Problemy wyjścia dla procesów typu Lévy'ego oraz ich zastosowania

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Exit problems for Lévy type models and their applications

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by

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Największe podziękowania kieruje w stronę mojej partnerki Nikoli, która przeżywała ze mną te dobre i te trudniejsze chwile. Jestem przekonany, że bez jej oparcia nie byłbym w stanie przejść drogi, którą przeszedłem. Dziękuje również rodzicom, którzy nauczyli mnie bycia dobrym człowiekiem i którzy wspierali mnie przez całe moje życie. Kieruje też moje podziękowania mojemu rodzeństwu, całej mojej rodzinie, przyjaciołom i wszystkim, którzy dobrze życzyli mi podczas tej drogi.

Z wielką wdzięcznością dla Wszystkich,

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Contents

Introduction

The Crámer-Lundberg process is an example of the most popular and basic stochastic processes used in practice, especially in risk theory. The case of a non-life insurance company is a classic example of the application. The company starts its operations with an initial capital of $x > 0$. Then, the insured pay the premiums with a certain intensity $p > 0$. At random times, claims of random size occur. Let the homogeneous Poisson process $N = \{N_t : t \geq 0\}$ with the intensity parameter $\lambda > 0$ be responsible for the modelling of the moments of claims. Moreover, for claims' size we assume that they are modelled by the sequence $\{U_i\}_{i=1,2,\ldots}$, *i.i.d.*[1] random variables with a common distribution F. The classical distribution of F is the exponential distribution with intensity $\mu > 0$ which will also be the case for this introduction. The problem posed this way allows writing the process of the financial surplus of this company (or this line of business) as

$$X_t = x + pt - \sum_{i=1}^{N_t} U_i.$$

Such a simple model can solve many problems with explicit results. One such result is the probability of classical ruin, i.e. the first time the process goes below zero. Let us set this stopping time as

$$\tau_0^- := \inf\{t > 0 : X_t < 0\}.$$

For this process, such probability is equal to

$$\mathbb{P}(\tau_0^- < \infty | X_0 = x) = \begin{cases} \frac{\lambda}{\mu p} e^{\frac{\lambda - \mu p}{p} x} & \text{if } p > \frac{\lambda}{\mu}, \\ 1 & \text{if } p \leq \frac{\lambda}{\mu}. \end{cases}$$

Condition $p > \frac{\lambda}{\mu}$ implies that the average payout exceeds the average loss. It is also known as the net profit condition. Having the above result, the appropriate value $x > 0$ can be determined so that the probability of bankruptcy is sufficiently small. Another problem that we can answer in such a model is the issue of optimal dividend payments. With the financial surplus process, it is natural to establish some form of repayment of the invested money to the investors (both external and internal). However, the payouts must, on the one hand, be large enough to attract future investors; on the other hand, they must not significantly affect the company's financial condition. In other words, one needs to do payouts optimally to balance these issues. This problem initially comes from the work of de Finetti [19], who considered the dividend payment problem for a simple random walk with increments of ± 1. In his work, de Finetti obtained the result that the barrier strategy is optimal, that is, for a fixed level $a > 0$, whenever the surplus process reaches this

[1] *i.i.d.* stands for collection of random variables which are independent and identically distributed

level, one reflects the process and pays all excess above a as dividends. One can also find different strategies across the literature. In particular, one can assume that the divided rate should be bounded. For example, in the Brownian motion setting, Jeanblanc-Picqué and Shiryaev [30], and Asmussen and Taksar [3] obtained that the optimal dividend strategy should be a threshold strategy. That is, the dividends should be paid out at the maximal admissible dividend rate as soon as the surplus process exceeds a given threshold. A similar result for the Crámer-Lundberg model with exponentially distributed jumps was obtained in Gerber and Shiu [22].

One can observe some disadvantages of using the Crámer-Lundberg process. In particular, the process is deterministic between claims, not affected by market fluctuations. Moreover, the process does not distinguish between small and large claims, a natural practice in non-life insurance mathematics. Therefore, a natural step could be to extend the model with a Gaussian component or a second compound Poisson process along with a different distribution of the claims. However, it is worthwhile to consider a class of processes so that one can choose "tailor-made" models for the specific situation. The root class for our considerations will be the class of spectrally negative Lévy processes. These are processes with independent and stationary increments, with a.s. cádlág[2] trajectories and with a jump measure concentrated on the negative half-line. The condition concerning only negative jumps is in line with the example described at the beginning, where the only jumps were related to the claims. One can find Lévy processes in risk theory, financial mathematics, environmental problems, queueing, etc. This class includes compound Poisson processes, linear Brownian motion, stable processes or gamma processes. In addition, the simple fact that this class is closed under finite summation enlarges the flexibility of model creation.

For spectrally negative Lévy processes, it is natural to ask the same questions we asked for the Crámer-Lundberg process. However, to answer, we need to introduce some tools to express these results in a general way. The so-called scale functions play this role. Namely, many practical and theoretical problems boil down to solving some exit problems, i.e. the analysis of the fluctuation of the process in a specific interval. One can express these problems in the language of scale functions, which often reduces the issue to an explicit or numerical calculation evolving these functions. Chapter 1 will formally introduce the concept of the Lévy processes and the scale functions. Then, we will give examples of the processes along with the formulas of their scale functions for those processes for which we can calculate these functions explicitly. We will also show how one can solve the problem of the optimal dividend payment in the case of spectrally negative Lévy processes by referring to the articles Avram *et al.* [5], Renaud and Zhou [53], and Loeffen [44]. We will also recall some results from the theory of Markov processes and stochastic calculus, as we will need these results in the following chapters.

Despite the wide variety of applications, the above setting cannot respond to many natural practical problems. Therefore, this dissertation will deal with a generalisation of two assumptions, i.e. the choice of bankruptcy time and the problem of stationarity of the increments. Our goal will be to explore two models, each with a different approach to extending these two assumptions. Before we discuss these models, let us look at these two assumptions from a practical point of view. First, the classical ruin time implies that we do not allow a company to operate when the surplus is below zero. In practice, this is not true. For example, consider the risk management theory in the banking industry. The Basel Committee, one of the regulatory committees, specifies in its Basel II/III documents that the moment of default is standardised at 90 days after unpaid commitment.

[2] i.e. $\mathbb{P}$-almost surely right continuous with left limits, for some given probability measure $\mathbb{P}$.

After this period, a given company is called insolvent. This example shows that the moment of technical bankruptcy is too restrictive, even from a regulatory point of view. Another example is government-run companies (such as hospitals) for which it is natural to operate on a deficit. Thus, as a generalisation, the literature considers, among others, the following times of ruin:

- Parisian time of ruin (see e.g. Chesney *et al.* [12], Czarna and Palmowski [17] and Czarna *et al.* [16]). In this case, we allow the company's budget to stay below zero but no longer than a certain fixed time $r > 0$. In this definition, we are only interested in the time when the process is below zero, not where exactly it is located. Moreover, there is no limit to the number of times the considered process goes below zero. This time of ruin corresponds to the beforementioned Basel committee definition of the default.

- Unlike the previous case, one can consider r as a random variable, not as a constant (see e.g. Landriault *et al.* [40]). In other words, each time a process crosses zero, an independent random variable, usually exponential, is "started" that determines how much time the process is allowed to stay below zero.

- Another approach deals with the problem mentioned earlier. That is, the so-called Omega bankruptcy time allows for tracking the whole path of the process when defining bankruptcy (see, e.g. Albrecher *et al.* [1] and Gerber *et al.* [23]). For some process, X, one can define such a time as
$$\tau^{\omega} := \{t > 0 : \int_0^t \omega(X_s)ds > e_1\},$$
where ω is a non-negative measurable function and e_1 is an exponential random variable with the parameter 1. One can treat the ω function as imposing penalties, for example, for staying below a certain level, called *red zone*. In particular, one can consider the functions with the following shape
$$\omega(x) = \begin{cases} \omega_1(x) & \text{if } x > d, \\ +\infty & \text{if } x \leq d. \end{cases}$$
Thanks to this structure, one can immediately declare bankruptcy when the process drops below critical level d. Such a definition of ruin generates a lot of new control possibilities.

The second generalised assumption is the stationarity of the increments. Some phenomena change their behaviour due to changes in the environment or specific events. An example of this could be different modelling of the claims in the case of different seasons. Another example is when the process changes its drift term when crossing some level. For instance, it can be that the process has some additional funding due to its situation.
Hence, in the second chapter, we will consider a class of the refracted Lévy processes. This class of processes introduces the idea of an additional drift when staying in the negative half-line. Such an assumption intends to imitate saving a company from bankruptcy. Additionally, we assume that the time of bankruptcy is the Parisian ruin time. In such a model, we will solve the problem of optimal dividend payment assuming fixed transaction cost $\beta > 0$. This cost makes it impossible to consider barrier strategy as a candidate for the optimal strategy. Thus, we will consider impulse control strategy instead. This is to bring the process down to level c_1 when it reaches level c_2 ($c_2 > c_1$). Moreover, we will show the scale functions generalized to this setting for two underlying

processes: the linear Brownian motion and the Crámer-Lundberg process with exponential claims. This chapter is based on the article [14] co-written with Irmina Czarna.
In the third chapter, we will consider Markov additive processes. An element of this class can be seen as a two-dimensional stochastic process, where the first component X is responsible for the position of the process, and the second J for the environment. Depending on the state of the environment, the process X has a different distribution of the increments. One can observe that the process X evolves as some Lévy process until the state of J changes. In the case of these processes, the generalisation of scale functions comes down to considering scale matrices. We will start this chapter by briefly introducing Markov additive processes, scale matrices and corresponding exit problems. Next, we will introduce the concept of the ω-killing and related exit problems. We will show some basic facts related to this concept for a deeper understanding. In particular, as an example, we will show that for the Crámer-Lundberg process with the exponential claims, the probability of an Omega ruin is a linear function of classical probability. Assuming such a class of processes and such a moment of bankruptcy, we will show, among others, the form of the value function for the optimal dividend problem in the Omega model. However, before that, one needs to solve the exit problems. Thus, as the first step, we will solve exit problems in the new generalised ω-scale matrix language. Moving away from the motivation associated with the Omega model, we will see that the solutions to the exit problems may lead to applications going in a different direction. For example, the ω function can be responsible for the structure of interest rates depending on the state of the process X and J. Moreover, one can also observe that the results represent a new way of killing the stochastic process. Furthermore, we will show the forms of these matrices assuming that the process (X, J) is a Markov-modulated Brown motion, with different selections of the ω function. The vast majority of Chapter 3 was published in [15] with co-authors Irmina Czarna, Shu Li and Zbigniew Palmowski. Except for the Sections 3.2.3, 3.5.4 which were published in [32].

My contribution

This thesis consists of three chapters. The first one is an introduction to the theory and is based on well-known books or articles. The second and third are based on:

- **Chapter 2:**
  - Czarna I., Kaszubowski A., Optimality of impulse control problem in refracted Lévy model with Parisian ruin and transaction costs, *Journal of Optimization Theory and Applications* **185**, 982-1007 (2020) [14].
- **Chapter 3:**
  - Czarna I., Kaszubowski A., Li S., Palmowski Z., Fluctuation identities for Omega-killed spectrally negative Markov additive processes and dividend problem, *Advances in Applied Probability* **52** (2), 404-432 (2020) [15],
  - Kaszubowski A., Omega bankruptcy for different Lévy models, *Silesian Statistical Review*, **17** (23), 31-57 (2019) [32].

The content of these chapters differs from the original articles. Mostly it is due to expanding reasoning to give more detailed proofs or intuitions. Moreover, one can find different numbering of theorems, lemmas or equations, editorial changes or content in a different order. I made all these changes to make it more comfortable for the reader. It is hard to determine which parts were done sole by me as we created them during constant work with co-authors. However, one can find below the list of the parts I can recognise as my significant contribution.

Chapter 2

Compared to the original article, one can find a necessary change in this chapter. Namely, in the article, we have the following definition of the controlled risk process

$$U_t^\pi := R_t - L_t^\pi,$$

where L^π represents cumulative sum of dividends payed until time t and R is a refracted Lévy process, i.e. the process that is a strong solution to the following SDE

$$dR_t = dX_t - \delta \mathbf{1}_{\{R_t>0\}} dt,$$

where X is a spectrally negative Lévy process, $\delta > 0$ and $R_0 = X_0$. While preparing this thesis, I found that this definition is incorrect with respect to the purpose. Namely, process U^π is responsible for modeling financial surplus after dividend payments. A company can pay dividends before the (Parisian) ruin, and with the use of the refraction, we would like to try to save this process from ruin. However, in the above definition, refraction is connected with the process R, not with U, which is counter-intuitive. Therefore, in this thesis, one can find that we define U^π as a strong solution to the following SDE

$$dU_t^\pi = dX_t - \delta \mathbf{1}_{\{U_t^\pi>0\}} dt - dL_t^\pi,$$

for $U_0^\pi = X_0$. This reflects some changes in the theoretical part of this chapter, and due to this, the authors will submit a correction of the article shortly after completing this dissertation.

