neq v^Y$ and for $s \leq N$ we have $\overline{v}_{s+1}^Y \neq g\overline{u}_s^Y \neq \overline{v}_s^Y$. In this case we have $g\overline{u}_N^Y = v_{N+1}^Y$.

Like in the proof of Lemma 6.3.2 we will show that each of these cases results in a contradiction.

In case 1) $\overline{u}_0^Y$ belongs to the link of $x$ in $Y$ and $d_{Y_x}(g\overline{u}_0^Y, \overline{u}_0^Y) < 3$, a contradiction.

In case 2) we have $g\overline{u}_p^Y = \overline{v}_p^Y$ and $gu_{p-1}^Y \neq v_{p-1}^Y$. It follows that there is a cycle of the length 4 in the link of $v_p^Y$. Figure 6.11 shows the complex $S$ in that case.

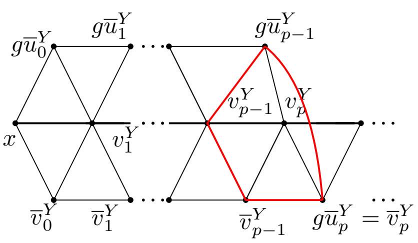



Figure 6.11: Link of $v_p^Y$ (red).

Since $Y$ is systolic $\overline{v}^Y_{p-1}$ is incident to $g\overline{u}^Y_{p-1}$, as otherwise $\overline{v}^Y_p$ would be incident to $v^Y_{p-1}$, which contradicts the geodesity of $\gamma_2$ .

Clearly, the tuple

$$(u_{p-1}, u_{p-2}, \ldots, u_1, x, v_1, \ldots, v_{p-1}, \overline{u}_{p-1}, \overline{u}_{p-2}, \ldots, \overline{u}_1, \overline{u}_0, \overline{v}_0, \overline{v}_1, \ldots, \overline{v}_{p-1})$$

satisfies the assumptions of Lemma 6.3.3. Therefore one of the following holds:

(i) for some $1 \leq s \leq p-1$ $(\overline{v}^Y_s, v^Y_{s-1}) \in Y$;

(ii) for some $1 \leq s \leq p-1$ $(\overline{u}^Y_s, u^Y_{s-1}) \in Y$;

(iii) $d_{Y_x}(g\overline{u}^Y_0, \overline{u}^Y_0) \leq 2$.

In case (i) both $\overline{v}_M$ and $v_{s-1}$ are vertices belonging to the geodesic $\gamma_2$. Therefore $(v_{s-1}, \ldots, v_N, \overline{v}_N)$ is a geodesic, but $(v_{s-1}, \overline{v}_s, \ldots, \overline{v}_N)$ is a shorter path between the same pair of vertices, a contradiction.

In case (ii) both $\overline{u}_N$ and $u_{s-1}$ are vertices belonging to the geodesic $\gamma_1$. Therefore $(u_{s-1}, \ldots, u_N, \overline{u}_N)$ is a geodesic, but $(u_{s-1}, \overline{u}_s, \ldots, \overline{u}_N)$ is a shorter path between the same pair of vertices, a contradiction.

In case (iii) $\overline{u}^Y_0$ belongs to the link $x$ in $Y$ and $d_{Y_x}(g\overline{u}^Y_0, \overline{u}^Y_0) < 3$, a contradiction.

In case 3) $g\overline{u}^Y_p = \overline{v}^Y_{p+1}$. But $g\overline{u}^Y_p$ is incident to $v^Y_{p-1}$. Both $\overline{v}_M$ and $v_{p-1}$ are vertices belonging to the geodesic $\gamma_2$. Therefore $(v_{p-1}, \ldots, v_N, \overline{v}_N)$ is a geodesic, but $(v_{p-1}, \overline{v}_{p+1}, \ldots, \overline{v}_N)$ is a shorter path between the same pair of vertices, a contradiction.

In case 4) we have a double ladder with a cap in $Y$:

$$\{g\overline{u}^Y_0, g\overline{u}^Y_1, \ldots, g\overline{u}^Y_N = v^Y_{N+1} | x^Y, v^Y_1, \ldots, v^Y_N | \overline{v}^Y_0, \overline{v}^Y_1 \ldots, \overline{v}^Y_N\}$$

(see Figure 6.12).

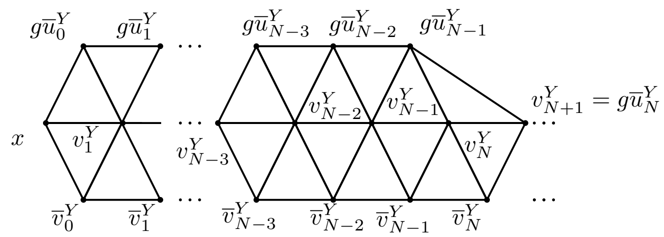



Figure 6.12: Double ladder with a cap in $Y$. The triangle $[v^Y_N, v^Y_{N+1}, \overline{v}^Y_N]$ is a cap.

Clearly, the set

$$\{u_N, u_{N-1}, \ldots, u_1, x, v_1, \ldots, v_N, \overline{u}_N, \overline{u}_{N-1}, \ldots, \overline{u}_1, \overline{u}_0, \overline{v}_0, \overline{v}_1, \ldots, \overline{v}_N\}$$

satisfies the assumptions of Lemma 6.3.3. Again, each case following from this lemma results in a contradiction.

Therefore between any two vertices $x_n < u < v < y_m$ such that $\kappa_D(u) = \kappa_D(v) = 1$ there is a vertex $w$, such that $u^Y < w^Y < v^Y$ and curvature of $w$ is at most $-1$. Therefore the sum of the curvature along any subpath of $\alpha$ is bounded by 1 and we can use Lemma 6.2.3 to show that $\alpha$ is a geodesic. $\square$

## 6.4 Rotations

Let $X$ be a simply connected $C(4)$–$T(4)$ or $C(6)$ small cancellation complex such that every 1-cell of $X$ is contained in the boundary of a 2-cell. Let $G$ be a finitely generated group acting on $X$ by automorphisms and assume that this action induces a free action on the 1-skeleton $X^1$ of $X$.

By the following proposition each group element acts by a rotation on each 2-cell fixed by it.

**Proposition 6.4.1.** *Let $g \in G$ and $\widehat{x} \in \mathrm{Fix}_X(g)$. Then $g$ acts on $\widehat{x}$ by a rotation of finite order.*

*Proof.* Group $G$ acts on $X$ by automorphisms, therefore $g$ acts on $\widehat{x}$ by an isometry. Only possible isometries of 2-cells are reflections and rotations. reflections does not act freely on 1-skeleton, therefore $g$ acts by a rotation. Since $\widehat{x}$ is an $n$-gon for some $n$, then by the freeness of the action on 1-skeleton $g$ has an order $m$ such that $m$ is a divisor of $n$. □

We now consider $C(4)$–$T(4)$ and $C(6)$ cases separately.

### 6.4.1 $C(4)$–$T(4)$ case.

Let $Y$ be the quadrization of $X$.

**Lemma 6.4.2.** *Let $\widehat{x} \in X$ be a 2-cell fixed by $f \in G \setminus \{id\}$ and $x$ be a corresponding vertex in $Y$. Then there exists $k = k(f)$ such that for any 0-cell $\widehat{y}$ belonging to $\widehat{x}$ we have $d_{Y_x}(f^k y, y) > 2$.*

*Proof.* First observe that $d_{Y_x}(f^k y, y) \leq 2$ means that for some piece $p$ the intersection $p \cap f^k p$ is non-empty.

By Proposition 6.4.1, $f$ is a rotation of some finite order $m$. Therefore, there exists $k_0$ such that $f^{k_0}$ is a 'clockwise rotation through $\frac{2\pi}{m}$'. We claim that $k = \frac{mk_0}{2}$ for even $m$ and $k = \frac{(m-1)k_0}{2}$ for odd $m$ is as required.

Observe that if $m$ is even then $f^2k$ is a rotation through $2\pi$ and if it is odd then $f^3k$ is a rotation through at least $2\pi$. Indeed, since $m \geq 3$, we have $3\frac{(m-1)\pi}{m} \geq 2\pi$. If for some piece $p$ the intersection $p \cap f^k p$ is non-empty, then we have three pieces $p, f^k p, f^{2k} p$ covering the whole boundary of $\widehat{x}$, see Figure 6.13. Contradiction with the $C(4)$ condition. □

### 6.4.2 $C(6)$ case.

Let $\widehat{x}$ be a 2-cell in $X$ and $v_1, v_2$ be a vertices from $\partial\widehat{x}$. By $(v_1, v_2)^{\circlearrowright}$ we denote the clockwise path between $v_1$ and $v_2$ in $\partial\widehat{x}$. For any piece $p$ we assume that $p = (p_l, p_r) = (p_l, p_r)^{\circlearrowright}$.

Let $Y$ be a Wise complex of $X$.

**Lemma 6.4.3.** *Let $\widehat{x} \in X$ be a 2-cell fixed by $f \in G$ and $x$ be a corresponding vertex in $Y$. Then either there exists $k = k(f)$ such that for any other 2-cell $\widehat{y}$ we have $d_{Y_x}(f^k y, y) \geq 3$ or $f^3 = id$.*

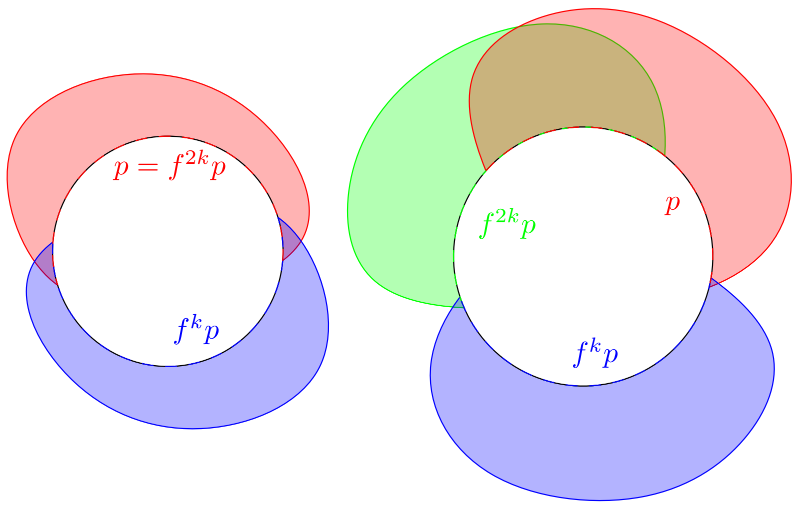



Figure 6.13: Pieces covering whole boundary of the cell $\widehat{x}$ in the cases of even and odd $m$.

*Proof.* First observe that $d_{Y_x}(f^k y, y) < 3$ means that for some piece $p = \widehat{x} \cap \widehat{y}$ there's a piece $p'$ covering one of two arcs $(p_r, f^k p_l)^{\circlearrowright}$, $(f^k p_r, p_l)^{\circlearrowright}$.

By Proposition 6.4.1 $f$ is a rotation of some finite order $m$. Therefore, there exists $k_0$ (coprime with $m$) such that $f^{k_0}$ is a 'clockwise rotation through $\frac{2\pi}{m}$'. We claim that $k = \frac{mk_0}{2}$ for even $m$ and $k = \frac{(m-1)k_0}{2}$ for odd $m$ is as required.

If for some piece $p$ there's a piece $p'$ covering one of two arcs $(p_r, f^k p_l)^{\circlearrowright}$, $(f^k p_r, p_l)^{\circlearrowright}$, then if either $m$ is even, or $p'$ covers $(f^k p_r, p_l)^{\circlearrowright}$ we have that $f^{-k}p', p, p'$ and $f^k p$ covers whole boundary of the cell, a contradiction to the $C(6)$ condition (see left part of the Figure 6.14).

If $m$ is odd and $p'$ covers $(p_r, f^k p_l)^{\circlearrowright}$. we have three possible cases:

1. $p \cap f^{2k}p \neq \emptyset$;
2. $f^q p \cap f^r p = \emptyset$ for any $q \not\equiv r \pmod m$ and $\frac{3(m-1)}{2} > m$;
3. $\frac{3(m-1)}{2} \leq m$.

In the first case $p, p', f^k p, f^k p'$ and $f^{2k}p$ cover the whole boundary, a contradiction (see central part of the Figure 6.14). In the second case we have $\frac{2(m-1)}{2} < m < \frac{3(m-1)}{2}$ and in particular, since $k_0$ is coprime with $m$, we have $2k \not\equiv 0 \pmod m$ and $3k \not\equiv 0 \pmod m$. It follows that $f^{2k}p \cap p = f^{3k}p \cap p = \emptyset$, hence $f^{2k}p' \supset p$. Thus $p', f^k p, f^k p', f^{2k}p$ and $f^{2k}p'$ cover the whole boundary, a contradiction (see right part of the Figure 6.14). In the third case $m = 3$. □

## 6.5 Lack of global fixed point implies existence of group element of infinite order

In this section we study closer the fixed points of the action of $G$ on $X$. First, we show that under assumptions from the previous section, each element of $G$ can fix at most one 2-cell in $X$, equivalently at most one vertex in $Y$.

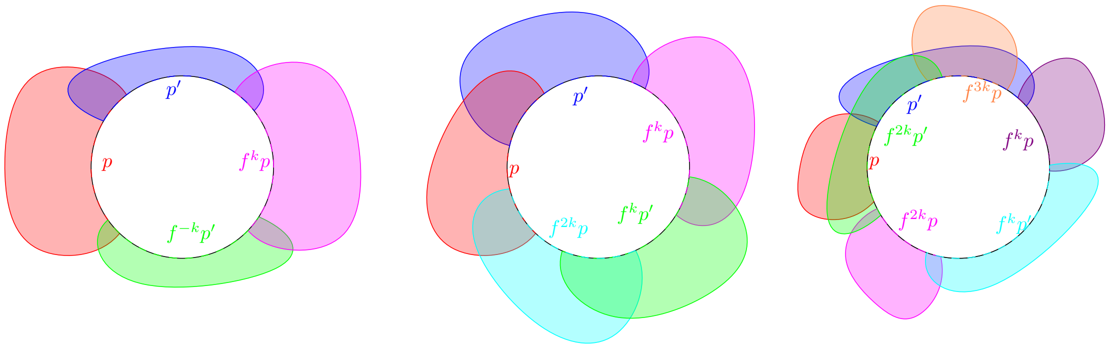



Figure 6.14: Pieces covering whole boundary of the cell $\widehat{x}$.

**Lemma 6.5.1.** *Let $g \neq 1$ be an element of $G$ such that $\mathrm{Fix}_Y(g) \neq \emptyset$. If $v \in \mathrm{Fix}_Y(g)$ then $\mathrm{Fix}_Y(g) = \{v\}$.*

*Proof.* Assume that there is another 2-cell such that $\widehat{v}' \in \mathrm{Fix}_X(g)$.

First, we consider the $C(4)$–$T(4)$ case. Let $\gamma := (v_0 := v, v_1, \ldots, v_n := v')$ be a geodesic in $Y$. Let $k$ be given by Lemma 6.4.2 and $\alpha = \gamma \cup g^k\gamma$. By the choice of $k$ for any $u$ incident to $v$ we have $d_{Y_v}(g^ku, u) > 2$. Therefore for every square $P \in Y$ containing $v$ in its boundary we have $P \cap g^kP = \{v\}$. By Lemma 6.3.2 $\alpha$ is a geodesic. Therefore $\gamma = g^k\gamma$. If $v \neq v'$ then $g^k$ fixes a vertex in $X$, contradiction with the freeness of the action on 1-skeleton.

Now we consider $C(6)$ case. We note here that because of case $g^3 \neq id$ this case cannot be proved analogously to $C(4)$–$T(4)$ one. Consider the set of all geodesics between $v$ and $v'$ in the complex $Y$. Since $Y$ is systolic, the set of all vertices incident to $v'$ belonging to these geodesics spans an $n$-simplex for some $n > 0$. This simplex is fixed by $g$. Since $Y$ is a Wise complex, $g$ fixes an intersection of all 2-cells corresponding to vertices of this simplex. That intersection is either a piece or a vertex. Contradiction with the freeness of the action on 1-skeleton. $\square$

The aim of the rest of this section is to prove that for any two elements of $G$ that do not fix the same 2-cell in $X$ there exists an element of $G$ of infinite order. By the following lemma such an element cannot fix any cell of $X$.

**Lemma 6.5.2.** *Let $g \in G$. If $\mathrm{Fix}_X(g) \neq \emptyset$ then $g$ has finite order.*

*Proof.* Let $\widehat{v} \in \mathrm{Fix}_X(g)$. It is a 2-cell in $X$, therefore for some $n$ it is an $n$-gon. Let $\widehat{v}'$ be a vertex belonging to the boundary of $\widehat{v}$. Since $G$ acts by automorphisms on $X$ and $g$ fixes $\widehat{v}$, there exists $m > 0$ such that $g^m\widehat{v}' = \widehat{v}'$. By the assumption of the freeness of the action on the 1-skeleton $g^m$ is trivial. $\square$

**Lemma 6.5.3.** *Let $X$ be either $C(4)$–$T(4)$ or $C(6)$ complex. If $\mathrm{Fix}_Y(f) \neq \mathrm{Fix}_Y(g)$ and:*

- *$X$ is a $C(4)$–$T(4)$ complex and $k = k(f), l = k(g)$ are given by Lemma 6.4.2;*
- *$X$ is a $C(6)$ complex and $k = k(f), l = k(g)$ are given by Lemma 6.4.3,*

*then $f^kg^l$ has infinite order.*

*Proof.* Let $\gamma := (x_0 := y, x_1, \ldots, x_n := x)$ be a geodesic in $Y$. Consider the images of $\gamma$ under $(f^k g^l)^i$ and $(f^k g^l)^i f^k$. Since $G$ acts by automorphisms and $\gamma$ is a geodesic, $(f^k g^l)^i \gamma$ and $(f^k g^l)^i f^k \gamma$ are also geodesics for any $i$.

We will prove that

$$\alpha_i := \bigcup_{0 \leq j \leq i} (f^k g^l)^j (\gamma \cup f^k \gamma)$$

is a geodesic for any $i$. In such a case $f^k g^l$ has an infinite order.

We argue by induction: first, observe that if $k, l$ are given by Lemma 6.4.2 (resp. by Lemma 6.4.3) then the conditions of Lemma 6.3.2 (resp. of Lemma 6.3.4) are satisfied. It follows that $\alpha_0$ is a geodesic.

Now, we will show that the geodesity of $\alpha_{i+1}$ follows from the geodesity of $\alpha_i$.

To do that we first need to show that $\alpha_i \cup (f^k g^l)^{i+1} \gamma$ is a geodesic, this follows from Lemma 6.3.2 in the $C(4)$–$T(4)$ case (resp. Lemma 6.3.4 in the $C(6)$ case). Indeed, $\alpha_i$ and $(f^k g^l)^{i+1} \gamma$ satisfy the conditions of Lemma then their concatenation is a geodesic.

But then $\alpha_i \cup (f^k g^l)^{i+1} \gamma$ and $(f^k g^l)^{i+1} f^k \gamma$ also satisfy the conditions of Lemma 6.3.2 (resp. Lemma 6.3.4). Therefore

$$\alpha_{i+1} = \alpha_i \cup (f^k g^l)^{i+1} \gamma \cup (f^k g^l)^{i+1} f^k \gamma$$

is a geodesic.

By induction, $\alpha_i$ is a geodesic for any $i$. $\square$

This Lemma finishes the case of $C(4)$–$T(4)$ complexes, and it remains to complete the case of $C(6)$ complexes. We remind here that in the case of $C(6)$ complexes, it is possible that for a given $f$, we cannot find $k$ such that $d_{Y_x}(f^k y, y) \geq 3$. In such a case by Lemma 6.4.3 we know that $f^3 = id$.

**Lemma 6.5.4.** *Let $X$ be $C(6)$ complex. If $G$ does not have a global fixed point, then there exists an element of infinite order in $G$.*

*Proof.* Assume that $G$ does not have a global fixed point. Let $f, g$ be a group elements such that $\{x\} = \mathrm{Fix}_Y(f) \neq \mathrm{Fix}_Y(g) = \{y\}$. We have three possible cases:

1. $f^3 \neq id \neq g^3$;
2. exactly one of $f, g$ has order 3;
3. $f^3 = g^3 = id$.

In the first case, by Lemma 6.4.3 there exist $k = k(f), l = k(g)$ such that the conditions of Lemma 6.5.3 are satisfied, therefore $f^k g^l$ has an infinite order.

In the second case without loss of generality we can assume that $f$ has the order 3 and $g$ does not. Consider the conjugation of $g$ by $f$. It has same the order as $g$ and fixes the vertex $fy$. If $fy = y$ then $f$ fixes $y$, a contradiction. By Lemma 6.4.3 there exists $l = k(g)$ such that the conditions of Lemma 6.5.3 are satisfied, therefore $fg^l f^{-1} g^l$ has an infinite order.

In the third case, observe that there is an element $h$ of the order not equal to 3 in the subgroup $\langle f, g \rangle$. Indeed, if all elements of $\langle f, g \rangle$ have order 3, then this subgroup is a quotient of the Free Burnside group $B(2, 3)$ which is finite. This subgroup acts on a Wise complex, which is systolic. By [CO15, Theorem C] a finite group acting on a systolic complex has a global fixed point, a contradiction to the fact that $\mathrm{Fix}_Y(f) \neq \mathrm{Fix}_Y(g)$.

Since $\mathrm{Fix}_Y(f) \neq \mathrm{Fix}_Y(g)$ then at least one of $\mathrm{Fix}_Y(f)$, $\mathrm{Fix}_Y(g)$ is not equal to $\mathrm{Fix}_Y(h)$. Thus clearly at least one of $\mathrm{Fix}_Y(fhf^{-1})$, $\mathrm{Fix}_Y(ghg^{-1})$ is not equal to $\mathrm{Fix}_Y(h)$. By Lemma 6.4.3 there exists $k$ such that the conditions of Lemma 6.5.3 are satisfied either for $h$ and $fhf^{-1}$ or for $h$ and $ghg^{-1}$. It follows that at least one of $fh^kf^{-1}h^k, gh^kg^{-1}h^k$ has infinite order. □

## 6.6 $C(3)$–$T(6)$ is CAT(0)

In this section we prove Theorem II.3 and Corollaries II.4-II.5.

**Definition 6.6.1.** A combinatorial 2-complex $X$ is called a *polygonal complex* if an intersection of any two closed cells of $X$ is either empty or exactly one closed cell.

**Proposition 6.6.2.** *Any simply connected $T(6)$ complex is a polygonal complex.*

*Proof.* By the result of Pride [Pri88], all pieces in $T(6)$ complex are of the length 1 therefore each piece is exactly a closed 1-cell. It follows that any non-empty intersection of two closed cells of $X$ consist of exactly one closed cell. □

We can view the edges of $X$ as segments of the length 1 and the closed 2-cells of $X$ as a regular Euclidean polygons of side length 1. This induces a metric in $X$. It gives us a criterion for a polygonal complex to be CAT(0). In the following definition the length of an edge in the link of $v$ is the angle in the corresponding polygon of $X$.

**Definition 6.6.3.** A polygonal complex $X$ with a metric $d$ satisfies *the link condition* if for each vertex $v \in X$ every injective cycle in the link of $v$ has length at least $2\pi$.

Observe that $C(3)$–$T(6)$ complex $X$ does not necessarily have bounded size of 2-cells, therefore we consider a complex $\mathfrak{X}$, which consists of barycentric subdivision of each cell in $X$. It is easy to see that $\mathfrak{X}$ is a triangle complex, and each triangle has one vertex corresponding to a center of a 2-cell from $X$, one to a center of a 1-cell and one which is a 0-cell in $X$.