- Sections 2.4.1 and 2.4.2 were re-written to face above mentioned correction. Especially, I have entirely re-written section 2.4.2.

- Section 2.4.3 was created by co-author and me. The reasoning is similar to one in Loeffen [43]; however, I did almost all calculations with the help of the co-author.

- Section 2.4.4 faced most of the corrections or expansions. Thus, I list them separately below:
  - Fact 2.4.5 was expanded with the detailed calculations. The reasoning is similar to Noba and Yano [49] and Noba [48], but some work was needed to adjust it in the context of this setting as we needed to take fluctuation identities from Kyprianou and Loeffen [37],
  - Lemma 2.4.6 was re-written as the underlying process has been changed. I find this lemma to be the most important part of this chapter. The idea of this lemma is standard and one can find similar calculations in e.g. Avram *et al.* [5], Loeffen [43] or Kyprianou *et al.* [39]. I did the whole calculation, but the co-author did a similar proof in the previous version,
  - Lemma 2.4.8 was written by me, but the reasoning is partially from Egami and Yamazaki [21],

  - Lemma 2.4.9 was expanded by me, but it was based on the previous version written by the co-author,
  - Theorem 2.4.10 was expanded by me, but it was based on the previous version written by the co-author.

- Section 2.4.5 was entirely written and derived by me. Also, numerical examples were done by me.

Chapter 3

This chapter, as mentioned at the beginning of this part, is based on the two articles. The first one was co-written with co-authors, and I wrote the second myself. Given that four authors did the first article, it is hard to separate some parts, as we worked together on the major elements of this article. Moreover, the article was based on the previous work of one of the co-authors, Li and Palmowski [41]. Thus, the proofs' ideas are similar to this article but expanded to the Markov additive framework. Below one can find parts where I can admit my significant contribution.

- Section 3.1.3 was done entirely by me,
- Section 3.2.3 was done entirely by me,
- Section 3.4 was done by one of the co-authors and me,
- Section 3.5 was done entirely by me. In particular, I would like to mention Proposition 3.5.3, which consists of an analytic solution of the so-called Sylvester equation, which one can generally solve only with numerical calculations. I did figures and numerical calculations with the partial help of the numerical package from Ivanovs [25].

Chapter 1

Lévy processes

Lévy processes will be the primary class from which we want to derive processes tailored to specific problems. Therefore, we must spend some time deriving the basic concepts related to the Lévy processes. We will start by defining this class. Then we will tell how one can decompose any Lévy process into three parts, each responsible for the different behaviour of the trajectories. Next, we will give some examples of the Lévy processes and define what scale functions are. Then, we will show semi-explicit representations of the exit problems in their language. Finally, we will describe the problem of optimal dividend payment and how this problem was approached in the literature. We will also recall some results from the theory of Markov processes and stochastic calculus, as we will need these results in the following chapters. This chapter is based on mostly Kyprianou [36] and also Bertoin [6], Kuznetsov *et al.* [34], and other works directly mentioned in the specific parts.

1.1 Definition

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtrated probability space, with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ which satisfies usual conditions (i.e. right-continuous and complete). Let us start with the main definition of this thesis.

Definition 1.1.1 (Lévy Process)**.** *A process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it possesses the following properties:*

(i) the paths of X are cádlág,

(ii) $\mathbb{P}(X_0 = 0) = 1$,

(iii) for $0 \leq s \leq t, X_t - X_s$ is equal in distribution to X_{t-s},

(iv) for $0 \leq s \leq t, X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

Note that one can take $\mathbb{F}$ as a natural filtration of the above process. It does not follow from the above definition how large this class of processes is. Several processes, such as the Brownian motion, the Poisson process, or the compound Poisson process, belong to this class, which is evident at first glance. However, to see how complex this class is, let us introduce the following concept.
If X is a Lévy process, then for each t, X_t belongs to the infinite divisible class, which has a valuable representation.

Definition 1.1.2. *We say that a real-valued random variable Θ has an infinitely divisible distribution if for each $n = 1, 2, \dots$ there exists a sequence of i.i.d. random variables $\Theta_{1,n}, \dots, \Theta_{n,n}$ such that*

$$\Theta \overset{d}{=} \Theta_{1,n} + \cdots + \Theta_{n,n},$$

where $\overset{d}{=}$ is equality in distribution. Alternatively, we could have expressed this relation in terms of probability laws. That is to say, the law μ of a real-valued random variable is infinitely divisible if for each $n = 1, 2, \dots$ there exists another law μ_n of a real-valued random variable such that $\mu = \mu_n^{*n}$. (Here μ_n^{*n} denotes the n-fold convolution of μ_n).

Theorem 1.1.3 (Lévy-Khintchine formula)**.** *A probability law μ of a real-valued random variable is infinitely divisible with characteristic exponent Ψ,*

$$\int_{\mathbb{R}} e^{i\theta x} \mu(dx) = e^{-\Psi(\theta)}, \quad \text{for } \theta \in \mathbb{R},$$

if and only if there exists a triple (a, σ, Π), where $a \in \mathbb{R}, \sigma \geq 0$ and Π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2)\Pi(dx) < \infty$, such that

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\theta x} + i\theta x \mathbf{1}_{|x|<1}\right) \Pi(dx)$$

for every $\theta \in \mathbb{R}$.

Definition 1.1.4. *The measure Π is called the Lévy (characteristic) measure.*

If X is a Lévy process and $\Psi_t(\theta)$ is characteristic exponent of X_t then

$$\Psi_t(\theta) = t\Psi_1(\theta).$$

Therefore it is convenient to state the following remark.

Remark 1.1.5. *In the sequel we call $\Psi(\theta) := \Psi_1(\theta)$ as the characteristic exponent of the Lévy process.*

All above turns to the following theorem. For every infinitely divisible distribution there exists a probability space on which exists corresponding Lévy process. This makes it easier to understand how extensive this class is.

Theorem 1.1.6 (Lévy-Khintchine formula for Lévy processes)**.** *Suppose that $a \in \mathbb{R}, \sigma \geq 0$ and Π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2)\Pi(dx) < \infty$. Define for each $\theta \in \mathbb{R}$,*

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\theta x} + i\theta x \mathbf{1}_{|x|<1}\right) \Pi(dx).$$

Then there exists a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which a Lévy process is defined having characteristic exponent Ψ.

The proof of this theorem can be found in Kyprianou [36]. However, it is worth mentioning some details. At first, one can observe that Ψ can be divided into three pieces.

$$\Psi(\theta) = \Psi_1(\theta) + \Psi_2(\theta) + \Psi_3(\theta),$$

where

$$\Psi_1(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2,$$
$$\Psi_2(\theta) = \Pi(\mathbb{R} \setminus (-1,1)) \int_{|x|\geq 1} \left(1 - e^{i\theta x}\right) \frac{\Pi(dx)}{\Pi(\mathbb{R} \setminus (-1,1))},$$
$$\Psi_3(\theta) = \int_{0<|x|<1} \left(1 - e^{i\theta x} + i\theta x\right) \Pi(dx).$$

One can observe that every Ψ_i is the characteristic exponent of some Lévy process. The sum of the independent Lévy processes is a Lévy process itself. Thus, proof comes to proving the existence of these three processes. The first one associated with Ψ_1, let us call it $X^{(1)}$, corresponds to a linear Brownian motion with the drift term $-a$ and the standard deviation equal to σ. The second one is a compound Poisson process,

$$X_t^{(2)} = \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0,$$

where $N = \{N_t : t \geq 0\}$ is a Poisson process with intensity equal to $\Pi(\mathbb{R}\setminus(-1,1))$ and $\{\xi_i : i \geq 1\}$ is a sequence of $i.i.d.$ random variables with the distribution $\frac{\Pi(dx)}{\Pi(\mathbb{R}\setminus(-1,1))}$ concentrated on $\{x : |x| \geq 1\}$. Thus, we have one Gaussian part and a part with large jumps (i.e. jumps with a size of at least 1 a.s.). Proof that $X^{(1)}$ and $X^{(2)}$ exist is relatively standard. Thus, an idea of the proof comes down to proving that $X^{(3)}$ exists, with Ψ_3 as the characteristic exponent. It turns out that $X^{(3)}$ is a square integrable martingale with an almost surely countable number of jumps on each finite time interval of magnitude less than one. One can see it as a point process with "small" jumps or superposition of some compound Poisson processes with different arrival rates and some drift.
One can also observe that using the word "large" jumps for jumps at least one and the word "small" jumps for jumps less than one is somehow arbitrary. The crucial idea is to split the jump structure near to the origin ("small" jumps) and far from the origin ("large" jumps). There is nothing special about point 1, we could chose any $\epsilon > 0$ and divide $\mathbb{R} \setminus \{0\}$ into $(-\epsilon, 0) \cup (0, \epsilon)$ and $(-\infty, -\epsilon] \cup [\epsilon, \infty)$.
One can observe that the identification of the Lévy process is related to the triple (a, σ, Π). In the next part, we will present some known examples. However, before we do that, we would like to cite one more result, which is often essential when working with Lévy processes.

Lemma 1.1.7. *A Lévy process with Lévy–Khintchine exponent corresponding to the triple* (a, σ, Π) *has paths of bounded variation if and only if*

$$\sigma = 0 \quad \textit{and} \quad \int_{\mathbb{R}\setminus\{0\}} (1 \wedge |x|)\Pi(dx) < \infty.$$

1.2 Examples

1.2.1 Linear Brownian motion

Let us start examples with one of the most important stochastic processes.

Definition 1.2.1. *Stochastic process $W = \{W_t : t \geq 0\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called standard Wiener process (standard Brownian Motion) if it satisfy the following conditions:*

(i) *trajectories of W are $\mathbb{P}$-almost continuous,*

(ii) $\mathbb{P}(W_0 = 0) = 1$,

(iii) *for $0 \leq s \leq t$, $W_t - W_s$ has the same distribution as W_{t-s},*

(iv) *for $0 \leq s \leq t$, $W_t - W_s$ is independent of $\{W_u : u \leq s\}$,*

(v) *for every $t > 0, W_t \sim N(0, t)$.*

Again, one can take $\mathbb{F}$ as the natural filtration of the above process. One can see that the standard Wiener process belongs to the class of the Lévy processes. Moreover, one can obtain another process, called a linear Brownian motion, as follows. For every $t \geq 0$ we define

$$B_t := \mu t + \sigma W_t,$$

where $\mu \in \mathbb{R}, \sigma > 0$ and $W = \{W_t : t \geq 0\}$ is the standard Wiener process. It is also the Lévy process. Its decomposition triple is $(-\mu, \sigma, 0)$.

1.2.2 Poisson process

Definition 1.2.2. *Process $N = \{N_t : t \geq 0\}$ defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called Poisson process (with the intensity $\lambda > 0$) if possess the following properties:*

- *trajectories of N are cádlág,*
- $\mathbb{P}(N_0 = 0) = 1$,
- *for $0 \leq s \leq t$, $N_t - N_s$ has the same distribution as N_{t-s},*
- *for $0 \leq s \leq t$, $N_t - N_s$ is independent from $\{N_u : u \leq s\}$,*
- *for every $t > 0$, N_t has Poisson distribution with parameter λt.*

Again, one can take $\mathbb{F}$ as the natural filtration of the above process. The above definition implies that the Poisson process belongs to the class of the Lévy Processes. Its decomposition is equal to $(0, 0, \lambda\delta_1)$, where δ_1 is a Dirac measure on $\{1\}$.

1.2.3 Compound Poisson process

Another example comes from the previous one. Namely, let us start with the random variable

$$X = \sum_{i=1}^{M} \xi_i,$$

where $M \sim \text{Poiss}(\lambda)$[1], $\lambda > 0$ and $\{\xi_i : i \geq 1\}$ is a sequence of i.i.d random variables independent from M, with the same distribution μ (which does not have mass at zero). We can see that its characteristic exponent is equal to

$$\mathbb{E}\left(e^{i\theta \sum_{i=1}^{M} \xi_i}\right) = e^{-\lambda \int_{\mathbb{R}} (1-e^{i\theta x})\mu(dx)}.$$

Thus, one can observe that this random variable is infinite divisible and has decomposition $(a, 0, \Pi)$, where $a = -\lambda \int_{|x|<1} x\mu(dx)$ and $\Pi(dx) = \lambda\mu(dx)$. Now, let $N = \{N_t : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Let us define compound Poisson process $X = \{X_t : t \geq 0\}$ as

$$X_t = \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0.$$

Thus, from above, Lévy triplet is equal to $(a, 0, \lambda\mu)$. Moreover, one can observe that this process is a part of the Crámer-Lundberg process mentioned in the introductory part of this thesis.

1.2.4 Inverse Gaussian process

Let W be a standard Brownian motion, and let us define

$$\tau_s := \inf\{t > 0 : W_t + bt > s\}.$$

That is the first time when W with drift $b > 0$ crosses the level s. It turns out that this random variable is also infinity divisible. Its triple is equal to $(a, 0, \Pi)$ where

$$a = -2sb^{-1} \int_0^b (2\pi)^{-1/2} e^{-y^2/2} dy,$$

$$\Pi(dx) = s\frac{1}{\sqrt{2\pi x^3}} e^{-\frac{b^2 x}{2}} dx \mathbf{1}_{\{x>0\}}(x).$$

1.2.5 Stable Process

Let us start with the definition of stable distribution.

Definition 1.2.3. *Random variable Y has a stable distribution, if for every $n \geq 1$ there exist constants $a_n > 0, b_n \in \mathbb{R}$, such that*

$$Y_1 + \cdots + Y_n \stackrel{D}{=} a_n Y + b_n,$$

where $(Y_i)_{i=1}^n$ are independent copies of Y. If $b_n = 0$ we say that Y is strictly stable.

[1]Here Poiss(λ) means Poisson distribution with parameter λ

Lemma 1.2.4. *Coefficient a_n is of the form $a_n = n^{\frac{1}{\alpha}}$ for $\alpha \in (0, 2]$. As for $\alpha = 2$ the distribution of Y is normal, we will consider only $\alpha \in (0, 2)$.*

Lemma 1.2.5. *Let Y be a stable random variable. Its characteristic exponent is of the form*

$$\Psi(\theta) = \begin{cases} c|\theta|^\alpha[1 - i\beta sign(\theta)\tan(\frac{\pi\alpha}{2})] + i\xi\theta, & \text{when } \alpha \neq 1, \\ c|\theta|[1 + i\beta sign(\theta)\frac{2}{\pi}\log(|\theta|)] + i\xi\theta, & \text{when } \alpha = 1, \end{cases}$$

where $\alpha \in (0, 2), \beta \in [-1, 1], c > 0, \xi \in \mathbb{R}$.

Remark 1.2.6. *As it is clear from the definition of stable distribution, this variable is also infinitely divisible. To connect characteristic exponent with decomposition triple, one needs to set $\sigma = 0$, and its Lévy measure has the following form*

$$\Pi(dx) = \frac{c_1}{x^{1+\alpha}}\mathbf{1}_{(0,\infty)}(x)dx + \frac{c_2}{|x|^{1+\alpha}}\mathbf{1}_{(-\infty,0)}(x)dx,$$

where $c = -(c_1+c_2)\Gamma(-\alpha)\cos(\pi\alpha/2), c_1, c_2 \geq 0$, Γ is the Gamma function and $\beta = (c_1-c_2)/(c_1+c_2)$ if $\alpha \neq 1$ and $c_1 = c_2$ if $\alpha = 1$. The choice of a, from Lévy triplet, is then implicit.

1.3 Markov processes and Feller semigroups

For now, we saw a definition of the Lévy process through the Lévy-Khintchine formula. There are few possibilities on how one can proceed to define this class, each beneficial. The former approach underlines the probabilistic structure of every Lévy process. But, as the Lévy process is (strong) Markov process, one can start reasoning from this point of view. It will be beneficial in terms of understanding objects like $\mathbb{P}_x, \mathbb{E}_x$ or the infinitesimal generator of the process. As the infinitesimal generator plays a crucial role in applications, it is also natural to ask how large the class is for which we can define a similar operator. It will be important in Chapter 2, where we will show that the so-called refracted Lévy process belongs to the family of the Feller processes. Therefore, this section will give a rough introduction to the Markov processes and Feller semigroups. This section is quoted from Khoshnevisan and Schilling [54][2]. In addition, we also recommend Chapter 17 from Kallenberg [31].

Definition 1.3.1. *A (temporally homogeneous) Markov transition function is a measure kernel $p_t(x, B), t \geq 0, x \in \mathbb{R}, B \in \mathcal{B}(\mathbb{R})$[3] such that*

a) $B \to p_s(x, B)$ is a probability measure for every $s \geq 0$ and $x \in \mathbb{R}$,

b) $(s, x) \to p_s(x, B)$ is a Borel measurable function for every $B \in \mathcal{B}(\mathbb{R})$,

c) the Chapman-Kolmogorov equations hold

$$p_{s+t}(x, B) = \int p_t(y, B)p_s(x, dy), \quad \text{for all } s, t \geq 0, x \in \mathbb{R}, B \in \mathcal{B}(\mathbb{R}).$$

[2]Clearly, we need this part only for 1-dimension Markov processes. However, a more general setting can be found in Khoshnevisan and Schilling [54]

[3]Here $\mathcal{B}(\mathbb{R})$ stands for the family of the Borel sets on $\mathbb{R}$

Definition 1.3.2. *A stochastic process* $X = \{X_t : t \geq 0\}$ *is called a (temporally homogeneous) Markov process if there exists a Markov transition function* $p_t(x, B)$ *such that*

$$\mathbb{P}(X_t \in B|\mathcal{F}_s) = p_{t-s}(X_s, B) \quad \text{a.s. for all } s \leq t, B \in \mathcal{B}(\mathbb{R}).$$

Definition 1.3.3. *A (universal) Markov process is a tuple* $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, X = \{X_t : t \geq 0\}, \{\mathbb{P}_x : x \in \mathbb{R}\})$ *such that* $p_t(x, B) = \mathbb{P}_x(X_t \in B)$ *is a Markov transition function and* X *is for each* $\mathbb{P}_x$ *a Markov process in the sense of Definition 1.3.2 such that* $\mathbb{P}_x(X_0 = x) = 1$. *In particular,*

$$\mathbb{P}_x(X_t \in B|\mathcal{F}_s) = \mathbb{P}_{X_s}(X_{t-s} \in B) \quad \mathbb{P}_x\text{-a.s. for all } s \leq t, B \in \mathcal{B}(\mathbb{R}).$$

Lemma 1.3.4. *Let* $X = \{X_t : t \geq 0\}$ *be a Lévy process on* $\mathbb{R}$. *Then*

$$p_t(x, B) := \mathbb{P}_x(X_t \in B) := \mathbb{P}(X_t + x \in B), \quad t \geq 0, x \in \mathbb{R}, B \in \mathcal{B}(\mathbb{R}),$$

is a Markov transition function.

From the proof of this lemma, we can also deduce that

$$p_t(x, B) = \int \mathbf{1}_B(x + y)\mathbb{P}(X_t \in dy) = \int \mathbf{1}_{B-x}(y)\mathbb{P}(X_t \in dy) = p_t(0, B - x).$$

Thus, one can observe that kernels $p_t(x, B)$ are invariant under shifts in $\mathbb{R}$ (translation invariant). In case of the Lévy process we will write $p_t(B - x) := p_t(x, B)$. Let $B_b(\mathbb{R})$ be a set of Borel bounded measurable functions. Using the Markov transition function $p_t(x, B)$ we can define a linear operators as the following:

$$P_t f(x) := \int f(y)p_t(x, dy) = \mathbb{E}_x f(X_t), \quad f \in B_b(\mathbb{R}), t \geq 0, x \in \mathbb{R}. \tag{1.1}$$

Such operators can satisfy a lot of essential properties. Therefore, we state the following definition.

Definition 1.3.5. *Let* $C_0(\mathbb{R})$ *denotes the space of continuous functions vanishing at infinity and* $C_b(\mathbb{R})$ *denotes set of the continuous bounded functions. Moreover, let* $(P_t)_{t\geq 0}$ *be defined by* (1.1). *The operators are said to be*

a) action on $B_b(\mathbb{R})$, *if* $P_t : B_b(\mathbb{R}) \to B_b(\mathbb{R})$,

b) operator semigroup if $P_{t+s} = P_t \circ P_s$ *for all* $s, t \geq 0$ *and* $P_0 = 1$,

c) sub-Markovian if $0 \leq f \leq 1 \implies 0 \leq P_t f \leq 1$,

d) contractive if $||P_t f||_\infty \leq ||f||_\infty$ *for all* $f \in B_b(\mathbb{R})$,

e) conservative if $P_t 1 = 1$,

f) Feller operators, if $P_t : C_0(\mathbb{R}) \to C_0(\mathbb{R})$,

g) strongly continuous on $C_0(\mathbb{R})$ *if* $\lim_{t\to 0} ||P_t f - f||_\infty = 0$ *for all* $f \in C_0(\mathbb{R})$, *h) strong Feller operators, if $P_t : B_b(\mathbb{R}) \to C_b(\mathbb{R})$.*

Lemma 1.3.6. *Let $(P_t)_{t\geq 0}$ be defined by (1.1). The properties a) − e) from Definition 1.3.5 hold for any Markov process, a) − g) for any Lévy process, and a) − h) hold for any Lévy process such that all $p_t(dy) = \mathbb{P}(X_t \in dy), t > 0$, are absolutely continuous with respect to Lebesgue measure.*

Through this section, we will be working with the following family

Definition 1.3.7. *A Feller semigroup is a family of linear operators*

$$P_t : B_b(\mathbb{R}) \to B_b(\mathbb{R})$$

satisfying the properties a) − g) of Definition 1.3.5.

Remark 1.3.8. *If $(P_t)_{t\geq 0}$ is a Feller semigroup then there exists a unique stochastic process (a Feller process) with $(P_t)_{t\geq 0}$ as a transition semigroup.*

Definition 1.3.9. *Let $(P_t)_{t\geq 0}$ be a Feller semigroup. The (infinitesimal) generator is a linear operator defined by*

$$\mathcal{D}(\Gamma) := \left\{ f \in C_0(\mathbb{R}) \middle| \exists g \in C_0(\mathbb{R}) : \lim_{t\to 0} \left|\left| \frac{P_t f - f}{f} - g \right|\right|_\infty = 0 \right\},$$

$$\Gamma f := \lim_{t\to 0} \frac{P_t f - f}{f}, \quad f \in \mathcal{D}(\Gamma).$$

Lemma 1.3.10. *Let $(P_t)_{t\geq 0}$ be a Feller semigroup with infinitesimal generator $(\Gamma, \mathcal{D}(\Gamma))$. Then $P_t(\mathcal{D}(\Gamma)) \subset \mathcal{D}(\Gamma)$ and*

$$\frac{d}{dt} P_t f = \Gamma P_t f = P_t \Gamma f, \quad \text{for all } f \in \mathcal{D}(\Gamma), t \geq 0.$$

Moreover, $\int_0^t P_s f ds \in \mathcal{D}(\Gamma)$ for any $f \in C_0(\mathbb{R})$, and

$$\begin{aligned} P_t f - f &= \Gamma \int_0^t P_s f ds, \quad f \in C_0(\mathbb{R}), t > 0 \\ &= \int_0^t P_s \Gamma f ds, \quad f \in \mathcal{D}(\Gamma), t > 0. \end{aligned}$$

Sometimes it is more beneficial to work with the so-called resolvent, as we will see in Section 2.4.4

Definition 1.3.11. *Let $(P_t)_{t\geq 0}$ be defined by (1.1). The resolvent is a linear operator on $B_b(\mathbb{R})$ given by*

$$R_\lambda f(x) := \int_0^\infty e^{-\lambda t} P_t f(x) dt, \quad f \in B_b(\mathbb{R}), x \in \mathbb{R}, \lambda > 0.$$

We call $(R_\lambda)_{\lambda>0}$ as the resolvent of a Feller semigroup if associated $(P_t)_{t\geq 0}$ is a Feller semigroup.

We can give probabilistic intuition behind resolvents. Let e_λ be a exponential random variable with the parameter $\lambda > 0$. Then

$$R_\lambda f(x) = \mathbb{E}_x(f(X_{e_\lambda})).$$

The following theorem connects resolvents with the generator of the Feller semigroup.

Theorem 1.3.12. *Let* $(\Gamma, \mathcal{D}(\Gamma))$ *and* $(R_\lambda)_{\lambda>0}$ *be the generator and the resolvent of a Feller semigroup. Then*

$$R_\lambda = (\lambda - \Gamma)^{-1} \quad \text{for all } \lambda > 0.$$

One can also use the resolvents to prove that the family $(P_t)_{t\geq 0}$ of linear operators is the Feller semigroup.

Theorem 1.3.13. *Let* $(R_\lambda)_{\lambda>0}$ *be a resolvent associated with* $(P_t)_{t\geq 0}$ *defined by* (1.1). *If the* $(R_\lambda)_{\lambda>0}$ *satisfy*

1. *for all* $q, p > 0$, $R_q - R_p = (p - q)R_q R_p$,
2. *for all* $q > 0$, $\|qR_q 1\| \leq 1$,
3. *for all* $q > 0$, R_q *is a map from* $C_0(\mathbb{R})$ *to* $C_0(\mathbb{R})$,
4. *for all* $f \in C_0(\mathbb{R})$, $\lim_{q\to\infty} \|qR_q f - f\| = 0$.

Then $(P_t)_{t\geq 0}$ *is a Feller semigroup.*

One can find proof of this theorem (in a slightly different form) in the Kostrykin *et al.* [33]. It is natural to ask about the shape of the infinitesimal generator for a given class of processes. Therefore, we will state below two propositions, one for Lévy processes and the second for Feller processes.