Clearly $\mathfrak{X}$ is a triangle complex, but it does not satisfy the link condition with metric induced by taking each 2-cell to be a regular Euclidean triangle of the side length 1. Therefore, we induce another metric $\mathfrak{d}$ in the following way: we take all 2-cells to be Euclidean triangles with angle $\frac{\pi}{2}$ adjacent to a center of an 1-cell of $X$, angle $\frac{\pi}{3}$ adjacent to a center of a 2-cell of $X$ and angle $\frac{\pi}{6}$ adjacent to a 0-cell from $X$.

*Proof of Theorem II.3.* It is enough to show that the complex $\mathfrak{X}$ with metric $\mathfrak{d}$ is CAT(0). Since $\mathfrak{X}$ with that metric has only one shape of 2-cells, therefore by [BH09, Lemma 5.6] $\mathfrak{X}$ is CAT(0) as long as it satisfies the link condition i.e. we have to show that every injective cycle in each link has the length at least $2\pi$.

Let $v$ be a vertex from $\mathfrak{X}$. If $v$ is a 0-cell in $X$ then each corner has length at least $\frac{\pi}{6}$. Each cycle in the link of $v$ consists of at least 12 corners. Indeed, $X$ satisfies the condition $T(6)$, and each 2-cell adjacent to $v$ in $X$ is replaced in the link by two triangles in $\mathfrak{X}$. Each corner has length at least $\frac{\pi}{6}$, thus each cycle has length at least $2\pi$.

If $v$ is a center of an 1-cell, then each cycle in the link of $v$ has at least four corners, each of length $\frac{\pi}{2}$, so clearly each cycle has length at least $2\pi$.

If $v$ is a center of a 2-cell, then each cycle in the link of $v$ has at least six corners, each of length $\frac{\pi}{3}$, so clearly each cycle has length at least $2\pi$. □

*Proof of Corollary II.4.* Let a finitely generated group $G$ act locally elliptically on a simply connected $C(3)$–$T(6)$ small cancellation complex $X$. Define $\mathfrak{X}$ and $\mathfrak{d}$ as above. Then $\mathfrak{X}$ has *rational angles* with respect to $G$ in the sense of [NOP22, Definition 2.3]. This follows from the fact that all triangles of $\mathfrak{X}$ have angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$ and from an observation on [NOP22, page 9] just after [NOP22, Definition 2.3]. Corollary follows from [NOP22, Theorem 1.1(iii)]. □

*Proof of Corollary II.5.* Let $X$ be a simply connected $C(3)$–$T(6)$ small cancellation complex acted upon by $G$. Then, by Theorem II.3, $G$ acts almost freely on the CAT(0) complex $\mathfrak{X}$. By [OP22, Theorem A], the group $G$ is virtually cyclic, or virtually $\mathbb{Z}^2$, or contains a nonabelian free subgroup. □

*Remark* 6.6.4. Another way of proving Corollary II.5 is applying [OP21, Main Theorem]. One observes that $\mathfrak{X}$ is *reccurent* with respect to $G$ in the sense of [OP21, Definition 2.1]. This is by Theorem II.3 and [OP21, Remark 2.3], because $\mathfrak{X}$ satisfies [OP21, Definition 2.1(v)]. Moreover, $\mathfrak{X}$ admits a simplicial map to one triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$, whose restriction to each triangle of $\mathfrak{X}$ is an isometry. It follows that by [OP21, Example 2.5] the complex $\mathfrak{X}$ satisfies [OP21, Definition 2.1(i)-(iv)].

Thanks to $\mathfrak{X}$ being CAT(0) we can also deduce the following Lemma that will be used in the proof of Theorem II.2.

**Lemma 6.6.5.** *Let $G$ act on $C(3)$–$T(6)$ complex $X$ by automorphisms such that the action induces a free action on the 1-skeleton $X^1$ of $X$. Let $g \neq 1$ be an element of $G$ such that* $\mathrm{Fix}_X(g) \neq \emptyset$*. If $v \in \mathrm{Fix}_X(g)$ then* $\mathrm{Fix}_Y(g) = \{v\}$.

*Proof.* Assume that $g$ has two fixed points $v \neq v'$. It is clear that $v$ and $v'$ are both centers of 2-cells. The (unique) geodesic $\gamma$ between $v$ and $v'$ in $\mathfrak{X}$ is fixed by $g$. Since $\gamma$ has non-empty intersection with $X^1$, we get a contradiction. □

## 6.7 Proofs of Theorems II.2 and II.6

In our case of $G$ acting on $X$ by automorphisms in such a way that the action induces a free action on the 1-skeleton $X^1$ of $X$, if the action is additionally locally elliptic, each element fixes a 2-cell, equivalently, the center of a 2-cell.

*Proof of Theorem II.2.* Assume that the action of $G$ on $X$ does not have a global fixed point.

In the $C(4)$–$T(4)$ and $C(6)$ cases we first observe the following. Each 1-cell of $X$ that is not contained in the boundary of a 2-cell can be thickened to a 2-cell to obtain a new 2-complex $X'$ which deformation retracts to $X$. The complex $X'$ is a $C(4)$–$T(4)$ (or $C(6)$) small cancellation complex. Moreover, the complex $X$ embeds into $X'$ and the action of $G$ is preserved, therefore the action of $G$ induces a free action on the 1-skeleton of $X'$ and does not have a global fixed point. This allows us to use Lemmas from Sections 6.3-6.5.

By Lemma 6.5.1 an element of the group $G$ can fix at most one 2-cell of $X'$. Assume that $f, g$ are elements of $G$ such that $\mathrm{Fix}(f) \neq \mathrm{Fix}(g)$. Then in $C(4)$–$T(4)$ case by Lemma 6.5.3 there exist $k$, $l$ such that $f^k g^l$ has an infinite order. In $C(6)$ case by Lemma 6.5.4 there exist an element of an infinite order. By Lemma 6.5.2 only elements of finite order can fix a 2-cell, hence $G$ is not locally elliptic.

In the $C(3)$–$T(6)$ case by Corollary II.4 any finitely generated subgroup of $G$ has a global fixed point. From Lemma 6.6.5 each nontrivial element of $G$ has exactly one fixed point. As a consequence, any two nontrivial elements of $G$ fix the same point.

Therefore, in the $C(6)$, $C(4)$–$T(4)$, or $C(3)$–$T(6)$ cases there exists a 2-cell fixed by all elements of $G$. It is an $n$-gon for some $n$. Then, by freeness of the group action on the 1-skeleton of $X$, $G$ is finite and cyclic. □

We now pass to the proof of Theorem II.6. For a group $G$ and a metric space $X$, a group action $\Phi : G \to Isom(X)$ is called *proper* if for each $x \in X$ there exist a real number $r > 0$ such that the set $\{g \in G | (B_r(\Phi(g)(x)) \cap B_r(x)) \neq 0\}$ is finite. The group action $\Phi$ is called *cocompact* if there exists a compact subset $K \subseteq X$ such that $\Phi(G)(K) = X$. We say that $\Phi$ is a *geometric* action if it is both proper and cocompact.

A group $G$ is called *artinian* if any descending chain of subgroups $G_1 \supset G_2 \supset \dots$ becomes stationary, that is, $G_n = G_{n+1} = \dots$ from some $n$ onwards.

Let $\mathcal{G}$ denote the class of all groups $G$ with the following three properties:

(i) $G$ does not include $\mathbb{Q}$ or the $p$-adic integers $\mathbb{Z}_p$ for any prime as a subgroup;

(ii) $G$ does not include the Prüfer $p$ group $\mathbb{Z}(p^\infty)$ for any prime as a subgroup.

**Proposition 6.7.1.** *If G:*

1. *is a CAT(0) group; or*
2. *is a Helly group; or*
3. *is a systolic group;*

*then $G$ is in the class $\mathcal{G}$.*

*Proof.* Cases (1) and (2) are known by Prop 5.2 of [KMV22].

Case (3). A systolic group is finitely generated, in particular it is countable, so it cannot contain an uncountable subgroup $\mathbb{Z}_p$. It is known that an abelian subgroup of a systolic group is finitely generated, see [OP18], thus systolic group cannot have $\mathbb{Q}$ as a subgroup. Furthermore, any systolic group contains only finitely many conjugacy classes of finite subgroups: [Prz08, Corollary 1.3.], also [CO15]. Therefore, there is a bound on the order of finite order elements in a systolic group, and therefore this group can not have Prüfer $p$ group $\mathbb{Z}(p^\infty)$ as a subgroup. □

*Proof of Theorem II.6.* By Theorem B of [KMV22] the statement holds for subgroups of any group belonging to the class $\mathcal{G}$ whose torsion subgroups are artinian. In the $C(4)$–$T(4)$ case, by Theorem 6.18 of [Cha+ a], $G$ is Helly because it acts geometrically on $X$. In the $C(6)$ case, $G$ is systolic because it acts geometrically on the Wise Complex, which is systolic. In the $C(3)$–$T(6)$ case, by Theorem II.3, $G$ acts geometrically on a CAT(0) complex, therefore it is CAT(0).

Since $H$ is a torsion subgroup of $G$, its action is locally elliptic on $X$, and by Theorem II.2 we know that the subgroup $H$ has to be finite. It is clear that finite groups are artinian. □

## 6.8 Proof of Theorem II.1

Let $\langle X|R\rangle$ be a presentation of a group $G$. The *presentation complex* of $\langle X|R\rangle$ is formed by taking a unique 0-cell, adding a labeled oriented 1-cell for each generator, and then attaching a 2-cell along the closed combinatorial path corresponding to each relator.

The *Cayley complex* of $G$ with respect to the presentation $\langle X|R\rangle$ (denoted $Cayley(G,X,R)$) is constructed in the following way. Let the set of vertices of $Cayley(G,X,R)$ consist of all the elements of $G$. Then, at each vertex $g \in G$, insert a directed edge from $g$ to $gx$ for each of the generators $x \in X$. The translation of any relator $r \in R$ by any element of the group $G$ gives a loop in the graph. We attach a 2-cell to each such loop. The 1-skeleton of the Cayley complex is a directed graph known as the *Cayley graph*. It is known [Hat00, Section 1.3] that the Cayley complex is a universal cover of the presentation complex.

The presentation $\langle X|R\rangle$ of the group $G$ is a $C(p)$–$T(q)$ *small cancellation presentation* if its presentation complex is a $C(p)$–$T(q)$ complex. In such a case, since the Cayley complex with respect to the presentation $\langle X|R\rangle$ is a universal cover of the presentation complex, it is a $C(p)$–$T(q)$ complex as well [MW02].

A *spherical diagram* $S$ is a 2-sphere $\mathbb{S}^2$ with a structure of a combinatorial 2-complex. As in the case of disc diagrams, a diagram $S$ *in* $X$ is $S$ along with a combinatorial map from $S$ to $X$ denoted by $S \to X$.

A presentation of a group is *aspherical* if there are no 'non-trivial' spherical diagrams in $Cayley(G,X,R)$ in the sense of [LS01, III.10, p.156]. The following theorem states one of the known properties of groups with a small cancellation presentation.

**Theorem 6.8.1.** *[Hue79, Theorem 4] Any $C(6)$, $C(4)$–$T(4)$, or $C(3)$–$T(6)$ small cancellation presentation is aspherical.*

Before giving a proof of Theorem II.1 we need to state a theorem of Huebschmann concerning groups with aspherical presentations.

**Theorem 6.8.2.** *[Hue79, Theorem 3] Let $G$ be a group with an aspherical presentation $\langle X|R\rangle$. If $x \in G$ is an element of order $1 < s < \infty$, then there is a relator $r = z_r^{q_r}$ with $s|q_r$ such that $x$ is conjugate to $z_r^{q_r/s}$.*

*Proof of Theorem II.1.* Let $Cayley(G,X,R)$ be either a $C(3)$–$T(6)$ , a $C(4)$–$T(4)$ or a $C(6)$ complex. Assume that $H$ is a torsion subgroup of $G$. The action of the group $G$ on the 0-skeleton of the Cayley complex is free. As the 1-skeleton of $Cayley(G,X,R)$ is a directed graph, the action of $G$ on the 1-skeleton is free. This property is inherited by any subgroup of $G$, in particular $H$. Since $H$ is a torsion group, by Theorem 6.8.2 each of its elements is conjugate to a root of some relator. Obviously, a root of a relator fixes the 2-cell corresponding to this relator in the

Cayley complex, therefore the action of $H$ on $Cayley(G, X, R)$ is locally elliptic. It follows from Theorem II.2 that $H$ is a finite cyclic group. $\square$

# Bibliography


[Aha88] Ron Aharoni. “Matchings in infinite graphs”. *Journal of Combinatorial Theory, Series B* 44.1 (1988), pp. 87–125.

[AJ77] Mohammed A.-R. Al-Janabi. *The SQ universality of some small cancellation groups.* PhD thesis, Queen Mary College. 1977.

[BT24] Stefan Banach and Alfred Tarski. “Sur la décomposition des ensembles de points en parties respectivement congruentes”. *Fundamenta Mathematicae* 6 (1924), pp. 244–277.

[BK96] Howard Becker and Alexander S. Kechris. *The descriptive set theory of Polish group actions.* Vol. 232. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1996, pp. xii+136.

[Bes00] Mladen Bestvina. *Questions in geometric group theory.* `https://www.math.utah.edu/~bestvina/eprints/questions.pdf.`. 2000.

[BFH00] Mladen Bestvina, Mark Feighn, and Michael Handel. “The Tits alternative for $\mathrm{Out}(F_n)$. I. Dynamics of exponentially-growing automorphisms”. *Ann. of Math. (2)* 151.2 (2000), pp. 517–623.

[BFH05] Mladen Bestvina, Mark Feighn, and Michael Handel. “The Tits alternative for $\mathrm{Out}(F_n)$. II. A Kolchin type theorem”. *Ann. of Math. (2)* 161.1 (2005), pp. 1–59.

[Bie+12] Laurent Bienvenu, Adam R. Day, Mathieu Hoyrup, Ilya Mezhirov, and Alexander Shen. “A constructive version of Birkhoff’s ergodic theorem for Martin-Löf random points”. *Inform. and Comput.* 210 (2012), pp. 21–30.

[Bol79] Béla Bollobás. *Graph theory, an introductory course.* Graduate Texts in Mathematics, Vol. 63. Springer, 1979.

[BH09] Martin Bridson and Andr’e Haefliger. *Metric Spaces of Non-Positive Curvature.* Vol. 319. Jan. 2009.

[Bri06] Martin R. Bridson. “Non-positive curvature and complexity for finitely presented groups”. In: *International Congress of Mathematicians.* Vol. II. 2006, pp. 961–987.

[Bri07] Martin R. Bridson. *Problems concerning hyperbolic and CAT(0) groups.* 2007.

[Cap14] Pierre-Emmanuel Caprace. “Lectures on proper CAT(0) spaces and their isometry groups”. In: *Geometric group theory.* Vol. 21. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2014, pp. 91–125.

[Cav17] Matteo Cavaleri. “Computability of Følner sets”. *Internat. J. Algebra Comput.* 27.7 (2017), pp. 819–830.

[Cav18] Matteo Cavaleri. “Følner functions and the generic word problem for finitely generated amenable groups”. *J. Algebra* 511 (2018), pp. 388–404.

[CSC10] Tullio Ceccherini-Silberstein and Michel Coornaert. *Cellular automata and Groups*. Springer Monographs in Mathematic. Springer-Verlag, Berlin, 2010.

[CSGH99] Tullio Ceccherini-Silberstein, Rostislav I. Grigorchuk, and Pierre de la Harpe. “Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces”. *Proc. Steklov Inst. Math.* 224 (1999), pp. 57–97.

[Cha+ a] Jérémie Chalopin, Victor Chepoi, Anthony Genevois, Hiroshi Hirai, and Damian Osajda. “Helly groups”. *Geom. Topol.* (to appear).

[CK18] Ruiyuan Chen and Alexander S. Kechris. “Structurable equivalence relations”. *Fund. Math.* 242.2 (2018), pp. 109–185.

[Che00] Victor Chepoi. “Graphs of Some CAT(0) Complexes”. *Advances in Applied Mathematics* 24.2 (2000), pp. 125–179.

[CO15] Victor Chepoi and Damian Osajda. “Dismantlability of weakly systolic complexes and applications”. *Trans. Amer. Math. Soc.* 367.2 (2015), pp. 1247–1272.

[Col73] Donald J. Collins. “Free subgroups of small cancellation groups”. *Proc. London Math. Soc. (3)* 26 (1973), pp. 193–206.

[Day57] Mahlon M. Day. “Amenable semigroups”. *Illinois Journal of Mathematics* 1.4 (1957), pp. 509 –544.

[Deh11] Max Dehn. “Über unendliche diskontinuierliche Gruppen”. *Mathematische Annalen* 71 (1911), pp. 116–144.

[Die97] Reinhard Diestel. *Graph Theory (Graduate Texts in Mathematics)*. Springer, 1997.

[Dud21a] Karol Duda. *A new computable version of Hall’s harem theorem.* "`https://arxiv.org/abs/2105.06304`". 2021.

[Dud21b] Karol Duda. “Torsion subgroups of small cancellation groups” (2021). `https://arxiv.org/abs/2112.01912`.

[DI22a] Karol Duda and Aleksander Ivanov. “Computable paradoxical decompositions”. *International Journal of Algebra and Computation* 32 (2022), pp. 953–967.

[DI22b] Karol Duda and Aleksander Ivanov. “On Decidability of Amenability in Computable Groups”. *Arch. Math. Log.* 61.7–8 (2022), 891–902.

[EH88] Martin Edjvet and James Howie. “Star Graphs, Projective Planes and Free Subgroups in Small Cancellation Groups”. *Proceedings of the London Mathematical Society* s3-57.2 (1988), pp. 301–328.

[FHT11] Benson Farb, Chris Hruska, and Anne Thomas. "Problems on automorphism groups of nonpositively curved polyhedral complexes and their lattices". In: *Geometry, rigidity and group actions.* 2011, pp. 515–560.

[Fø55] Erling Følner. "On groups with full Banach mean value". *Mathematica Scandinavica* 3 (1955), 243–254.

[Gen21] Anthony Genevois. "Coning-off CAT(0) cube complexes". to appear in Ann. Inst. Fourier (Grenoble). 2021.

[Gro87] M. Gromov. "Hyperbolic groups". In: *Essays in group theory.* Vol. 8. Math. Sci. Res. Inst. Publ. Springer, New York, 1987, pp. 75–263.

[HO21] Thomas Haettel and Damian Osajda. "Locally elliptic actions, torsion groups, and nonpositively curved spaces" (2021). `https://arxiv.org/abs/2110.12431`.

[Hat00] Allen Hatcher. *Algebraic topology.* Cambridge: Cambridge Univ. Press, 2000.

[Hod20] Nima Hoda. "Quadric Complexes". *Michigan Mathematical Journal* 69.2 (2020), pp. 241 –271.

[HKP] Ehud Hrushovski, Krzysztof Krupiński, and Anand Pillay. "On first order amenability" (). `https://arxiv.org/abs/2004.08306`.

[HKP22] Ehud Hrushovski, Krzysztof Krupiński, and Anand Pillay. "Amenability, connected components, and definable actions". *Selecta Mathematica* 28 (Feb. 2022).

[HPP08] Ehud Hrushovski, Ya'acov Peterzil, and Anand Pillay. "Groups, measures, and the NIP". *J. Amer. Math. Soc.* 21.2 (2008), pp. 563–596.

[Hue79] Johannes Huebschmann. "Cohomology theory of aspherical groups and of small cancellation groups". *Journal of Pure and Applied Algebra* 14.2 (1979), pp. 137–143.

[Iva84] Nikolai V. Ivanov. "Algebraic properties of the Teichmüller modular group". *Dokl. Akad. Nauk SSSR* 275.4 (1984), pp. 786–789.

[JS06] Tadeusz Januszkiewicz and Jacek Świątkowski. "Simplicial nonpositive curvature". *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 104.1 (2006), pp. 1–85.

[Kec91] Alexander S. Kechris. "Amenable equivalence relations and Turing degrees". *J. Symbolic Logic* 56.1 (1991), pp. 182–194.

[KPT05] Alexander S. Kechris, Vladimir G. Pestov, and Stevo Todorcevic. "Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups". *Geom. Funct. Anal.* 15.1 (2005), pp. 106–189.

[KMV22] Daniel Keppeler, Philip Möller, and Olga Varghese. "Automatic continuity for groups whose torsion subgroups are small". *Journal of Group Theory* 25.6 (2022), pp. 1017–1043.

[Kie83] Henry A. Kierstead. "An effective version of Hall's theorem". *Proc. Amer. Math. Soc.* 88.1 (1983), pp. 124–128.

[LV20] Nils Leder and Olga Varghese. "A note on locally elliptic actions on cube complexes". *Innov. Incidence Geom.* 18.1 (2020), pp. 1–6.

[LP86] László Lovász and Michael D. Plummer. *Matching Theory*. North-Holland Mathematics Studies. North-Holland, 1986.

[LS01] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Reprint of the 1977 edition. Springer-Verlag, Berlin, 2001, pp. xiv+339.

[MU16] Andrew Marks and Spencer Unger. "Baire measurable paradoxical decompositions via matchings". *Adv. Math.* 289 (2016), pp. 397–410.

[MW02] Jonathan P. McCammond and Daniel T. Wise. "Fans and Ladders in Small Cancellation Theory". *Proceedings of the London Mathematical Society* 84.3 (2002), pp. 599–644.

[McC85] John McCarthy. "A "Tits-alternative" for subgroups of surface mapping class groups". *Trans. Amer. Math. Soc.* 291.2 (1985), pp. 583–612.

[Mor18] Nikita Moriakov. "On effective Birkhoff's ergodic theorem for computable actions of amenable groups". *Theory Comput. Syst.* 62.5 (2018), pp. 1269–1287.

[Neu29] John von Neumann. "Zur allgemeinen Theorie des Masses". ger. *Fundamenta Mathematicae* 13.1 (1929), pp. 73–116.

[NOP22] Sergey Norin, Damian Osajda, and Piotr Przytycki. "Torsion groups do not act on 2-dimensional CAT(0) complexes". *Duke Mathematical Journal* 171.6 (2022), pp. 1379 –1415.

[Ol'81] Aleksandr Y. Ol'shanskiĭ. "An infinite group with subgroups of prime orders". *Mathematics of The Ussr-izvestiya* 16 (1981), pp. 279–289.

[OP18] Damian Osajda and Tomasz Prytuła. "Classifying spaces for families of subgroups for systolic groups". *Groups Geom. Dyn.* 12.3 (2018), pp. 1005–1060.

[OP21] Damian Osajda and Piotr Przytycki. "Tits alternative for groups acting properly on 2-dimensional recurrent complexes". *Adv. Math.* 391 (2021). With an appendix by J. McCammond, Osajda and Przytycki, Paper No. 107976, 22.

[OP22] Damian Osajda and Piotr Przytycki. "Tits Alternative for 2-dimensional CAT(0) complexes". *Forum Math. Pi* (2022).

[Pri88] Stephen J. Pride. "Star-complexes, and the dependence problems for hyperbolic complexes". *Glasgow Mathematical Journal* 30.2 (1988), 155–170.

[Prz08] Piotr Przytycki. "The fixed point theorem for simplicial nonpositive curvature". *Mathematical Proceedings of the Cambridge Philosophical Society* 144.3 (2008), 683–695.

[Sag95] Michah Sageev. "Ends of group pairs and non-positively curved cube complexes". *Proc. London Math. Soc. (3)* 71.3 (1995), pp. 585–617.

[SW05] Michah Sageev and Daniel T. Wise. "The Tits alternative for CAT(0) cubical complexes". *Bull. London Math. Soc.* 37.5 (2005), pp. 706–710.

[SST20] Jeroen Schillewaert, Koen Struyve, and Anne Thomas. "Fixed points for group actions on 2-dimensional affine buildings" (2020). https://arxiv.org/abs/2003.00681.

[Sch18] Friedrich Martin Schneider. "About von Neumann's problem for locally compact groups". *J. Noncommut. Geom.* 12 (2018), pp. 1531–1549.

[Ser03] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. Springer-Verlag, Berlin, 2003, pp. x+142.