Proposition 1.3.14. *Let* X *be a Lévy process with the Lévy-Khintchine triple* (a, σ, Π). *Then, the infinitesimal operator* Γ *is of the form*

$$\Gamma f(x) = af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}\setminus\{0\}} [f(x+y) - f(x) - f'(x)y\mathbf{1}_{\{|y|<1\}}]\Pi(dy),$$

for sufficiently smooth f. [4].

Proposition 1.3.15. *Let* X *be a Feller process. Then, the infinitesimal operator* Γ *is of the form*

$$\Gamma f(x) = a(x)f'(x) + \frac{q(x)}{2} f''(x) + \int_{\mathbb{R}\setminus\{0\}} [f(x+y) - f(x) - f'(x)y\mathbf{1}_{\{|y|<1\}}]\Pi(x, dy),$$

where $(a(x), q(x), \Pi(x, \dot{}))$ *is a Lévy-Khintchine triple for every fixed* $x \in \mathbb{R}$ *and* f *is sufficiently smooth.*

Intuitively, Proposition 1.3.15 outlines that the Feller process behaves as the Lévy process in a short period. The following result allows the creation of the new martingales using the infinitesimal operator.

[4]Here and after sufficiently smooth depends on the specific Lévy process in the consideration. This we will explain later when this operator will be in use.

Corollary 1.3.16. *Let $X = \{X_t : t \geq 0\}$ be a Feller process with generator $(\Gamma, \mathcal{D}(\Gamma))$ and semigroup $(P_t)_{t\geq 0}$. For every $f \in \mathcal{D}(\Gamma)$ the process*

$$M_t^{[f]} := f(X_t) - \int_0^t \Gamma f(X_r)dr, \quad t \geq 0,$$

is a martingale for the canonical filtration $\mathcal{F}_t^X := \sigma(X_s : s \leq t)$ and any $\mathbb{P}_x, x \in \mathbb{R}$.

In the end, we give the Dynkin's formula.

Lemma 1.3.17 (Dynkin's formula)**.** *Let $X = \{X_t : t \geq 0\}$ be a Feller process with generator $(\Gamma, \mathcal{D}(\Gamma))$ and semigroup $(P_t)_{t\geq 0}$. For every stopping time σ with $\mathbb{E}_x\sigma < \infty$ one has*

$$\mathbb{E}_x f(X_\sigma) - f(x) = \mathbb{E}_x \left[\int_{[0,\sigma)} \Gamma f(X_r)dr \right], \quad f \in \mathcal{D}(\Gamma).$$

1.4 Scale functions for spectrally negative Lévy processes

Definition 1.4.1. *We call Lévy process X spectrally negative if its Lévy measure Π is concentrated only on the negative half-line i.e.* $\Pi((0,\infty)) = 0$.

From now we assume that X is a spectrally negative Lévy process. The assumption about one-sided jumps allows us to work with the Laplace exponent rather than the characteristic exponent. Namely,

$$\psi(\lambda) := \log \mathbb{E}(e^{\lambda X_1}) = -\Psi(-i\lambda),$$

which is finite at least for all $\lambda \geq 0$. Then one can write it as

$$\psi(\theta) = \log(\mathbb{E}[e^{\theta X_1}]) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \Big(e^{\theta x} - 1 - \theta x \mathbf{1}_{\{-1<x<0\}}\Big)\Pi(dx),$$

The function $\psi : [0, \infty) \to \mathbb{R}$ is zero at zero and tends to infinity at infinity. Further, it is infinitely differentiable and strictly convex. In particular $\psi^{'}(0+) = \mathbb{E}(X_1) \in [-\infty, \infty)$, which determines long term behaviour of the process. Namely, when $\pm\psi^{'}(0+) > 0$, then $\lim_{t\to\infty} X_t = \pm\infty$ and if $\psi^{'}(0+) = 0$, then $\limsup_{t\to\infty} X_t - \liminf_{t\to\infty} X_t = \infty$, which means that process oscillates. Define the right inverse

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\},$$

for each $q \geq 0$. If $\psi^{'}(0+) \geq 0$ then $\lambda = 0$ is the unique solution to $\psi(\lambda) = 0$ and otherwise there are two solutions to the latter with $\lambda = \Phi(0) > 0$ being the larger of the two, the other is $\lambda = 0$. As we investigate exit problems of Lévy processes, we will need a definition of the first passage times. Namely, define for $a \in \mathbb{R}$

$$\tau_a^- = \inf\{t > 0 : X_t < a\}$$

and

$$\tau_a^+ = \inf\{t > 0 : X_t > a\}.$$

Note that due to working with the class of the spectrally negative Lévy processes, the only possibility to upper cross level a is by the continuous passing. As in the Section 1.3, we shall endow

X with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ such that under $\mathbb{P}_x$, we have $X_0 = x$ with probability one. Furthermore, $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$. We will use a convention that $\mathbb{P} = \mathbb{P}_0$ and $\mathbb{E} = \mathbb{E}_0$. One can be interested in obtaining a semi-analytical representation of the following expression (the so-called two-sided exit problem) for $0 \leq x \leq c$

$$\mathbb{E}_x\Big[e^{-q\tau_c^+}\mathbf{1}_{\{\tau_c^+ < \tau_0^-\}}\Big]. \tag{1.2}$$

Namely, one would like to examine a unit payment made when the process reaches level c before the first moment the process go below zero. This payment is additionally discounted by a discount factor of $q > 0$.

There is another interpretation from a more theoretical point of view. Assume that the state space is enlarged with an absorbing state ϑ. Let us call it *cemetery* state. Moreover, let e_q be a random variable with distribution $Exp(q)$, independent from X. If $t > e_q$ we put X into state ϑ. We call this behaviour a killing of X. Then, expression (1.2) is the probability that the process reaches level c before crossing 0 from above and before being killed by e_q. There are some situations where this duality of view is convenient. It will be clear in Chapter 3, where results can be used in various applications, independent at first glance. To obtain an analytical expression for the expectation in (1.2), define the following family of functions.

Theorem 1.4.2. *For each $q \geq 0$, there exists a function $W^{(q)} : \mathbb{R} \to [0, \infty)$, called the q-scale function, that satisfies $W^{(q)}(x) = 0$ for $x < 0$ and is characterised on $[0, \infty)$ as a strictly increasing and continuous function whose Laplace transform is given by*

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)dx = \frac{1}{\psi(\theta) - q}, \quad \text{for } \theta > \Phi(q).$$

Moreover, we define the second scale function by

$$Z^{(q)}(x) := 1 + q\int_0^x W^{(q)}(y)dy, \quad \text{for } x \in \mathbb{R}.$$

It turns out that for $a \leq x \leq c$ and $q \geq 0$ (see e.g., Kyprianou [36])

$$\mathbb{E}_x\Big[e^{-q\tau_c^+}\mathbf{1}_{\{\tau_c^+ < \tau_a^-\}}\Big] = \frac{W^{(q)}(x-a)}{W^{(q)}(c-a)},$$

and also for $q > 0$

$$\mathbb{E}_x\left[e^{-q\tau_a^-}\mathbf{1}_{\{\tau_a^- < \tau_c^+\}}\right] = Z^{(q)}(x-a) - \frac{Z^{(q)}(c-a)}{W^{(q)}(c-a)}W^{(q)}(x-a).$$

Above expectations are called two-sided exit problems. Moreover, one can also consider one-sided counterparts. For $x \in \mathbb{R}$ and $q \geq 0$

$$\mathbb{E}_x\left[e^{-q\tau_0^-}\mathbf{1}_{(\tau_0^- < \infty)}\right] = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x),$$

where we understand $\frac{q}{\Phi(q)}$ in the limiting sense for $q = 0$, so that

$$\mathbb{P}_x(\tau_0^- < \infty) = \begin{cases} 1 - \psi'(0+)W(x) & \text{if } \psi'(0+) > 0, \\ 1 & \text{if } \psi'(0+) \leq 0. \end{cases} \tag{1.3}$$

Let us note that the above is a probability of a classical ruin time in the infinite time interval. One can express the occupation measures for X in a given Borel set A in terms of scale functions. In particular, we are interested in not only whole lifetime of the process but as well time restricted up to the times τ_a^+, τ_0^- and $\tau := \tau_a^+ \wedge \tau_0^-$. One can find the following theorem in e.g. Kuznetsov *et al.* [34].

Theorem 1.4.3. *(i) For all $a \geq x \geq 0, q \geq 0$ and Borel set $A \subseteq [0, a]$*

$$\mathbb{E}_x \left[\int_0^\infty e^{-qt} \mathbf{1}_{\{X_t \in A, t < \tau\}} dt \right] = \int_A \left\{ \frac{W^{(q)}(x) W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \right\} dy.$$

(ii) For all $a \geq x$ and Borel set $A \subseteq (-\infty, a]$,

$$\mathbb{E}_x \left[\int_0^\infty e^{-qt} \mathbf{1}_{\{X_t \in A, t < \tau_a^+\}} dt \right] = \int_A \left\{ e^{-\Phi(q)(a-x)} W^{(q)}(a-y) - W^{(q)}(x-y) \right\} dy.$$

(iii) For all $x \geq 0$ and Borel set $A \subseteq [0, \infty)$,

$$\mathbb{E}_x \left[\int_0^\infty e^{-qt} \mathbf{1}_{\{X_t \in A, t < \tau_0^-\}} dt \right] = \int_A \left\{ e^{-\Phi(q)y} W^{(q)}(x) - W^{(q)}(x-y) \right\} dy.$$

(iv) For all $x \in \mathbb{R}$ and Borel set $A \subseteq \mathbb{R}$,

$$\mathbb{E}_x \left[\int_0^\infty e^{-qt} \mathbf{1}_{\{X_t \in A\}} dt \right] = \int_A \left\{ \Phi'(q) e^{-\Phi(q)y} - W^{(q)}(-y) \right\} dy.$$

Scale functions occur in many fluctuation identities. Thus, the natural question is if it is possible to calculate them explicitly. For some particular examples, like Brownian motion with drift or Cramér-Lundberg process with exponential jumps, the form of functions $W^{(q)}$ and $Z^{(q)}$ can be obtained explicitly (see Bertoin [6], Kyprianou [36], Hubalek and Kyprianou [24] and Kuznetsov *et al.* [34]). We also present some explicit results in Section 1.4.2. For all others, there is a need to use numerical methods for inverting the Laplace transform. We recommend Section 5 from the article Kuznetsov *et al.* [34] for more details about numerical methods.

1.4.1 Properties of scale functions

Previously, we saw that one could translate some exit problems into the language of the scale functions. Thus, it is rewarding to understand some analytical properties of these functions. Let us cite from Kuznetsov *et al.* [34] the following lemmas and corollary.

Lemma 1.4.4. *For any $q \geq 0$, the scale function $W^{(q)}$ is almost everywhere differentiable.*

Moreover, for some special cases of the Lévy processes, one can prove that $W^{(q)}$ is even smoother.

Lemma 1.4.5. *For each $q \geq 0$, the scale function $W^{(q)}$ belongs to $C^1(0, \infty)$ if and only if at least one of the following criteria holds,*

- $\sigma \neq 0$,

- $\int_{(-1,0)} |x|\Pi(dx) = \infty$,
- $\overline{\Pi}(x) := \Pi(-\infty, -x)$ *is continuous.*

The last of these types of results will be the following corollary.

Corollary 1.4.6. *When X has paths of bounded variation the scale function $W^{(q)}$ does not posses a derivative at $x > 0$ (for all $q \geq 0$) if and only if Π has an atom at $-x$. In particular, if Π has a finite number of atoms supported by the set $\{-x_1, \dots, -x_n\}$ then, for all $q \geq 0$, $W^{(q)} \in C^1((0,\infty) \setminus \{x_1, \dots, x_n\})$.*

Next, we would like to understand how the scale function behaves at its origin. Let us set $p := a + \int_0^1 x\Pi(dx)$ when process X is of bounded variation (this quantity then represents drift of the process). Then,

$$W^{(q)}(0+) = \begin{cases} \frac{1}{p}, & \text{when X has bounded variation paths,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

Similarly, one can obtain a value of the right limit of $W^{(q)\prime}$ at 0. For $q \geq 0$

$$W^{(q)\prime}(0+) = \begin{cases} \frac{2}{\sigma^2}, & \text{when } \sigma \neq 0, \\ \frac{\Pi(-\infty,0)+q}{a^2}, & \text{when } \sigma = 0 \text{ and } \Pi(-\infty,0) < \infty, \\ +\infty, & \text{when } \sigma = 0 \text{ and } \Pi(-\infty,0) = \infty. \end{cases}$$

There is much more that one can say about the behaviour of the scale functions. But, we will cite one last result, namely the following theorem from Loeffen and Renaud [47].