[Soa16] Robert I. Soare. *Turing computability*. Theory and Applications of Computability. Theory and applications. Springer-Verlag, Berlin, 2016, pp. xxxvi+263.

[Swe99] Eric L. Swenson. "A cut point theorem for CAT(0) groups". *J. Differential Geom.* 53.2 (1999), pp. 327–358.

[Tar38] Alfred Tarski. "Algebraische Fassung des Maßproblems". *Fundamenta Mathematicae* 31 (1938), pp. 207–223.

[Tit72] Jacques Tits. "Free subgroups in linear groups". *J. Algebra* 20 (1972), pp. 250–270.

[Why99] Kevin Whyte. "Amenability, bilipschitz equivalence, and the von Neumann conjecture". *Duke Mathematical Journal* 99.1 (1999), pp. 93 –112.

[Wis03] Daniel T. Wise. *Sixtolic complexes and their fundamental groups*. Unpublished manuscript. 2003.

# Part III

# Appended paper

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# Computable paradoxical decompositions

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We prove a computable version of Hall's Harem theorem and apply it to computable versions of Tarski's alternative theorem.

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## 1. Introduction

The Hall Harem theorem describes a condition which is equivalent to the existence of a perfect $(1, k)$-matching of a bipartite graph, see [4, Theorem H.4.2]. When $k = 1$ this is exactly Hall's marriage theorem, see [1, Sec. III.2]. These theorems are useful in amenability. For example, some versions of Tarski's alternative theorem can be obtained in this way, see [4, Chap. 4; 5, Sec. III.1]. In [13], Kierstead found a computable version of Hall's marriage theorem. In this paper, we generalize his theorem for arbitrary $k$ and give an application of this generalization to effective amenability.

To introduce the reader to the subject we recall the following definition.

**Definition 1.1.** Let $X$ be a set and let $G$ be a group which acts on $X$ by permutations. The $G$-space $(G, X)$ has a paradoxical decomposition, if there exists a

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finite set $K \subset G$ and two families $(A_k)_{k\in K}$ and $(B_k)_{k\in K}$ of subsets of $X$ such that

$$X = \left(\bigsqcup_{k\in K} k(A_k)\right) \bigsqcup \left(\bigsqcup_{k\in K} k(B_k)\right) = \left(\bigsqcup_{k\in K} A_k\right) = \left(\bigsqcup_{k\in K} B_k\right).$$

We call $(K, (A_k)_{k\in K}, (B_k)_{k\in K})$ a paradoxical decomposition of $X$.

Here we use a version of the definition given in [4], where some members $A_k$ or $B_k$ can be empty. It is equivalent to the traditional one. A well-known theorem of Tarski [23] states that the existence of such a paradoxical decomposition is opposite to amenability of the $G$-space $(G, X)$. In particular, a group is amenable if and only if it does not admit a paradoxical decomposition.

It is worth noting that there is a variety of versions of this theorem in different contexts, see for example [16, 19–21]. In this paper, we will study ones which are natural from the point of view of computability theory, [22]. In the situation when $X = \mathbb{N}$ and $G$ acts by computable permutations one can additionally demand that the families $(A_k)_{k\in K}$ and $(B_k)_{k\in K}$ consist of computable sets. We call such a paradoxical decomposition *computable*.

One of the versions of Tarski's theorem concerns a very general situation of *pseudogroups of transformations*. The following definition is taken from [5, 9].

**Definition 1.2.** A pseudogroup $\mathcal{G}$ of transformations of a set $X$ is a set of bijections $\rho : S \to T$ between subsets $S$ and $T \subseteq X$ which satisfies the following conditions:

(i) the identity $id_X$ is in $\mathcal{G}$,
(ii) if $\rho : S \to T$ is in $\mathcal{G}$, so is the inverse $\rho^{-1} : T \to S$,
(iii) if $\rho_1 : S \to T$ and $\rho_2 : T \to U$ are in $\mathcal{G}$, so is their composition $\rho_2 \circ \rho_1 : S \to U$,
(iv) if $\rho : S \to T$ is in $\mathcal{G}$ and if $S_0$ is a subset of $S$, the restriction $\rho|S_0$ is in $\mathcal{G}$,
(v) if $\rho : S \to T$ is a bijection between two subsets $S, T$ of $X$ and if there exists a finite partition $S = \bigcup_{j\leq n} S_j$ with $\rho|S_j \in \mathcal{G}$ for $j \in 1, \ldots, n$, then $\rho$ is in $\mathcal{G}$.

For $\gamma : S \to T$ in $\mathcal{G}$, we write $\alpha(\gamma)$ for the domain $S$ of $\gamma$ and $\omega(\gamma)$ for its range $T$.

**Definition 1.3.** When $X$ is countable, after identifying $X$ with $\mathbb{N}$, we say that a transformation $\rho : S \to T$ from $\mathcal{G}$ is computable if $S$ and $T$ are computable subsets of $\mathbb{N}$ and $\rho$ is a computable function.

Note that for any tuples $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ with pairwise distinct coordinates where each $b_i$ is in the same $\mathcal{G}$-orbit with the corresponding $a_i$, the map $(a_1, \ldots, a_k) \to (b_1, \ldots, b_k)$ is a computable transformation from $\mathcal{G}$.

A typical illustration of these notions appears in the case of discrete metric spaces. We remind the reader that given a metric space $(X, d)$ and a subset $F \subseteq X$ the set $N_m(F) = \{x \in X \mid d(x, F) \leq m\}$ is called the *m-ball* of $F$. A metric space $X$ is called *discrete* if the 1-ball of every finite subset is finite.

**Definition 1.4.** For a metric space $X$, the pseudogroup $W(X)$ of bounded perturbations of the identity consists of bijections $\rho : S \to T$ such that $\mathsf{sup}_{x\in S}(d(\rho(x), x))$

is bounded by some natural number (depending on $\rho$). It is called the pseudogroup of wobbling bijections.

When $X$ is infinite and discrete the values $\mathsf{sup}_{x\in S}(d(\rho(x),x))$ for $\rho \in W(X)$ are not uniformly bounded by a natural number.

**Definition 1.5.** When $X$ is countable, then after identifying $X$ with $\mathbb{N}$, the effective wobbling pseudogroup $W_{\text{eff}}(X)$ of $X$ is a subset of $W(X)$ consisting of computable transformations of $X$.

We now formulate one of the definitions of amenability. Let $\mathcal{G}$ be a pseudogroup of transformations of $X$. For $R \subset \mathcal{G}$ and $A \subset X$ we define the $R$-boundary of $A$ as

$$\partial_R A = \{x \in X \backslash A : \exists\, \rho \in R \cup R^{-1}(x \in \alpha(\rho) \text{ and } \rho(x) \in A)\}.$$

**Definition 1.6.** The pseudogroup $\mathcal{G}$ satisfies the Følner condition if for any finite subset $R$ of $\mathcal{G}$ and any natural number $n$ there exists a finite non-empty subset $F = F(R,n)$ of $X$ such that $|\partial_R F| < \frac{1}{n}|F|$.

The following theorem is a version of Tarski's theorem mentioned above, see [5, Theorems 7 and 25].

- The pseudogroup $\mathcal{G}$ satisfies the Følner condition if and only if there is no tuple $(X_1, X_2, \gamma_1, \gamma_2)$ consisting of a non-trivial partition $X = X_1 \sqcup X_2$ and $\gamma_i \in \mathcal{G}$ with $\alpha(\gamma_i) = X_i$ and $\omega(\gamma_i) = X$ for $i = 1, 2$.

**Remark 1.7.** Definition 1.6 can be applied to an action of a group $G$ on a set $X$ by permutations. In this case, we will say that the $G$-space $(G, X)$ satisfies Følner's condition.

The motivation for computable versions of this theorem comes from recent investigations in effective amenability theory, [2, 3, 17], where some effective versions of Følner's condition were suggested. Our main result connects this approach with paradoxical decompositions.

In Sec. 2, we generalize the work of Kierstead [13] concerning an effective version of Hall's theorem. These results will be applied in Sec. 3 to some computable versions of Tarski's alternative theorem. In Sec. 4, we study some complexity issues which are naturally connected with the main results of the paper.

We do not demand any special education of the reader in computability theory. Facts which we use are well known and easily available in [22]. Following trends in logic, we say computable instead of recursive.

## 2. A Computable Version of Hall's Harem Theorem

A graph $\Gamma = (V, E)$ is called a *bipartite graph* if the set of vertices $V$ is partitioned into sets $A$ and $B$ in such way, that the set of edges $E$ is a subset of $A \times B$. We denote such a bipartite graph by $\Gamma = (A, B, E)$. The set $A$ (respectively, $B$)

is called the set of *left* (respectively, *right*) *vertices*. From now on we concentrate on bipartite graphs. Although our definitions concern this case they usually have obvious extensions to all ordinary graphs.

Let $\Gamma = (A, B, E)$. When $(a, b)$ is an edge from $E$, it is called *adjacent* to vertices $a$ and $b$. In this case, we say that $a$ and $b$ are adjacent too. When two edges $(a, b), (a', b') \in E$ have a common adjacent vertex we say that $(a, b), (a', b')$ are also *adjacent*. A sequence $(a_1, a_2, \ldots, a_n)$ of vertices is called a *path* if each pair $(a_i,\, a_{i+1})$ is adjacent for $1 \leq i \leq n$.

Given a vertex $x \in A \cup B$ the neighborhood of $x$ is the set

$$N_\Gamma(x) = \{y \in A \cup B : (x, y) \in E\}.$$

For subsets $X \subseteq A$ and $Y \subseteq B$, we define the neighborhood $N_\Gamma(X)$ of $X$ and the neighborhood $N_\Gamma(Y)$ of $Y$ by

$$N_\Gamma(X) = \bigcup_{x \in X} N_\Gamma(x) \subseteq B \quad \text{and} \quad N_\Gamma(Y) = \bigcup_{y \in Y} N_\Gamma(y) \subseteq A.$$

The subscript $\Gamma$ is dropped if it is clear from the context.

In this section, we always assume that $\Gamma$ is *locally finite*, i.e. the set $N(x)$ is finite for all $x \in A \cup B$.

A subset $X$ of $A$ (respectively, of $B$) is called *connected* if for all $x, x' \in X$ there exist a path $(p_1, \ldots, p_k)$ in $\Gamma$ with $x = p_1$ and $x' = p_k$ such that $p_i \in X \cup N_\Gamma(X)$ for all $i \leq k$.

For a given vertex $v \in A \cup B$ the *star* of $v$ is a subgraph $S = (V', E')$ of $\Gamma$, with $V' = \{v\} \cup N_\Gamma(v)$ and $E' = (V' \times V') \cap E$.

**Definition 2.1.** A matching ($(1, 1)$-matching) for $\Gamma$ is a subset $M \subset E$ of pairwise nonadjacent edges. A matching $M$ is called left-perfect (respectively, right-perfect) if for all $a \in A$ (respectively, $b \in B$) there exists (exactly one) $b \in B$ (respectively, $a \in A$) with $(a, b) \in M$. The matching $M$ is called perfect if it is both right and left-perfect.

We now introduce perfect $(1, k)$-matchings for $\Gamma$ without defining $(1, k)$-matchings. We will use only perfect ones.

**Definition 2.2.** A perfect $(1, k)$-matching for $\Gamma$ is a subset $M \subset E$ satisfying the following conditions:

(1) for all $a \in A$ there exist exactly $k$ vertices $b_1, \ldots b_k \in B$ such that $(a, b_1), \ldots, (a, b_k) \in M$;
(2) for all $b \in B$ there is a unique vertex $a \in A$ such that $(a, b) \in M$.

Given a $(1, k)$-matching $M$ and a vertex $a \in A$ the *M-star* of $a$ is the graph consisting of all vertices and edges adjacent to $a$ in $M$.

The following theorem is known as *the Hall Harem theorem*, and the first of equivalent conditions below is known as *Hall's k-Harem condition*, see [4, Theorem H.4.2].

**Theorem 2.3.** *Let* $\Gamma = (A, B, E)$ *be a locally finite graph and let* $k \in \mathbb{N}, k \geq 1$. *The following conditions are equivalent*:

(1) *For all finite subsets* $X \subset A$, $Y \subset B$ *the following inequalities hold*: $|N(X)| \geq k|X|$, $|N(Y)| \geq \frac{1}{k}|Y|$.
(2) $\Gamma$ *has a perfect* $(1, k)$*-matching.*

In order to define computable versions of these conditions, we follow Kierstead's paper [13]. Definitions 2.4–2.6 are due to Kierstead. Definitions 2.7 and 2.8 are natural generalizations of the corresponding ones from [13].

**Definition 2.4.** A graph $\Gamma = (V, E)$ is computable if there exists a bijective function $\nu : \mathbb{N} \to V$ such that the set

$$R := \{(i, j) : (\nu(i), \nu(j)) \in E\}$$

is computable.

**Definition 2.5.** A bipartite graph $\Gamma = (A, B, E)$ is computably bipartite if $\Gamma$ is computable as a graph with respect to some $\nu$ and the set $\nu^{-1}(A) = \{n \in \mathbb{N} : \nu(n) \in A\} \subset \mathbb{N}$ is computable.

To simplify the matter below we will always identify $A$ and $B$ with $\mathbb{N}$. Thus, $A$ (respectively, $B$) will be called the left (respectively, right) copy of $\mathbb{N}$ and the function $\nu$ will be the identity map.

**Definition 2.6.** A locally finite (bipartite) graph $\Gamma$ is called highly computable if it is computable and the function $n \to |N_\Gamma(n)|$ for $n \in \mathbb{N}$ is computable.

**Definition 2.7.** Let $\Gamma = (A, B, E)$ be a computably bipartite graph. A perfect $(1, k)$-matching $M$ for $\Gamma$ is called computable if the set $\{(i, j) : (\nu(i), \nu(j)) \in M\} \subset \mathbb{N} \times \mathbb{N}$ is computable.

Note that computable perfectness exactly means that there is an algorithm which

- for each $i \in A$, finds the tuple $(i_1, i_2, \ldots, i_k)$ such that $(i, i_j) \in M$, for all $j = 1, 2, \ldots, k$;
- when $i \in B$ it finds $i' \in A$ such that $(i', i) \in M$.

The remainder of this section will be devoted to a proof that the following condition implies the existence of a computable perfect $(1, k)$-matching.

**Definition 2.8.** A highly computable bipartite graph $\Gamma = (A, B, E)$ satisfies the computable expanding Hall's harem condition with respect to $k$ (denoted *c.e.H.h.c.*$(k)$), if and only if there is a computable function $h : \mathbb{N} \to \mathbb{N}$ with domain $\mathbb{N}$ such that

- $h(0) = 0$,
- for all finite sets $X \subset A$, the inequality $h(n) \leq |X|$ implies $n \leq |N(X)| - k|X|$,
- for all finite sets $Y \subset B$, the inequality $h(n) \leq |Y|$ implies $n \leq |N(Y)| - \frac{1}{k}|Y|$.

Clearly, if the graph $\Gamma$ satisfies the *c.e.H.h.c.*$(k)$, then it satisfies Hall's $k$-harem condition. We emphasize that the requirements that $h$ is total and computable, essentially strengthen the latter ones. Moreover, [13, Theorems 2 and 5] state that the natural effective version of Hall's marriage theorem (i.e. when $k = 1$) does not hold without the assumptions that $h$ exists and is computable. It is worth noting that [13, Theorem 2] is a citation of a result of Manaster and Rosenstain from [15].

**Theorem 2.9.** *If* $\Gamma = (A, B, E)$ *is a highly computable bipartite graph satisfying the c.e.H.h.c.*$(k)$*, then* $\Gamma$ *has a computable perfect* $(1, k)$*-matching.*

**Proof.** We extend the proof of Theorem 3 of Kierstead's paper [13]. Let $h$ witness the *c.e.H.h.c.*$(k)$ for $\Gamma$. Let us fix computable enumerations of $A$ and $B$. We build a perfect $(1, k)$-matching $M$ by induction. The idea of the construction is as follows. At step 0 put $M = \emptyset$. At step $s$ we update the already constructed $M$ in the following way. For the first vertex $x_s$ from the remaining part of $A$ or $B$ we construct some finite subgraph $\Gamma_s$ and a matching $M_s$ in $\Gamma_s$. The matching $M$ is updated by adding the elements of $M_s$ adjacent to $x_s$. The subgraphs $\Gamma_s$ and $M_s$ are constructed so that after removal of the $M_s$-star of $x_s$ from $\Gamma$, the remaining part still is a highly computable bipartite graph satisfying the *c.e.H.h.c.*$(k)$.

At the first step of the algorithm, we choose $a_0$, the first element of the set $A$. We construct the induced subgraph $\Gamma_0 = (A_0, B_0, E_0)$ so that $A_0 \cup B_0$ is the set of vertices of distance of at most $\max\{2h(k)+1, 3\}$ from $a_0$. Since the graph $\Gamma$ is locally finite (respectively, highly computable) the graph $\Gamma_0$ is finite and can be found effectively. It is clear that for all vertices $v$ from $A_0$, $N_{\Gamma_0}(v) = N_\Gamma(v)$. Therefore, for every subset $X \subset A_0$ the inequality $h(n) \leq |X|$ implies $n \leq |N_{\Gamma_0}(X)| - k|X|$.

Let $B_{S_0}$ denote the set of vertices $v \in B_0$ at distance $\max\{2h(k) + 1, 3\}$ from $a_0$. It is clear that $N_{\Gamma_0}(B_0 \backslash B_{S_0}) = N_\Gamma(B_0 \backslash B_{S_0}) = A_0$. On the other hand since it may happen that $N_\Gamma(B_{S_0})$ is not contained in $A_0$, it is possible that there exists a subset $Y \subset B_{S_0}$, such that $|N_{\Gamma_0}(Y)| \leq \frac{1}{k}|Y|$.

Since $\Gamma$ contains a perfect $(1, k)$-matching, there exists a $(1, k)$-matching in $\Gamma_0$, that satisfies the conditions of perfect $(1, k)$-matchings for all $a \in A_0$ and $b \in B_0 \backslash B_{S_0}$. We denote it by $M_0$. Since $\Gamma_0$ is finite, the matching $M_0$ can be obtained effectively. Let $\{(a_0, b_{0,1}), \ldots, (a_0, b_{0,k})\}$ be the set of all edges from $a_0$ which belong to $M_0$. At step 1 we define $M$ to be the set of all these pairs.

Let $\Gamma'$ be the subgraph (yet bipartite) obtained from $\Gamma$ through removal of the $M_0$-star of $a_0$. Since the sets $A \cup B$, $A$ and $E$ are computable, and the matching $M_0$ is found effectively, the sets $A'$, $B'$ and $E'$ are also computable. Therefore,

$\Gamma'$ is a computably bipartite graph. Since $\Gamma$ is highly computable, the graph $\Gamma'$ is highly computable too. To finish this step it suffices to show that $\Gamma'$ satisfies *c.e.H.h.c.*$(k)$.

Define $h' : \mathbb{N} \to \mathbb{N}$ by setting

$$h'(n) = \begin{cases} 0 & \text{if } n = 0, \\ h(n+k) & \text{if } n > 0. \end{cases}$$

We claim that $h'$ works for $\Gamma'$. We start with the case when $X \subset A'$ and $n > 0$. Since $|N_{\Gamma'}(X)| \geq |N_\Gamma(X)| - k$, then for $n \geq 1$ the inequality $|X| > h'(n)$ implies $|N_{\Gamma'}(X)| - k|X| \geq |N_\Gamma(X)| - k|X| - k \geq n$.

Let us consider the case when $n = 0$ and $X$ is still a subset of $A'$. If $X$ is not connected, then its neighborhood would be the union of neighborhoods of its connected subsets. Therefore, without loss of generality, we may assume that $X$ is connected. If $X \subset A_0$, then $|N_{\Gamma'}(X)| - k|X| \geq 0$, since $M_0$ was a $(1,k)$-matching for $\Gamma_0$ that was perfect for subsets of $A_0$.

Now, let $a' \in X \backslash A_0$. If $b_{0,1}, \ldots, b_{0,k} \notin N_\Gamma(X)$, then $N_{\Gamma'}(X) = N_\Gamma(X)$, so $|N_{\Gamma'}(X)| - k|X| \geq 0$. Assume that for some $i \leq k$ and some $a \in X$, there exists $(a, b_{0,i}) \in E$. Since the distance between $a$ and $a'$ is at least $2h(k)$ we have $|X| \geq h(k) + 1$. Thus, $|N_\Gamma(X)| - k|X| \geq k$ and it follows that $|N_{\Gamma'}(X)| - k|X| \geq 0$. We conclude that the Hall condition for finite subsets of $A'$ is verified.

Now we need to show that $\Gamma'$ satisfies *c.e.H.h.c.*$(k)$ for finite sets $Y \subset B'$. We have to show that the inequality $h'(n) \leq |Y|$ implies $n \leq |N_{\Gamma'}(Y)| - \frac{1}{k}|Y|$. Note $Y \subset B' = B \backslash \{b_{0,1}, \ldots, b_{0,k}\}$ and $|N_{\Gamma'}(Y)| \geq |N_\Gamma(Y)| - 1$.

In the case $n > 0$ the inequality $|Y| > h'(n)$ implies $|N_{\Gamma'}(Y)| - \frac{1}{k}|Y| \geq |N_\Gamma(Y)| - \frac{1}{k}|Y| - 1 \geq n + k - 1 \geq n$.

Let us consider the case $n = 0$. As before, we may assume that $Y$ is connected. If $Y \subset B_0 \backslash B_{S_0}$, then $|N_{\Gamma'}(Y)| - \frac{1}{k}|Y| \geq 0$, since $M_0$ satisfied the conditions of a perfect $(1,k)$-matching for elements of $B_0 \backslash B_{S_0}$. If $a_0 \notin N_\Gamma(Y)$, then $N_{\Gamma'}(Y) = N_\Gamma(Y)$ and again $|N_{\Gamma'}(Y)| - \frac{1}{k}|Y| \geq 0$.

Assume that there exists $b' \in Y \backslash (B_0 \backslash B_{S_0})$ and there exists $b \in Y$ with the edge $(a_0, b) \in E$. Since the distance between $b$ and $b'$ is at least $2h(k)$ we have $|Y| \geq h(k) + 1$. It follows that $|N_\Gamma(Y)| - \frac{1}{k}|Y| \geq k$ and $|N_{\Gamma'}(Y)| - \frac{1}{k}|Y| \geq k - 1 \geq 0$. As a result we have that the graph $\Gamma'$ satisfies *c.e.H.h.c.*$(k)$.

To force $M$ to be a perfect $(1,k)$-matching, we use back and forth. Therefore, we start the next step of our algorithm by choosing the first element of $B'$, say $b_{1,1}$. We construct the induced subgraph $\Gamma_1 = (A_1, B_1, E_1)$ so that $A_1 \cup B_1$ is a set of vertices of $\Gamma'$ at distance at most $\max\{2h'(k) + 2, 4\}$ from $b_{1,1}$. Let $B_{S_1}$ denote the set of vertices at distance $\max\{2h'(k) + 2, 4\}$ from $b_{1,1}$. Since $\Gamma'$ contains a perfect $(1,k)$-matching, there exist a $(1,k)$-matching in $\Gamma_1$ that satisfies the conditions of a perfect $(1,k)$-matching for all $a \in A_1$ and $b \in B_1 \backslash B_{S_1}$. We denote it by $M_1$. We choose $a_1$ with $(a_1, b_{1,1}) \in M_1$. Let $\{(a_1, b_{1,2}), \ldots, (a_1, b_{1,k})\}$ be all remaining edges of the $M_1$-star of $a_1$. We update $M$ by adding all edges of this star.

Let $\Gamma''$ be a subgraph obtained from $\Gamma'$ through removal of the $M_1$-star of $a_1$. Then $\Gamma''$ is also a highly computable computably bipartite graph. We need to show that $\Gamma''$ satisfies $c.e.H.h.c.(k)$.