Theorem 1.4.7. *Suppose that $\overline{\Pi}$ is log-convex function. Then for all $q \geq 0, W^{(q)}$ has a log-convex first derivative.*

1.4.2 Examples of the scale functions

Linear Brownian motion

Recall that we defined linear Brownian motion as process $X = \{X_t : t \geq 0\}$ such that for $t \geq 0$

$$X_t := \mu t + \sigma B_t,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and B is a standard Brownian motion. Its characteristic function is equal to

$$\varphi(\theta) = e^{i\theta\mu - \frac{1}{2}\sigma^2\theta^2}.$$

Thus, its Laplace exponent is equal to

$$\psi(\lambda) = \lambda\mu + \frac{\sigma^2\lambda^2}{2}.$$

From the definition of $W^{(q)}$, one can obtain that

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)dx = \frac{1}{\theta\mu + \frac{\sigma^2\theta^2}{2} - q}.$$

Thus, after simple calculations, we have that

$$W^{(q)}(x) = \frac{2}{\sigma^2\rho}\Big(e^{\rho_2 x} - e^{-\rho_1 x}\Big),$$

where

$$\rho_1 = \frac{\sqrt{\mu^2+2q\sigma^2}+\mu}{\sigma^2},\ \rho_2 = \frac{\sqrt{\mu^2+2q\sigma^2}-\mu}{\sigma^2},\ \rho = \rho_1+\rho_2 = \frac{2\sqrt{\mu^2+2q\sigma^2}}{\sigma^2}.$$

In particular, for $q = 0$ we have that

$$W^{(0)}(x) = \frac{1}{\mu}\left(1 - e^{-\frac{2\mu x}{\sigma^2}}\right).$$

Cramér-Lundberg process with exponential claims

In the second example, we will consider the Cramér-Lundberg process with exponential claims

$$X_t = x + pt - \sum_{i=1}^{N_t} U_i,$$

where $x \in \mathbb{R}$, $p > 0$, $\{U_i\}_{i=1}^{\infty}$ is an *i.i.d.* sequence of exponential random variables with the parameter $\mu > 0$, and $N = \{N_t\}_{t\geq 0}$ is a homogeneous Poisson process with the intensity $\lambda > 0$. We also assume that the Poisson process and the exponential random variables are mutually independent.
For this process, the scale function is of the following form (see, e.g. Czarna [13])

$$W^{(q)}(x) = \frac{1}{p}\Big(A^+ e^{q^+ x} - A^- e^{q^- x}\Big),$$

with

$$q^{\pm} = \frac{q+\lambda-\mu p \pm \sqrt{(q+\lambda-\mu p)^2 + 4pq\mu}}{2p}, \quad \text{and} \quad A^{\pm} = \frac{\mu + q^{\pm}(q)}{q^+ - q^-}.$$

In case of $q = 0$ we have

$$W^{(0)}(x) = \frac{1}{p}\left(\frac{\lambda-\mu p}{p} e^{\frac{\lambda-\mu p}{p}x} - \frac{\mu p}{\lambda - \mu p}\right).$$

1.4.3 Spectrally negative stable process

Let X be a spectrally negative stable process of index $\alpha \in (1,2)$. The Laplace exponent is then

$$\psi(\theta) = \theta^{\alpha}.$$

Bertoin in [7] found that the scale function is of the form

$$W^{(q)}(x) = \alpha x^{\alpha-1} E_{\alpha}^{'}(qx^{\alpha}), \quad x \geq 0,$$

where $E_{\alpha}^{'}$ is the derivative of the Mittag-Leffler function of index α given by $E_\alpha(x) = \sum_{n=0}^{\infty} x^n \frac{1}{\Gamma(1+\alpha n)}$.

1.5 Problem of the optimal dividend payments for spectrally negative Lévy processes

In this thesis, we will deal with the problem of optimal dividend payments. This problem has been widespread in applied mathematics since de Finetti [19], who was the first to introduce the dividend model in risk theory. In his work, he proved that for a simple random walk with ± 1 increments and under the rule of maximising the expected discounted dividends before the classical ruin time, the optimal strategy is the barrier strategy described as follows. For a fixed level $a > 0$, whenever the surplus process reaches this level, one reflects the process and pays all excess above a as dividends. In the literature, there is a rich set of articles that studied this problem in the continuous-time framework; see, e.g., Avram *et al.* [5], Loeffen [44] and Loeffen and Renaud [47], where the value function of the barrier strategy and the optimal barrier level were described in terms of the scale functions. In the introductory part, we presented the practical foundations of this problem. Let us recall that paying dividends is the problem of compensating those who have invested in the company. Usually, in these issues, there must be a balance between paying as much money as possible for investors to be satisfied and managing the risk of bankruptcy.

Our goal here is to provide a pattern of behaviour in solving this problem without going into technical details. Of course, as the models tend to be more complicated, the reasoning differs in some way from the one presented here. One can find an example of complete steps in Section 2.4.4.

Let us assume that X is a spectrally negative Lévy process. Let π be a dividend strategy, such that non-decreasing, left-continuous $\mathbb{F}$-adapted process $L^\pi = \{L^\pi_t : t \geq 0\}$ represents cumulative sum of dividends paid until time t. As the next step, define for $t \geq 0$

$$U^\pi_t := X_t - L^\pi_t.$$

We call process $U^\pi = \{U^\pi_t : t \geq 0\}$ a controlled risk process. It is a process that controls the amount of money left after dividend payments, which will interest us from a practical and theoretical point of view. For U^π, let us define the moment of classical ruin as

$$\sigma^\pi = \inf\{t > 0 : U^\pi_t < 0\}.$$

Naturally, this moment of ruin is not equal to the first time when X goes below zero. However, these times will be equal given the event that no dividend payment ever occurs. This fact allows us to leverage the X's exit problems into the U's exit problems. Before we present a candidate for a dividend payout strategy, we need to state what conditions are set for a given strategy to be considered. We say a strategy is admissible if the controlled risk process does not cross zero from above after the dividend payment. Mathematically speaking we require that $L^\pi_{t+} - L^\pi_t < U^\pi_t$ for $t < \sigma^\pi$. Let $\mathcal{A}$ be the set of all admissible dividend strategies.

The essential tool to work with is the value function, defined as

$$v_\pi(x) := \mathbb{E}_x\left[\int_0^{\sigma^\pi} e^{-qt} dL^\pi_t\right],$$

where $q > 0$ is a discount factor and $x \geq 0$ is an initial capital of the company. Therefore, the whole problem comes down to maximising the value function, hence finding

$$v_*(x) := \sup_{\pi \in \Pi} v_\pi(x),$$

for every x and an optimal strategy $\pi \in \mathcal{A}$, such that

$$\upsilon_*(x) = \upsilon_\pi(x).$$

The candidate we will consider here is a barrier strategy at $a > 0$. From the practical point of view, it means to pay everything above level a. Theoretically, it reflects the process $X - a$ at its supremum. Namely, define $L^a = \{L_t^a : t \geq 0\}$ such that for $t \geq 0$

$$L_t^a := \sup_{s \leq t}[X_t - a] \vee 0.$$

Thus, let U_t^a and σ^a be the controlled risk process and the ruin time for the barrier strategy at the level a, respectively. In work Avram *et al.* [5], the authors proved that

Theorem 1.5.1. *Let $a > 0$. For $x \in [0, a]$ it holds that*

$$\mathbb{E}_x\left[\int_0^{\sigma^a} e^{-qt} dL_t^a\right] = \frac{W^{(q)}(x)}{W^{(q)\prime}(a)}.$$

The proof is based on the use of excursion theory, and in particular, the equality proved in work Avram *et al.* [4]

$$\mathbb{E}_0\left[\int_0^{\tilde{\tau}^a} e^{-qt} dS_t\right] = \frac{W^{(q)}(a)}{W^{(q)\prime}(a)},$$

for $S_t := \sup_{0 \leq s \leq t}(X_s \vee 0)$ and $\tilde{\tau}^a = \inf\{t > 0 : S_t - X_t > a\}$. The authors used also

$$\mathbb{E}_x\left[\int_0^{\sigma^a} e^{-qt} dL_t^a\right] = \mathbb{E}_{x-a}\left[\int_0^{\tilde{\tau}^a} e^{-qt} dS_t\right].$$

Getting a representation of the value function is crucial. Hence, it is worth to consider other proof techniques. In the article Renaud and Zhou [53], the proof is based on the simple properties of the spectrally negative Lévy process as well as the strong Markov property. They proved even more, namely for $k \geq 1$, that

$$\mathbb{E}_x\left[\left(\int_0^{\sigma^a} e^{-qt} dL_t^a\right)^k\right] = k!\frac{W^{(kq)}(x)}{W^{(kq)}(b)}\Pi_{i=1}^k \frac{W^{(iq)}(a)}{W^{(iq)\prime}(a)}.$$

They approached it in such a way that, at first, they obtained the result for the kth moment, assuming that the process start from the barrier level

$$\mathbb{E}_a\left[\left(\int_0^{\sigma^a} e^{-qt} dL_t^a\right)^k\right] = k!\Pi_{i=1}^k \frac{W^{(iq)}(a)}{W^{(iq)\prime}(a)}.$$

Then the idea was that to define for any $n \geq 1$

$$T_n = \inf\left\{t \geq : X_t \notin \left(\frac{1}{n}, a + \frac{1}{n}\right)\right\},$$

and next to obtain lower and upper bounds for the kth moment. Then, going with n to infinity, the result follows. Calculating lower and upper bound was crucial here, and the same approach

was used in other articles, e.g. Czarna *et al.* [16]. In particular, in this thesis, one can find a similar idea in Section 3.4.
Having a representation of the value function, it remains to prove the optimality of the barrier strategy. The first step is to select a candidate for the optimal barrier level. Loeffen, in work [44], proposed that the optimal level should be of the form

$$a^* = \sup\{a \geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(x) \text{ for all } x \geq 0\}. \tag{1.5}$$

The form of this level seems to be very natural when one looks at the representations of the value function. Then, in the same paper, the author proved the following theorem.

Theorem 1.5.2. *Suppose $W^{(q)}$ is sufficiently smooth and*

$$W^{(q)\prime}(a) \leq W^{(q)\prime}(b), \quad \text{for } a^* \leq a \leq b.$$

Then the barrier strategy at a^* is an optimal strategy.

Hence, checking the optimisation of the strategy comes down to analysing the shape of the scale function. The proof of this theorem was done in traditional way in this theory. First, for the spectrally negative Lévy process X, the infinitesimal operator Γ is of the form (for sufficiently smooth f)

$$\Gamma f(x) = af^{'}(x) + \frac{\sigma^2}{2} f^{''}(x) + \int_{(0,\infty)} [f(x-y) - f(x) + f^{'}(x) y \mathbf{1}_{\{0<y<1\}}]\Pi(dy),$$

where (a, σ, Π) are the parameters from the Lévy-Khintchine representation. Note that for convenience, from now we assume that the Lévy measure has a mass on the positive instead of the negative half line. This assumption leads to rewriting all respective equations to ensure that X has only negative jumps. Then, the following two lemmas are crucial in proving Theorem 1.5.2. Especially the first one, which is usually called the Verification Lemma. In this lemma, we will show sufficient conditions for the admissible dividend strategy to be the optimal one. Moreover, it involves the well-known Hamilton–Jacobi–Bellman (HJB) equation.

Lemma 1.5.3 (Verification Lemma)**.** *Suppose π is an admissible dividend strategy such that υ_π is sufficiently smooth and for all $x > 0$*

$$\max\{\Gamma \upsilon_\pi(x) - q\upsilon_\pi(x), 1 - \upsilon_\pi^{'}(x)\} \leq 0 \quad \textit{(HJB inequality)}.$$

Then $\upsilon_\pi(x) = \upsilon^*(x)$ for all $x \in \mathbb{R}$.

The proof is based on showing that if $\pi \in \mathcal{A}$ satisfies the lemma conditions then $\pi \geq \upsilon_\star$. Mainly, it is done with the use of Itô's Lemma. In more detail, we will show a similar proof in Section 2.4.4. The second lemma tells when the $\pi_{a^\star}$ strategy meets the conditions of the above lemma, i.e. when it is the optimal strategy.

Lemma 1.5.4. *Suppose $W^{(q)}$ is sufficiently smooth and suppose that*

$$(\Gamma - q)\upsilon_{a^\star} \leq 0 \quad \text{for } x > a^\star.$$

Then $\upsilon_\star(x) = \upsilon_{a^\star}$ for all $x \in \mathbb{R}$.

These two lemmas together are sufficient to prove Theorem 1.5.2.

1.6 Elements of the stochastic calculus

As we mentioned previously, some elements of the stochastic calculus are needed to prove that the dividend strategy is optimal. Therefore, in this section we will recall some basic facts and theorems from the stochastic analysis, mainly from the Lévy processes' point of view. This part is quoted from the book of Protter [51].
We will start with the theorem, which is strictly connected with the Lévy-Khintchine decomposition.