Define $h'' : \mathbb{N} \to \mathbb{N}$ by setting

$$h''(n) = \begin{cases} 0 & \text{if } n = 0, \\ h'(n+k) & \text{if } n > 0. \end{cases}$$

To prove that $h''(n)$ works for $\Gamma''$ we use the same method as in the case $h'(n)$ and $\Gamma'$.

We continue iteration by taking the elements of $A$ at even steps and the elements of $B$ at odd steps. At every step $n$, the graph $\Gamma^{(n)}$ satisfies the conditions for the existence of perfect $(1, k)$-matchings and we update $M$ by adding $k$ edges adjacent to $a_n$. Every vertex $v$ will be added to $M$ at some step of the algorithm. It follows that $M$ is a perfect $(1, k)$-matching of the graph $\Gamma$. Effectiveness of our back and forth construction guarantees that $M$ is computable. □

## 3. Effective Paradoxical Decomposition

The following definition gives an effective version of a paradoxical decomposition. Assume that a pseudogroup $\mathcal{G}$ acts on a countable set $X$. We will identify $X$ with $\mathbb{N}$.

**Definition 3.1.** Let $\mathcal{G}$ be a pseudogroup of transformations of a set $X = \mathbb{N}$. An effective paradoxical $\mathcal{G}$-decomposition of $(\mathcal{G}, X)$ is a tuple $(X_1, X_2, \gamma_1, \gamma_2)$ consisting of a non-trivial partition $X = X_1 \sqcup X_2$ into computable sets and computable $\gamma_i \in \mathcal{G}$ with $\alpha(\gamma_i) = X_i$ and $\omega(\gamma_i) = X$ for $i = 1, 2$.

We now formulate the main theorem of this section.

**Theorem 3.2.** *Let $(\mathcal{G}, X)$ be a pseudogroup of computable transformations defined on $\mathbb{N}$ which does not satisfy Følner's condition. Then $X$ has an effective paradoxical $\mathcal{G}$-decomposition.*

**Proof.** This proof is an effective version of [4, Theorem 4.9.2]. Let $R$ be a non-empty finite subset of $\mathcal{G}$ and let $n$ be a natural number such that for any non-empty finite subset $F$ of $X$ one has $|\partial_R F| \geq \frac{1}{n}|F|$. Define a function $d_R$ on $X$ by setting, for all $x, y \in X$,

$$d_R(x, y) = \mathsf{min}\{n \in \mathbb{N} : \exists\, \rho_1, \ldots, \rho_n \in R \cup R^{-1} (\rho_n \circ \ldots \circ \rho_1(x) \text{ is defined} \\ \text{and is equal to } y)\},$$

where in the case when there exists no $n$ as in the formula above we put $d_R(x, y) = \infty$. The function $d_R$ satisfies the triangle inequality for any triple from $X$. Hence, we use it as a metric. Since $R$ is a finite set of computable transformations, the set $\{(x, y) : d_R(x, y) \leq k\}$ is computable uniformly on $k$. Therefore, there is a computable enumeration of the set

$$\{(x, y, l) \in X \times X \times \mathbb{N} : d_R(x, y) \leq l\}.$$

Let $k$ be an integer such that $(1 + \frac{1}{n})^k \geq 3$. By the choice of $R$, for any finite subset $F$ of the space $(X, d_R)$ we have $|N_1(F)| \geq (1 + \frac{1}{n})|F|$. Thus in this space, the size of the $k$-neighborhood $N_k(F)$ is at least $3|F|$.

To find the corresponding effective paradoxical decomposition consider the bipartite graph $\Gamma(X) = (\mathbb{N}, \mathbb{N}, E)$, where the set $E \subset \mathbb{N} \times \mathbb{N}$ consists of all pairs $(x, y)$ with $d_R(x, y) \leq k$, with $x, y$ viewed as elements of $X$. By discreteness of $(X, d_R)$ and computability properties of $d_R$, the graph $\Gamma(X)$ is highly computable.

If $F$ be a finite subset of $\mathbb{N}$ then $|N_\Gamma(F)| = |N_k(F)| \geq 3|F|$. It follows that

$$|N_\Gamma(F)| - 2|F| \geq 3|F| - 2|F| = |F|.$$

Therefore, for any $n \in \mathbb{N}$ and a finite subset $F$ of the left side of $\Gamma(X)$ the inequality $n \leq |F|$ implies that $n \leq |N_\Gamma(F)| - 2|F|$. On the other hand viewing $F$ as a subset of the right side we have

$$|N_\Gamma(F)| - \frac{1}{2}|F| \geq 3|F| - \frac{1}{2}|F| \geq |F|.$$

Since the function $h = \mathsf{id}$ is computable, the graph $\Gamma(X)$ satisfies *c.e.H.h.c.*(2) with respect to $h$. By virtue of the Effective Hall Harem Theorem (Theorem 2.9), we deduce the existence of a computable perfect $(1, 2)$-matching $M$ in $\Gamma(X)$. In other words, there is a computable surjective map $\phi : \mathbb{N} \to \mathbb{N}$ which is a 2-to-1 map with the condition that $d_R(x, \phi(x)) \leq k$ for all $x \in X$.

We now define functions $\psi_1, \psi_2$ as follows:

$$\begin{cases} \psi_1(n) = \min(n_1, n_2), \\ \psi_2(n) = \max(n_1, n_2), \end{cases} \quad \text{where } \phi(n_1) = n = \phi(n_2), n_1 \neq n_2.$$

Since the function $\phi$ realizes a computable perfect $(1, 2)$-matching, both $\psi_1$ and $\psi_2$ are computable.

Let $X_i$ be the range of $\psi_i$, $i \in \{1, 2\}$. Clearly, both of them are computable sets and $X_1 \sqcup X_2 = X$. We define $\gamma_i : X_i \to X$ by $\gamma_i(n) = \phi(n)$. Since $d_R(x, \gamma_i(x)) \leq k$ for all $x \in X$, we have $\gamma_i \in \mathcal{G}$. Therefore, $(X_1, X_2, \gamma_1, \gamma_2)$ is an effective paradoxical decomposition of $X$. $\square$

**Corollary 3.3.** *Let $(X, d)$ be a countable discrete metric space. Assume that $W_{\mathrm{eff}}(X)$ does not satisfy Følner's condition. Then $(X, d)$ has an effective paradoxical $W_{\mathrm{eff}}(X)$-decomposition.*

In the case of an action of a group $G$ on $X$ we will consider a more precise condition.

**Definition 3.4.** Let $X$ be a set identified with $\mathbb{N}$ and let $G$ be a group which acts on $X$ by computable permutations. The space $(G, X)$ has a computable paradoxical decomposition, if there exists a finite set $K \subset G$ and two families of computable sets $(A_k)_{k\in K}$, $(B_k)_{k\in K}$ such that

$$X = \left( \bigsqcup_{k\in K} k(A_k) \right) \bigsqcup \left( \bigsqcup_{k\in K} k(B_k) \right) = \left( \bigsqcup_{k\in K} A_k \right) = \left( \bigsqcup_{k\in K} B_k \right).$$

We call $(K, (A_k)_{k\in K}, (B_k)_{k\in K})$ a computable paradoxical decomposition of $X$.

Observe that this definition makes sense without the assumption that any element of $G$ realizes a computable permutation of $X$. In fact, one may demand this only for elements of $K$. Since Theorem 3.5 does not transcend the assumptions of Definition 3.4 we do not consider the extended version. This theorem is a natural development of Theorem 3.2.

**Theorem 3.5.** *Let $G$ be a group of computable permutations on a countable set $X$ which does not satisfy Følner's condition. Then there is a finite subset $K \subset G$ which defines a computable paradoxical decomposition as in Definition* 3.4.

**Proof.** In the beginning of the proof we repeat the argument of Theorem 3.2.

We denote by $\circ$ the action of $G$ on $X$. Find a finite subset $K_0 \subset G$ and a natural number $n$ such that for any finite subset $F \subset X$, there exists $g \in K_0$ such that $\frac{|F\setminus g\circ F|}{|F|} \geq \frac{1}{n}$. We may assume that $K_0$ is symmetric. Let $R = K_0 \cup \{1\}$ and let a function $d_R$ be defined exactly as in the proof of Theorem 3.2:

$$d_R(x,y) = \mathsf{min}\{n \in \mathbb{N} : \exists\, \rho_1, \ldots, \rho_n \in R(\rho_n \circ \ldots \circ \rho_1(x) = y)\},$$

where in the case when there exists no $n$ as in the formula above we put $d_R(x,y) = \infty$. Then viewing $d_R$ as a metric, for any finite $F \subset X$ we have

$$|N_1(F)| = |R \circ F| \geq \left(1 + \frac{1}{n}\right)|F|.$$

Choose $n_1 \in \mathbb{N}$ such that $(1+\frac{1}{n})^{n_1} \geq 3$ and set $K = R^{n_1}$. So for any finite $F \subset X$ we have $|N_{n_1}(F)| = |K \circ F| \geq 3|F|$.

Now note that the set of edges of the bipartite graph $\Gamma(X) = (\mathbb{N}, \mathbb{N}, E)$, defined in the proof of Theorem 3.2 consists of all pairs $(x,y) \in \mathbb{N}\times\mathbb{N}$ with $y \in K\circ x$, where $x, y$ are viewed as elements of $X$ under the identification $X = \mathbb{N}$. Since $G$ consists of computable permutations and $K$ is finite, the graph $\Gamma(X)$ is computably bipartite. Since the degree of every vertex is computable (by application of $K$), the graph is highly computable. Exactly as in the proof of Theorem 3.2 we see that the graph $\Gamma(X)$ satisfies *c.e.H.h.c.*(2) with respect to $h = \mathsf{id}$. By virtue of the Effective Hall Harem Theorem, we deduce the existence of a computable perfect $(1,2)$-matching $M$ in $\Gamma_R(X)$. In other words, there is a computable surjective 2-to-1 map $\phi : \mathbb{N} \to \mathbb{N}$ such that for any $n \in \mathbb{N}$ there is $g \in K$ with $n = g \circ \phi(n)$.

Repeating the proof of Theorem 3.2 define functions $\psi_1, \psi_2$ as follows:

$$\begin{cases} \psi_1(n) = \min(n_1, n_2), \\ \psi_2(n) = \max(n_1, n_2), \end{cases} \quad \text{where } \phi(n_1) = n = \phi(n_2), n_1 \neq n_2.$$

Since the function $\phi$ realizes a computable perfect $(1,2)$-matching, both $\psi_1$ and $\psi_2$ are computable. Moreover, they preserve $\langle K\rangle$-orbits.

Define $\theta_1(n)$ to be $g \in K$ with $\psi_1(n) = g \circ n$ and $\theta_2(n)$ to be $h \in K$ with $\psi_2(n) = h \circ n$. Observe that $\theta_1$, $\theta_2$ can be chosen computable and $\theta_1(n), \theta_2(n) \in K$ for all $n \in \mathbb{N}$.

For each $k \in K$ define sets $A_k$ and $B_k$ in the following way:

$$A_k = \{n \in \mathbb{N} : \theta_1(n) = k\}, \quad B_k = \{n \in \mathbb{N} : \theta_2(n) = k\}.$$

It is clear that these sets are computable and

$$X = \bigsqcup_{k\in K} A_k = \bigsqcup_{k\in K} B_k.$$

For each $n \in A_k$, the value $\psi_1(n)$ is $k \circ n$. Thus, $\psi_1(\mathbb{N}) = \bigsqcup_{k\in K} k \circ A_k$. Similarly, we can show that $\psi_2(\mathbb{N}) = \bigsqcup_{k\in K} k \circ B_k$. Since $\mathbb{N} = \psi_1(\mathbb{N}) \bigsqcup \psi_2(\mathbb{N})$, we have

$$X = \left(\bigsqcup_{k\in K} k \circ A_k\right) \bigsqcup \left(\bigsqcup_{k\in K} k \circ B_k\right).$$

Therefore, $(K, (A_k)_{k\in K}, (B_k)_{k\in K})$ is an effective paradoxical decomposition of the action of $G$ on $X$. □

**Remark 3.6.** Groups of computable permutations of $\mathbb{N}$ are becoming an attractive object of investigations in computable algebra. We recommend the survey paper [18] and the recent paper of the second author [10]. Theorem 3.5 shows how naturally these groups appear in computable amenability.