Theorem 1.6.1. *Let X be a Lévy process. Then X has a decomposition*

$$X_t = \sigma B_t + Y_t,$$

where B is a (standard) Brownian motion and $Y = \{Y_t : t \geq 0\}$ is of the form

$$Y_t = \int_{|x|<1} x(N_t(\cdot, dx) - t\Pi(dx)) + at + \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s|\geq 1\}}, \qquad \text{for } t \geq 0,$$

where for any set Λ, $0 \notin \overline{\Lambda}$, $N_t^\Lambda = \int_\Lambda N_t(\cdot, dx)$ is a Poisson process independent of B; N_t^Λ is independent of N_t^Γ whenever Λ and Γ are disjoint; N_t^Λ has parameter $\Pi(\Lambda)$; and $\Pi(dx)$ is a measure on $\mathbb{R} \setminus \{0\}$ such that $\int \min(1, x^2)\Pi(dx) < \infty$.

Next, we have the result that every Lévy process is a semimartingale.

Definition 1.6.2. *We will say an adapted process X witch cádlág paths is decomposable if it can be decomposed $X_t = X_0 + M_t + A_t$, where $M_0 = A_0 = 0$, M is a locally square integrable martingale, and A is cádlág, adapted, with paths of finite variation on compacts.*

Theorem 1.6.3. *A decomposable process is a semimartingale.*

Corollary 1.6.4. *A Lévy process is a semimartingale.*

The quadratic variation process of a semimartingale, also called the bracket process, will play an essential role in changing the variable formula.

Definition 1.6.5. *Let X, Y be semimartingales. The quadratic variation process of X, denoted $[X, X] = ([X, X]_t)_{t\geq 0}$, is defined by*

$$[X, X] := X^2 - 2\int X_- dX,$$

with $X_{0-} = 0$. The quadratic covariation of X, Y, also called the bracket process of X, Y, is defined by

$$[X, Y] := XY - \int X_- dY - \int Y_- dX.$$

Corollary 1.6.6 (Integration by parts)**.** *Let X, Y be two semimartingales. Then XY is a semimaartingale and*

$$XY = \int X_- dY + \int Y_- dX + [X, Y].$$

We will need to divide the bracket process into two parts: continuous and jumps ones. Thus, we have the following definition.

Definition 1.6.7. *For a semimartingale X, the process $[X,X]^c$ denotes the path-by-path continuous part of $[X,X]$. We can then write*

$$[X,X]_t = [X,X]_t^c + X_0^2 + \sum_{0<s\le t} (\Delta X_s)^2.$$

Analogously, $[X,Y]^c$ denotes the path-by-path continuous part of $[X,Y]$.

Definition 1.6.8. *A semimartingale X will be called quadratic pure jump process if $[X,X]^c = 0$.*

Note that if X is a quadratic pure jump, then $[X,X]_t = X_0^2 + \sum_{0<s\le t}(\Delta X_s)^2$. From the Theorem 1.6.1 one can get that Y_t from this theorem is a quadratic pure jump semimartingale. Therefore, for the Lévy process X we have that $[X,X]_t^c = [\sigma B, \sigma B]_t^c = \sigma^2 t$. Now, we can proceed to the well-known Itó's formula, a change of variables formula for the semimartingales. At first, we will start with the version that deals with the functions from C^2(see [5])

Theorem 1.6.9. *Let X be a semimartingale and f be a C^2 real function. Then $f(X)$ is again a semimartingale, and the following formula holds:*

$$f(X_t)-f(X_0) = \int_{0+}^t f^{'}(X_{s-})dX_s + \frac{1}{2}\int_{0+}^t f^{''}(X_{s-})d[X,X]_s^c + \sum_{0<s\le t}\{f(X_s)-f(X_{s-})-f^{'}(X_{s-})\Delta X_s\}.$$

Also important is a multi-dimensional analogue. In particular, to have the change of variable formula in case of time and space functions.

Theorem 1.6.10. *Let $X = (X^1, \dots, X^n)$ be a n-tuple of semimartingales, and let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous second order partial derivatives. Then $f(X)$ is a semimartingale, and the following formula holds:*

$$\begin{aligned} f(X_t) - f(X_0) = \sum_{i=1}^n \int_{0+}^t \frac{\partial f}{\partial x_i}(X_{s-})dX_s^i + \frac{1}{2}\sum_{1\le i,j\le n} \int_{0+}^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-})d[X^i, X^j]_s^c \\ + \sum_{0<s\le t}\{f(X_s) - f(X_{s-}) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{s-})\Delta X_s^i\}. \end{aligned}$$

The above versions of the Itó formula are sufficient for many applications. However, sometimes one can also need to use a change of variable formula for less smooth functions. For example, we can consider functions whose first or second derivative has a finite number of single discontinuities. It turns out that the so-called local times are proper tools for developing such formulas. We need to start with the following theorem, which also says that the semimartingales are preserved under convex transformations.

[5]We denote C^2 as a set of the twice continuously differentiable functions

Theorem 1.6.11. *Let $f : \mathbb{R} \to \mathbb{R}$ be convex and X be a semimartingale. Then $f(X)$ is a semimartingale and one has*

$$f(X_t) - f(X_0) = \int_{0+}^{t} f^{'}(X_{s-})dX_s + A_t,$$

where $f^{'}$ is the left derivative of f and A is an adapted, right continuous, increasing process. Moreover, $\Delta A_t = f(X_t) - f(X_{t-}) - f^{'}(X_{t-})\Delta X_t$.

Definition 1.6.12. *The sign function is defined to be*

$$sign(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x \leq 0. \end{cases}$$

We further define $h_0(x) := |x|$ and $h_a(x) := |x-a|$.

Since $h_a(x)$ is convex by Theorem 1.6.11 we have for a semimartingale X

$$h_a(X_t) = |X_t - a| = |X_0 - a| + \int_{0+}^{t} \operatorname{sign}(X_{s-} - a)dX_s + A_t^a,$$

where A_t^a is the increasing process of Theorem 1.6.11.

Definition 1.6.13. *Let X be a semimartingale. The local time at a of X, denoted by $L_t^a = L^a(X)_t$, is defined to be the process given by*

$$L_t^a = A_t^a - \sum_{0<s\leq t} \{h_a(X_s) - h_a(X_{s-}) - h_a^{'}(X_{s-})\Delta X_s\}.$$

One can see that the local time L^a is the continuous part of the increasing process A^a. The following theorem is a generalization (in some sense) of the Theorem 1.6.9.

Theorem 1.6.14 (Meyer-Itó formula)**.** *Let f be the difference of two convex functions, $f^{'}$ be its left derivative, and μ be the signed measure (when restricted to the compacts) which is the second derivative of f in the generalized function sense. Then the following equation holds:*

$$f(X_t) - f(X_0) = \int_{0+}^{t} f^{'}(X_{s-})dX_s + \sum_{0<s\leq t}\{f(X_s) - f(X_{s-}) - f^{'}(X_{s-})\Delta X_s\} + \frac{1}{2}\int_{-\infty}^{\infty} \mu(da)L_t^a,$$

where X is a semimartingale and $L_t^a = L_t^a(X)$ is its local time at a.

The following two corollaries will help us to work with the local times.

Corollary 1.6.15. *Let X be a semimartingale with local time $(L^a)_{a\in\mathbb{R}}$. Let g be a bounded Borel measurable function. Then a.s.*

$$\int_{-\infty}^{\infty} L_t^a g(a)da = \int_0^t g(X_s)d[X,X]_s^c.$$

Corollary 1.6.16. *Let X be a semimartingale with local time $(L^a)_{a\in\mathbb{R}}$. Then*

$$[X,X]^c_t = \int_{-\infty}^{\infty} L^a_t da.$$

Having this, one can deduce a helpful version of Theorem 1.6.14.

Theorem 1.6.17 (Extant second derivative Meyer-Itó formula)**.** *Let $f \in C^1$ (see [6]) with an absolutely continuous derivative f' such that $f'(b) - f'(a) = \int_a^b f''(u)du$, with f'' being locally in L^1. Then*

$$f(X_t)-f(X_0) = \int_{0+}^t f'(X_{s-})dX_s + \frac{1}{2}\int_{0+}^t f''(X_{s-})d[X,X]^c_s + \sum_{0<s\le t}\{f(X_s)-f(X_{s-})-f'(X_{s-})\Delta X_s)\}.$$

The last theorem we would like to quote is the version for the functions that are not in C^1. But, at first, we need to state the following hypothesis.

Hypothesis A 1.6.18. *For the remainder of this section we let X denote a semimartingale with the restriction that $\sum_{0<s\le t}|\Delta X_s| < \infty$ a.s., for each $t > 0$.*

Again, the following corollary simplifies work with the local times.

Corollary 1.6.19. *Let X be a semimartingale satisfying Hypothesis A. Then for every (a,t) we have*

$$L^a_t = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t \mathbf{1}_{\{a\le X_s\le a+\epsilon\}} d[X,X]^c_s, \quad a.s.$$

and

$$L^{a-}_t = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t \mathbf{1}_{\{a-\epsilon\le X_s\le a\}} d[X,X]^c_s, \quad a.s.$$

Now, we can quote the last theorem.

Theorem 1.6.20 (Bouleau-Yor Formula)**.** *Let X be a semimartingale satisfying Hyphothesis A, U a positive random variable, f a bounded, Borel function, and $F(x) = \int_0^x f(u)du$. Then*

$$F(X_U) - F(X_0) = \int_{0+}^U f(X_{s-})dX_s - \frac{1}{2}\int f(a)d_a L^a_U + \sum_{0<s\le U}\{F(X_s) - F(X_{s-}) - f(X_{s-})\Delta X_s\}.$$

[6]We denote C^1 as a set of the continuously differentiable functions

Chapter 2

Refracted Lévy process & Parisian ruin time

In the previous chapter, we introduced the concept of the spectrally negative Lévy processes. For this class, we have shown solutions to the exit problems and the classical ruin time in terms of scale functions. Moreover, we have demonstrated the application of these results to the optimal dividend problem. The first thought that arises when analysing this setting is a restrictive approach to the ruin. In practice, it takes a particular time for a company to be declared bankrupt (either by itself or by legal entities dealing with companies' bankruptcy).

Hence, the basic model is well suited to the problems where results' conservatism plays a significant role. However, if one is interested in the accuracy of the results, there is a need to consider more accurate modelling of the phenomenon. Therefore, this chapter aims to analyse a conceptually more accurate setting than the classical one from Chapter 1. Moreover, we will apply the proposed setting to the slightly modified problem of optimal dividend payments.

In particular, it is possible to separate the moment of technical ruin (i.e. reaching zero) from the moment of actual bankruptcy. In this chapter, we will consider the Parisian ruin time for bankruptcy time. This stopping time allows us to continue the life of the process even when it goes below zero. However, if the process's excursion time is greater than the fixed value of $r > 0$, we say that the process is killed (or bankrupt). To complete the model's accuracy, we will also introduce a change to the class of the processes considered for modelling. Namely, we want to assume additional cash injection while the process is below zero. This behaviour will play the role of saving the company from bankruptcy. To achieve this property, we will consider a controlled risk process that behaves similarly to refracted Lévy processes near the origin.

We will begin this chapter by introducing the concept of the refracted Lévy processes. Then, we will show solutions to the exit problems for these processes obtained in the literature. Next, we will introduce the definition of the Parisian ruin time and immerse this concept in the literature. Finally, we will combine these two concepts and show the current results of this setting's exit problems. Again, scale functions will be key tools for this theory. Finally, we will proceed to the concept of the optimal dividend problem, as it is the central part of this chapter. Besides, we will consider fixed transaction costs for every dividend to bring this concept closer to reality.

In Section 2.4.1, we will briefly describe the dividend problem and explain what we mean by the dividend strategy to be optimal. Sections 2.4.2-2.4.4 contain the essential results. We will introduce the impulse (c_1, c_2) policy and provide sufficient conditions that the Parisian refracted scale function's derivative needs to fulfil to ensure that the strategy is optimal. The last part of this chapter contains examples, where we will give new analytical formulas for the Parisian refracted scale functions in the case of the linear Brownian motion and the Crámer-Lundberg process with exponential claims. Using these formulas, we will show a unique impulse policy that is optimal for the impulse control problem for these models. We will also show numerical examples.