## 4. Complexity of Paradoxical Decompositions

The approach of this section is similar to that in [12]. Throughout the section, we assume that $G$ is a computable group. We then identify $G$ with $\mathbb{N}$ and regard multiplication of $G$ and the inverse as computable functions $\mathbb{N}^2 \to \mathbb{N}$ and $\mathbb{N} \to \mathbb{N}$, respectively. Such a realization of $G$ is called a *computable presentation* of $G$. For simplicity, we assume that 1 is the neutral element of $G$. The expression $x^{-1}$ means the inverse in $G$.

Note that for any $g \in G$ the function $g \cdot x$, $x \in G$, defines a computable permutation on $\mathbb{N}$. In particular, the left action of $G$ on $G$ is by computable permutations of $\mathbb{N}$.

**Definition 4.1.** The computable group $G$ has a computable paradoxical decomposition, if the left action of $G$ on $G$ has a computable paradoxical decomposition.

By Theorem 3.5 (and its proof) we have the following statement.

**Corollary 4.2.** *Let $K_0$ be a finite subset of $G$ and suppose there is $n \in \mathbb{N}\backslash\{0\}$ such that the following condition is satisfied*:

- *for any finite subset $F \subset G$, there exists $k \in K_0$ with $\frac{|F\backslash kF|}{|F|} \geq \frac{1}{n}$.*

*Let $n_1$ be such that $(1+\frac{1}{n})^{n_1} \geq 3$. Then the subset $K = (K \cup K^{-1})^{n_1}$ defines a computable paradoxical decomposition as in Definition* 3.4.

In particular, if $G$ is a computable non-amenable group then it has a computable paradoxical decomposition. This corollary leads to the following definition.

**Definition 4.3.** Let

$$\mathfrak{W}_{BT} = \left\{ K \subset G \text{ is finite} : \exists\, n \in \mathbb{N}(\forall \text{ finite } F \subset G)(\exists k \in K)\left(\frac{|F \backslash kF|}{|F|} \geq \frac{1}{n}\right)\right\}.$$

We call $\mathfrak{W}_{BT}$ the set of witnesses of the Banach–Tarski paradox.

**Proposition 4.4.** *For any computable group the family $\mathfrak{W}_{BT}$ belongs to the class $\Sigma^0_2$ of the Arithmetical Hierarchy.*

**Proof.** Since the group $G$ is computable, for any finite subsets $K$, $F$ of $G$, and any $n \in \mathbb{N}$, we can effectively check if the inequality $\frac{|F \backslash kF|}{|F|} < \frac{1}{n}$ holds for all $k \in K$. Therefore, the set of triples $(n, K, F)$ such that $\frac{|F \backslash kF|}{|F|} < \frac{1}{n}$ holds for all $k \in K$ is computably enumerable, i.e. belongs to $\Sigma^0_1$.

Since the projection of this set to the first two coordinates is also computably enumerable, the set

$$\mathfrak{W}'_{BT} = \left\{ (K, n) : (\forall \text{ finite } F \subset \Gamma)(\exists k \in K)\left(\frac{|F \backslash kF|}{|F|} \geq \frac{1}{n}\right)\right\}$$

belongs to the class $\Pi^0_1$. The set $\mathfrak{W}_{BT}$ consists of all finite subsets $K \subset G$ such that there exists $n \in \mathbb{N}$ with $(K, n) \in \mathfrak{W}'_{BT}$. Thus, $\mathfrak{W}_{BT}$ belongs to the class $\Sigma^0_2$. □

It is well known that a finitely generated free group has a computable presentation. We consider the following theorem as the most natural example where the set $\mathfrak{W}_{BT}$ is computable.

**Theorem 4.5.** *The family $\mathfrak{W}_{BT}$ is computable for any finitely generated free group.*

Before the proof of this theorem we give some reformulation of witnessing. This observation belongs to Cavaleri. It simplifies our original argument.

**Proposition 4.6.** *Let $G$ be a group and $K$ be a finite subset of $G$. Then $K \in \mathfrak{W}_{BT}$ if and only if $\langle K \rangle$ is a non-amenable subgroup of $G$.*

**Proof.** The necessity holds by Følner's definition of amenability. Assume that $K \notin \mathfrak{W}_{BT}$. It follows that for every $n$ there exists a set $F_n$ such that

$$(\forall k \in K)\left(\frac{|F_n \backslash kF_n|}{|F_n|} \leq \frac{1}{n}\right).$$

In order to show that $\langle K \rangle$ is amenable we follow the proof of [6, Proposition 9.2.13]. Take any $n \in \mathbb{N}$. Put $m = n|K|$. Let us show that there exists $t_0 \in G$ such that the set $F_m t_0^{-1} \cap \langle K \rangle = \{k \in \langle K \rangle : kt_0 \in F_m\}$ is $\frac{1}{n}$-Følner for $K$. Let $T \subset G$ be a complete set of representatives of the right cosets of $\langle K\rangle$ in $G$. Clearly, every $g \in G$ can be uniquely written in the form $g = ht$ with $h \in \langle K\rangle$ and $t \in T$. We then have

$$|F_m| = \sum_{t\in T} |F_m t^{-1} \cap \langle K\rangle|. \tag{4.1}$$

For every $x \in K$, we have $xF_m = \bigsqcup_{t\in T}(xF_m t^{-1} \cap \langle K\rangle)t$, hence

$$xF_m \backslash F_m = \bigsqcup_{t\in T}((xF_m t^{-1} \cap \langle K\rangle)\backslash(F_m t^{-1} \cap \langle K\rangle))t.$$

This gives us

$$|xF_m \backslash F_m| = \sum_{t\in T} |(xF_m t^{-1} \cap \langle K\rangle)\backslash(F_m t^{-1} \cap \langle K\rangle)|. \tag{4.2}$$

Since for all $x \in K$,

$$|xF_m \backslash F_m| \le \frac{|F_m|}{m},$$

using (4.1) and (4.2), we get

$$\sum_{t\in T} |(KF_m t^{-1} \cap \langle K\rangle)\backslash(F_m t^{-1} \cap \langle K\rangle)|$$

$$= \sum_{t\in T} |\bigcup_{x\in K}((xF_m t^{-1} \cap \langle K\rangle)\backslash(F_m t^{-1} \cap \langle K\rangle))| \le \frac{|K|}{m} \sum_{t\in T} |F_m t^{-1} \cap \langle K\rangle|.$$

By the pigeonhole principle, there exists $t_0 \in T$ such that

$$|(KF_m t_0^{-1} \cap \langle K\rangle)\backslash(F_m t_0^{-1} \cap \langle K\rangle)| \le \frac{1}{n}|F_m t_0^{-1} \cap \langle K\rangle|.$$

Clearly, $F_m t_0^{-1} \cap \langle K\rangle$ is an $\frac{1}{n}$-Følner set with respect to $K$. Since $n$ was arbitrary, $\langle K\rangle$ is amenable. This finishes the proof. □

**Proof of Theorem 4.5.** Let $\mathbb{F}$ be a finitely generated free group under the standard presentation. Since it is computable, the equation $xy = yx$ can be effectively verified for every $x, y \in \mathbb{F}$. We will show that $K \in \mathfrak{W}_{BT}$ if and only if there exist $x, y \in K$ such that $xy \neq yx$. This will give the result.

($\Rightarrow$) Let us assume that $xy = yx$ for every $x, y \in K$. Since $\mathbb{F}$ is a free group, there exists $z \in \mathbb{F}$ such that all words from $K$ are powers of $z$ see [14, Sect. I.2]. Since the subgroup $\langle z\rangle$ is cyclic, the subgroup $\langle K\rangle$ is amenable and for every $n$ there is a finite set $F$, which is an $\frac{1}{n}$-Følner with respect to $K$. Clearly, $K \notin \mathfrak{W}_{BT}$.

($\Leftarrow$) Let us assume that there exist $x, y \in K$ with $xy \neq yx$. Then $x, y$ generate a free subgroup of $\mathbb{F}$ of rank 2. By Proposition 4.6 there is a natural number $n$ such that $\mathbb{F}$ does not contain $\frac{1}{n}$-Følner subsets with respect to both $\{x, y\}$ and $K$. □

We add few words concerning the following question:

- Are there natural examples with non-computable $\mathfrak{W}_{BT}$?

In [8] (see also [7]) we give an example of a finitely presented group, say $H_{nA}$, with decidable word problem such that detection of all finite subsets of $H_{nA}$ which generate amenable subgroups, is not decidable. Applying Proposition 4.6 we see that the set $\mathfrak{W}_{BT}$ is not computable in this group. In [8], we used slightly involved methods of computability theory. It can be also derived from [7, 8] that when a computable group $G$ is *fully residually free* [11], the corresponding set $\mathfrak{W}_{BT}$ is computable.

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## References

[1] B. Bollobás, *Graph Theory, an Introductory Course*, Graduate Texts in Mathematics, Vol. 63 (Springer, 1979).
[2] M. Cavaleri, Computability of Følner sets, *Int. J. Algebra Comput.* **27** (2017) 819–830.
[3] M. Cavaleri, Følner functions and the generic word problem for finitely generated amenable groups, *J. Algebra* **511** (2018) 388–404.
[4] T. Ceccherini-Silberstein and M. Coornaert, *Cellular Automata and Groups*, Springer Monographs in Mathematics (Springer-Verlag, 2010).
[5] T. Ceccherini-Silberstein, R. I. Grigorchuk and P. de la Harpe, Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces, *Proc. Steklov Inst. Math.* **224** (1999) 57–97.
[6] M. Coornaert, *Topological Dimension and Dynamical Systems*, Universitext (Springer International Publishing, 2015).
[7] K. Duda, Amenability and computability, preprint (2019), arXiv:1904.02640 [math.GT].
[8] K. Duda and A. Ivanov, On decidability of amenability in computable groups, *Arch. Math. Log.*, https://doi.org/10.1007/s00153-022-00819-5.
[9] P. de la Harpe and G. Skandalis, Un résultat de Tarski sur les actions moyennables de groupes et les partitions paradoxales, *Enseign. Math.* **32** (1986) 121–138.
[10] A. Ivanov, Sofic profiles of $S(\omega)$ and computability, *Arch. Math. Log.* **60** (2021) 477–494.
[11] I. Kapovich, Subgroup properties of fully residually free groups, *Trans. Amer. Math. Soc.* **354** (2001) 335–362.
[12] B. Khoussainov and A. Myasnikov, Finitely presented expansions of groups, semigroups, and algebras, *Trans. Amer. Math. Soc.* **366** (2014) 1455–1474.
[13] H. Kierstead, An effective version of Hall's Theorem, *Proc. Amer. Math. Soc.* **88** (1983) 124–128.
[14] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 89 (Springer-Verlag, 1977).
[15] A. Manaster and J. Rosenstein, Effective matchmaking (recursion theoretic aspects of a theorem of Philip Hall), *Proc. London Math. Soc.* **25** (1972) 615–654.

[16] A. Marks and S. Unger, Baire measurable paradoxical decompositions via matchings, *Adv. Math.* **289** (2016) 397–410.
[17] N. Moriakov, On effective Birkhoff's ergodic theorem for computable actions of amenable groups, *Theory Comput. Syst.* **62** (2018) 1269–1287.
[18] A. Morozov and P. Schupp, Computable permutations and word problems, *Enseign. Math.* (2) **64** (2018) 143–160.
[19] A. L. T. Paterson, Nonamenability and Borel paradoxical decompositions for locally compact groups, *Proc. Amer. Math. Soc.* **96** (1986) 89–90.
[20] F. M. Schneider, About von Neumann's problem for locally compact groups, *J. Noncommut. Geom.* **12** (2018) 1531–1549.
[21] F. M. Schneider and A. Thom, On Følner sets in topological groups, *Compos. Math.* **154** (2018) 1333–1361.
[22] R. I. Soare, *Turing Computability. Theory and Applications* (Springer-Verlag, 2016).
[23] A. Tarski, Algebraische fassung des massproblem, *Fund. Math.* **31** (1938) 47–66.