2.1 Refracted Lévy process

2.1.1 Definition

The definition of this process comes from the work of Kyprianou and Loeffen [37]. They introduced it as a uniquely strong solution $R = \{R_t\}_{t\geq 0}$ to the stochastic differential equation

$$dR_t = dX_t - \delta \mathbf{1}_{\{R_t > b\}} dt, \tag{2.1}$$

for X being spectrally negative Lévy process with the Lévy triplet (γ, σ, Π), $\delta > 0$ and $b \geq 0$. As in Lkabous *et al.* [42], we focus here on the case when the refraction level b equals zero. Moreover, to be compatible with Kyprianou and Loeffen [37] and Lkabous *et al.* [42], we subtract δ on the positive half-line instead of adding it on the negative half-line; however, the practical effect stays the same. Here we will assume that X is a spectrally negative Lévy process. Moreover, we assume that X does not have monotone paths.
From this equation, it is straightforward to observe that above the level 0, process R evolves as process $Y = \{Y_t : t \geq 0\}$ where for every $t \geq 0$

$$Y_t := X_t - \delta t.$$

Since the process Y is a spectrally negative Lévy process with the Lévy triplet $(\gamma - \delta, \sigma, \Pi)$, its Laplace exponent is given by $\psi_Y(\theta) = \psi(\theta) - \delta\theta$. In particular, process Y retains the probabilistic properties of the process X, e.g. the bounded/unbounded variation of the paths. Moreover, we want to emphasise here that the process R is no longer spatially homogeneous, which means that it is not a Lévy process. In Section 2.4.4, we will prove that the process R is a Feller process and present form of its infinitesimal generator.
It is interesting how Kyprianou and Loeffen [37] gave the solution of (2.1). At first, one needs to consider the processes with bounded variation. A pathwise construction for these processes divides the time period into moments when the process is below and above the point b. Namely, the times of T_n and S_n are defined recursively as follows. Let $S_0 = 0$ and for $n = 1, 2, \ldots$

$$T_n = \inf\{t > S_{n-1} : X_t - \delta \sum_{i=1}^{n-1} (S_i - T_i) \geq b\},$$

$$S_n = \inf\{t > T_n : X_t - \delta \sum_{i=1}^{n-1} (S_i - T_i) - \delta(t - T_n) < b\}.$$

Then we have

$$R_t = \begin{cases} X_t - \delta \sum_{i=1}^{n} (S_i - T_i), & \text{for } t \in [S_n, T_n) \text{ and } n = 0, 1, 2, \ldots \\ X_t - -\delta \sum_{i=1}^{n-1} (S_i - T_i) - \delta(t - T_n), & \text{for } t \in [T_n, S_n] \text{ and } n = 1, 2, 3, \ldots \end{cases}$$

Moreover, the times T_n and S_n for $n = 1, 2, \dots$ can then be identified as

$$T_n = \inf\{t > S_{n-1} : R_t \geq b\}, \quad S_n = \inf\{t > T_n : U_t < b\}.$$

For processes of unbounded variation, they used the fact (see Bertoin [6]) that for any Lévy process X of unbounded variation, one can find a sequence of bounded variation Lévy processes X_n such that, for each $t > 0$,

$$\lim_{n\to\infty} \sup_{s\in[0,t]} |X_n(s) - X(s)| = 0 \quad a.s.$$

Then the solution for the unbounded case is understood in the limiting sense.

2.1.2 Exit problems for refracted Lévy process

First, for a $\in \mathbb{R}$, we define the following first-passage stopping times

$$\kappa_a^- := \inf\{t > 0 : R_t < a\} \qquad \text{and} \qquad \kappa_a^+ := \inf\{t > 0 : R_t > a\}.$$

Recall that we denote $W^{(q)}$ and $Z^{(q)}$ as the first and second scale function for X, respectively. Analogously as for X, we can define the scale functions for the Lévy process Y, and we will use the notation $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ for the first and second scale functions for Y, respectively. Let us define the scale function for the refracted process R as follows. For $q \geq 0$ and $x, a \in \mathbb{R}$ set

$$w^{(q)}(x;a) := W^{(q)}(x-a) + \delta \int_0^x \mathbb{W}^{(q)}(x-y) W^{(q)\prime}(y-a) dy. \tag{2.2}$$

In particular, we write $w^{(q)}(\cdot) := w^{(q)}(\cdot;0)$ when $a = 0$. One can see that the above definition differs from the definition of scale functions for X and Y. However, in Kyprianou and Loeffen [37] and Renaud [52], it was proven that for $a \leq x \leq c$ and $q \geq 0$

$$\mathbb{E}_x\left[e^{-q\kappa_c^+} \mathbf{1}_{\{\kappa_c^+ < \kappa_a^-\}}\right] = \frac{w^{(q)}(x;a)}{w^{(q)}(c;a)},$$

and also for $a \geq x$ we have

$$\mathbb{E}_x\left[e^{-q\kappa_a^+} \mathbf{1}_{\{\kappa_a^+ < \infty\}}\right] = \frac{e^{\Phi(q)x} + \delta\Phi(q)\int_0^x e^{\Phi(q)z}\mathbb{W}^{(q)}(x-z)dz}{e^{\Phi(q)a} + \delta\Phi(q)\int_0^a e^{\Phi(q)z}\mathbb{W}^{(q)}(a-z)dz}. \tag{2.3}$$

Therefore, for process R, the function $w^{(q)}$ gives the same type of representation for the two-sided exit problem as the scale functions $W^{(q)}$ and $\mathbb{W}^{(q)}$ for the case of the spectrally negative Lévy processes. Moreover, for $w^{(q)}$, the following proposition was proved in Czarna *et al.* [18].

Proposition 2.1.1. *Scale function $w^{(q)}(\cdot;a)$ is a.e. continuously differentiable and its derivative is equal to*

$$w^{(q)\prime}(x;a) = \begin{cases} W^{(q)\prime}(x-a), & \text{for } x < 0, \\ \left(1 + \delta\mathbb{W}^{(q)}(0)\right) W^{(q)\prime}(x-a) + \delta\int_0^x \mathbb{W}^{(q)\prime}(x-y)W^{(q)\prime}(y-a)dy, & \text{for } x \geq 0. \end{cases}$$

In particular, if X is of unbounded variation, then $w^{(q)}(\cdot;a)$ is $C^1((a,\infty))$. In contrast, if we assume that $W^{(q)}(\cdot - a) \in C^1((a,\infty))$ and X is of bounded variation, then $w^{(q)}(\cdot;a)$ is also $C^1((a,\infty)\backslash\{0\})$.

Moreover, similarly one can also define the second scale function as

$$z^{(q)}(x;a) := Z^{(q)}(x-a) + \delta \int_0^x \mathbb{W}^{(q)}(x-y)W^{(q)}(y-a)dy.$$

Then, from Kyprianou and Loeffen [37] or Renaud [52] it also holds that for $a \le x \le c$ and $q \ge 0$

$$\mathbb{E}_x\left[e^{-q\kappa_a^-}; \kappa_a^- < \kappa_c^+\right] = z^{(q)}(x;a) - \frac{z^{(q)}(c;a)}{w^{(q)}(c;a)} w^{(q)}(x;a).$$

For other representations, we refer to Kyprianou and Loeffen [37].

2.2 Parisian ruin time for spectrally negative Lévy process

As we mentioned earlier, the Parisian stopping time is one of the alternatives to the classical ruin time. It allows for a less stringent and more practical approach to bankruptcy. Let us formally define it as

$$\tau^r := \inf\{t > 0 : t - \sup\{s < t : X_s \ge 0\} \ge r, X_t < 0\},$$

where $r > 0$ is the so-called Parisian delay, and X is the spectrally negative Lévy process. Next, let us define the following function as a Parisian scale function

$$G^{(q)}(x) := \int_0^\infty W^{(q)}(x+z)\frac{z}{r}\mathbb{P}(X_r \in dz).$$

For this function one can obtain the following representation of the two-sided exit problem (see e.g. Lkabous *et al.* [42] or Loeffen *et al.* [46]). For $x \le c$ and $q \ge 0$

$$\mathbb{E}_x\left[e^{-q\tau_c^+}\mathbf{1}_{\{\tau_c^+ < \tau^r\}}\right] = \frac{G^{(q)}(x)}{G^{(q)}(a)}.$$

2.3 Exit problems for refracted Lévy process and Parisian ruin time

We now introduce tools that will help investigate the optimal dividend strategy. Namely, let us define the Parisian ruin time for the process R, for $r > 0$

$$\kappa_R^r := \inf\{t > 0 : t - \sup\{s < t : R_s \ge 0\} \ge r, R_t < 0\}.$$

From Lkabous *et al.* [42], it is known that

$$\mathbb{E}_x\left[e^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+ < \kappa_R^r\}}\right] = \frac{V^{(q)}(x)}{V^{(q)}(a)}, \tag{2.4}$$

where

$$V^{(q)}(x) := \int_0^\infty w^{(q)}(x;-z)\frac{z}{r}\mathbb{P}(X_r \in dz).$$

From (2.2), one can see that the initial value of $w^{(q)}$ equals $W^{(q)}(-a)$. In contrast, for $V^{(q)}$, one can find in Lkabous *et al.* [42] or Loeffen *et al.* [45] that

$$V^{(q)}(0) = \int_0^\infty W^{(q)}(z)\frac{z}{r}\mathbb{P}(X_r \in dz) = e^{qr}. \tag{2.5}$$

2.4 Dividend problem

2.4.1 Mathematical model

Surplus process

From the practical point of view, we want the surplus process to behave like the spectrally negative refracted Lévy process, which means that we allow injecting (in a continuous way) a certain amount of money with intensity $\delta > 0$ when reserves are below zero. However, as we will see later, cash injection will be directly connected with controlled risk process.

Dividend definition

We will formally introduce the problem studied in this chapter; in particular, we define the optimisation criterion and then define a candidate for the optimal strategy. Denote π as a dividend or control strategy, where $L^\pi = \{L^\pi_t : t \geq 0\}$ is a non-decreasing, left-continuous $\mathbb{F}$-adapted process that starts at zero. We will assume that process L^π is a pure jump process, i.e.

$$L^\pi_t = \sum_{0 \leq s < t} \Delta L^\pi_s, \quad \text{for all } t \geq 0. \tag{2.6}$$

Here, by $\Delta L^\pi_s = L^\pi_{s+} - L^\pi_s$, we mean the jump of the process L^π at time s. Therefore, the random variable L^π_t can be interpreted as an accumulated dividend up to the time t. Note that the pure jump assumption is taken directly from the presence of non-zero transaction costs, and such control strategies as (2.6) are known as *impulse controls*. Let us define the controlled risk process $U^\pi = \{U^\pi_t : t \geq 0\}$ by the dividend strategy π as the strong solution to the following stochastic differential equation

$$dU^\pi_t = dX_t - \delta \mathbf{1}_{\{U^\pi_t > 0\}} dt - dL^\pi_t, \tag{2.7}$$

with $U^\pi_0 = X_0$. Here, we additionally assume that $\delta < \gamma + \int_{(0,1)} x\Pi(dx)$, where we recall that the γ and Π come from X's Lévy triplet. One can observe that we cannot say that refracted process is our surplus process as the time when we introduce refraction depends on the dividends paid until this time. However, until the first dividend, U^π behaves like R. Thus, intuition is not so far from the truth.

The company pays dividends up to its bankruptcy moment; that is the Parisian ruin time in our model. Thus, we need to define Parisian ruin time for process U^π as

$$\kappa^r := \inf\{t > 0 : t - \sup\{s < t : U^\pi_s \geq 0\} \geq r, U^\pi_t < 0, \}, \quad \text{for } r > 0.$$

Denote the value function of a dividend strategy π as

$$\upsilon^{\kappa^r}_\pi(x) = \mathbb{E}_x \left[\int_0^{\kappa^r} e^{-qt} d\Big(L^\pi_t - \sum_{0 \leq s < t} \beta \mathbf{1}_{\{\Delta L^\pi_s > 0\}} \Big) \right], \quad \text{for } x \geq 0,$$

where $q > 0$ is the discount rate and $\beta > 0$ denotes the transaction cost that occurs whenever the company pays dividends. With the assumption (2.6), the above integral can be interpreted as the following sum

$$\upsilon^{\kappa^r}_\pi(x) = \mathbb{E}_x \left[\sum_{0 \leq t < \kappa^r} e^{-qt} \Big(\Delta L^\pi_t - \beta \mathbf{1}_{\{\Delta L^\pi_t > 0\}} \Big) \right], \text{where } x \geq 0.$$

We call a strategy π admissible if we do not get to the *red zone* due to dividend payments, i.e.

$$U_t^\pi - \Delta L_t^\pi \geq 0, \quad \text{for} \quad t < \kappa^r. \tag{2.8}$$

Let $\mathcal{A}$ be the set of all admissible dividend strategies. Our main goal is to find the optimal value function υ_* given by

$$\upsilon_*(x) := \sup_{\pi \in \mathcal{A}} \upsilon_\pi^{\kappa^r}(x)$$

and the optimal strategy $\pi_* \in \mathcal{A}$, such that

$$\upsilon_{\pi_*}^{\kappa^r}(x) = \upsilon_*(x), \quad \text{for all } x \geq 0.$$

2.4.2 Impulse strategy with the Parisian ruin

Let us present the candidate for an optimal strategy for the dividend problem described in Section 2.4.1. The so-called impulse strategy is to reduce the risk process to c_1 whenever the process exceeds level c_2. We assume that the distance between c_1 and c_2 must be greater than β because there must be something left for the shareholders after paying the transaction costs. Additionally, we will make the assumption that $c_1 \geq 0$ which is a consequence of (2.8). Given all said before, one can observe that, in the case of this strategy, risk process U^π needs to solve the following version of (2.7) in the form of an integral equation

$$U_t^\pi = X_t - \delta \int_0^t \mathbf{1}_{\{U_s^\pi > 0\}} ds - (c_2 - c_1) \sum_{0 < s \leq t} \mathbf{1}_{\{U_{s-}^\pi \geq c_2\}} - (\mathbf{1}_{\{X_0 \geq c_2 \wedge t > 0\}} \cdot (X_0 - c_1)), \tag{2.9}$$

where $U_0^\pi = X_0$, $c_2 - c_1 > \beta > 0$ and $c_1 \geq 0$. Formally, we will set π_{c_1,c_2} as the strategy and we will construct such U^{c_1,c_2} that solves (2.9). We start our construction by the simple observation that before the dividend payment process U^{c_1,c_2} should behave as the refracted process started from X_0. After dividend payment, but until next c_2 crossing, process U^{c_1,c_2} should behaves again like refracted process, but this time issued from c_1. This idea will be continued with each c_2 crossing. Having this in mind, we will proceed with the construction.
We define sequence of times $(\tau_k^{c_1,c_2})_{k \geq 0}$ and processes $(U^k = \{U_t^k : t \geq \tau_k^{c_1,c_2}\})_{k \geq 0}$, such that $\tau_0^{c_1,c_2} := 0$ and $U^0 := R$. For brevity of the description we will now define $\tau_1^{c_1,c_2}$ and U^1 separately to show the idea and then $(\tau_k^{c_1,c_2})_{k \geq 2}$ and processes $(U^k = \{U_t^k : t \geq \tau_k^{c_1,c_2}\})_{k \geq 2}$ which will follow the same pattern. Let

$$\tau_1^{c_1,c_2} := \inf\{t > 0 : U_t^0 > c_2\}.$$

Then, from the definition of the refracted process, one can observe that

$$U_t^0 = X_t - \delta \int_0^t \mathbf{1}_{\{U_s^0 > 0\}} ds.$$

Thus, one can deduce that

$$U_{\tau_1^{c_1,c_2}}^0 = X_{\tau_1^{c_1,c_2}} - \delta \int_0^{\tau_1^{c_1,c_2}} \mathbf{1}_{\{U_s^0 > 0\}} ds = [X_0 \vee c_2] \quad \text{a.s.}$$

Therefore, one need to define $U^1 = \{U^1_t : t \geq \tau_1^{c_1,c_2}\}$ as

$$U^1_t := X_t - \delta \int_0^t \mathbf{1}_{\{U^1_s > 0\}} ds - ([X_0 \vee c_2] - c_1),$$

with $U^1_{\tau_1^{c_1,c_2}} = c_1$. Such the process exists uniquely by the unique existence of refracted process and its strong Markov property. Namely, one can observe that U^1 solves

$$dU^1_t = dX_t - \delta \mathbf{1}_{\{U^1_t > 0\}} dt,$$

with $U^1_{\tau_1^{c_1,c_2}} = c_1$. Similarly one can set for $k \geq 2$

$$\tau_k^{c_1,c_2} := \inf\{t > \tau_{k-1}^{c_1,c_2} : U^{k-1}_t \geq c_2\}$$

and $U^k = \{U^k_t : t \geq \tau_k^{c_1,c_2}\}$ such that

$$U^k_t := X_t - \delta \int_0^t \mathbf{1}_{\{U^k_s > 0\}} ds - ([X_0 \vee c_2] - c_1) - (k-1)(c_2 - c_1),$$

with $U^k_{\tau_k^{c_1,c_2}} = c_1$. Again, the same argument about existence follows as for U^1. Then, we construct $U^{c_1,c_2} = \{U^{c_1,c_2}_t : t \geq 0\}$ as

$$U^{c_1,c_2}_t := \sum_{k=0}^{\infty} U^k_t \mathbf{1}(\tau_k^{c_1,c_2} \leq t < \tau_{k+1}^{c_1,c_2}).$$

One can observe that the process $L^{c_1,c_2} = \{L^{c_1,c_2}_t : t \geq 0\}$ is of the form

$$L^{c_1,c_2}_t = ([X_0 \vee c_2] - c_1)\mathbf{1}(t > \tau_1^{c_1,c_2}) + (c_2 - c_1) \sum_{k=2}^{\infty} \mathbf{1}(t > \tau_k^{c_1,c_2}).$$

Let us note that necessarily $\tau_k^{c_1,c_2} < \tau_{k+1}^{c_1,c_2}$ a.s. for $k = 1, 2, \ldots$ as the process U^k_t creeps upward and $c_2 - c_1 > \beta$. It is immediately that U^{c_1,c_2} is a strong solution for the equation (2.9), which can be seen when one divides time into periods $[\tau_k^{c_1,c_2}, \tau_{k+1}^{c_1,c_2})$ for $k = 0, 1, 2, \ldots$.
Before we give the necessary conditions for the (c_1, c_2) strategy to be optimal, we need to consider the form of the value function as a crucial tool for further investigations.

2.4.3 Representation of the value function

Proposition 2.4.1. *The value function $v^{\kappa^r}_{c_1,c_2}$ for the strategy π_{c_1,c_2} with the ruin time κ^r is of the form*

$$v^{\kappa^r}_{c_1,c_2}(x) = \begin{cases} (c_2 - c_1 - \beta)\frac{V^{(q)}(x)}{V^{(q)}(c_2) - V^{(q)}(c_1)}, & \text{for } x \leq c_2, \\ x - c_1 - \beta + (c_2 - c_1 - \beta)\frac{V^{(q)}(c_1)}{V^{(q)}(c_2) - V^{(q)}(c_1)}, & \text{for } x > c_2. \end{cases} \tag{2.10}$$

Proof. At the beginning of the proof, note that it is sufficient to prove this proposition only for $x \leq c_2$ because U^{c_1,c_2} is a Markov process. If we are above level c_2, we immediately put the process into level c_1. Assume that $x \leq c_2$.

The first time when we pay dividends is $\tau_1^{c_1,c_2}$, which means that we must wait until the first time when process U^{c_1,c_2} reach c_2. Using a strong Markov property, we have

$$\upsilon_{c_1,c_2}^{\kappa^r}(x) = \mathbb{E}_x\left[e^{-q\kappa_{c_2}^+}\mathbf{1}_{\{\kappa_{c_2}^+<\kappa^r\}}\right]\upsilon_{c_1,c_2}^{\kappa^r}(c_2) = \frac{V^{(q)}(x)}{V^{(q)}(c_2)}\upsilon_{c_1,c_2}^{\kappa^r}(c_2), \tag{2.11}$$

where the last equality follows from (2.4). If we are at point c_2, we pay $c_2 - c_1 - \beta$ and decrease U^{c_1,c_2} by $c_2 - c_1$. Again, by the strong Markov property, we have

$$\upsilon_{c_1,c_2}^{\kappa^r}(c_2) = c_2 - c_1 - \beta + \upsilon_{c_1,c_2}^{\kappa^r}(c_1) = c_2 - c_1 - \beta + \frac{V^{(q)}(c_1)}{V^{(q)}(c_2)}\upsilon_{c_1,c_2}^{\kappa^r}(c_2).$$

The next step is to solve the above equation with respect to $\upsilon_{c_1,c_2}^{\kappa^r}$. We obtain

$$\upsilon_{c_1,c_2}^{\kappa^r}(c_2) = \frac{V^{(q)}(c_2)}{V^{(q)}(c_2) - V^{(q)}(c_1)}(c_2 - c_1 - \beta).$$

Finally, to get the result, we must put the above formula into (2.11). □

Looking at (2.10) the idea of finding the optimal points (c_1, c_2) leads to finding the minimum of the function below

$$g(c_1, c_2) = \frac{V^{(q)}(c_2) - V^{(q)}(c_1)}{c_2 - c_1 - \beta}. \tag{2.12}$$

Let us denote the domain of g as $dom(g) = \{(c_1, c_2) : c_1 \geq 0, c_2 > c_1 + \beta\}$. Let C^* be a set of (c_1, c_2) from $dom(g)$ that minimizes function g, namely

$$C^* := \{(c_1^*, c_2^*) \in dom(g) : \inf_{(c_1,c_2)\in dom(g)} g(c_1, c_2) = g(c_1^*, c_2^*)\}.$$

Also, fix set $\mathcal{B} := \{(c_1, c_2) : (c_1, c_2) \in dom(g), c_1 \neq 0\}$.

Proposition 2.4.2. *For $W^{(q)} \in C^1((0,\infty))$, the set C^* is not empty and for each $(c_1^*, c_2^*) \in C^*$, we have*

$$V^{(q)\prime}(c_2^*) = \frac{V^{(q)}(c_2^*) - V^{(q)}(c_1^*)}{c_2^* - c_1^* - \beta}. \tag{2.13}$$

Also, we know that in this case there are the following possibilities:
(i) $V^{(q)\prime}(c_1^*) = V^{(q)\prime}(c_2^*)$ or (ii) $c_1^* = 0$.

Proof. At the beginning, we will show that if $c_1 \to \infty$ function g is not attaining its minimum.

$$\begin{aligned}
g(c_1, c_2) &= \int_0^\infty \Big(\frac{w^{(q)}(c_2; -z) - w^{(q)}(c_1; -z)}{c_2 - c_1 - \beta}\Big)\frac{z}{r}\mathbb{P}(X_r \in dz) \\
&\geq \int_0^\infty \left(\frac{W^{(q)}(c_2 + z) - W^{(q)}(c_1 + z)}{c_2 - c_1}\right)\left(\frac{c_2 - c_1}{c_2 - c_1 - \beta}\right)\frac{z}{r}\mathbb{P}(X_r \in dz) \\
&> \int_0^\infty \min_{x\in[c_1+z,c_2+z]} W^{(q)\prime}(x)\frac{z}{r}\mathbb{P}(X_r \in dz) \geq \int_0^\infty \min_{x\in[c_1,\infty)} W^{(q)\prime}(x)\frac{z}{r}\mathbb{P}(X_r \in dz) \\
&= \min_{x\in[c_1,\infty)} W^{(q)\prime}(x)\int_0^\infty \frac{z}{r}\mathbb{P}(X_r \in dz) \overset{c_1\to\infty}{\longrightarrow} \infty.
\end{aligned}$$

In the first inequality, we used

$$w^{(q)}(c_2;-z)-w^{(q)}(c_1;-z)\geq W^{(q)}(c_2+z)-W^{(q)}(c_1+z). \tag{2.14}$$

The next inequality follows from the mean value theorem ($W^{(q)}\in C^1((0,\infty))$) and the simple fact that $\frac{c_2-c_1}{c_2-c_1-\beta}>1$. The last inequality is a consequence of $[c_1+z,c_2+z]\subseteq[c_1,\infty)$ for all $(c_1,c_2)\in dom(g)$ and all $z>0$. Note that $\int_0^\infty \frac{z}{r}\mathbb{P}(X_r\in dz)>0$ and for that reason, the last statement follows. We get that $\inf_{(c_1,c_2)\in dom(g)} g(c_1,c_2)$ is not attained when $c_1\to\infty$; thus,

$$\inf_{(c_1,c_2)\in dom(g)} g(c_1,c_2)=\inf_{(c_1,c_2)\in dom(g)\wedge c_1\leq C_1} g(c_1,c_2),$$

for some $C_1>0$. In the next step, we will show the same for c_2. Namely

$$\begin{aligned}\inf_{c_1\in[0,C_1]} g(c_1,c_2)\geq&\inf_{c_1\in[0,C_1]}\int_0^\infty\left(\frac{W^{(q)}(c_2+z)-W^{(q)}(c_1+z)}{c_2-c_1-\beta}\right)\frac{z}{r}\mathbb{P}(X_r\in dz)\\ \geq&\Big(\frac{W^{(q)}(c_2)}{c_2-\beta}\int_0^\infty\frac{z}{r}\mathbb{P}(X_r\in dz)\\ &-\frac{1}{c_2-C_1-\beta}\int_0^\infty W^{(q)}(C_1+z)\frac{z}{r}\mathbb{P}(X_r\in dz)\Big)\overset{c_2\to\infty}{\longrightarrow}\infty.\end{aligned}$$

Note that we used only (2.14) (in the first inequality) and the property that $W^{(q)}$ is increasing (in the second inequality). The last step is to consider the case when (c_1,c_2) converges to the line $c_2=c_1+\beta$.

$$\begin{aligned}g(c_1,c_2)=&\int_0^\infty\left(\frac{w^{(q)}(c_2;-z)-w^{(q)}(c_1;-z)}{c_2-c_1-\beta}\right)\frac{z}{r}\mathbb{P}(X_r\in dz)\\ \geq&\int_0^\infty\min_{x\in[c_1+z,c_2+z]}W^{(q)\prime}(x)\left(\frac{\beta}{c_2-c_1-\beta}\right)\frac{z}{r}\mathbb{P}(X_r\in dz)\\ \geq&W^{(q)\prime}(a^*)\frac{\beta}{c_2-c_1-\beta}\int_0^\infty\frac{z}{r}\mathbb{P}(X_r\in dz)\to\infty.\end{aligned}$$

We used, again, the mean value theorem and fact that $c_2>c_1+\beta$. We checked that infimum of g is not reached when $c_1\to\infty$ or $c_2\to\infty$ or (c_1,c_2) converges to $c_2=c_1+\beta$. Because of it and the continuity of g, we get that C^* is not empty, and we are left with the following possibilities.

(a) First is that (c_1^*,c_2^*) belongs to the interior of $\mathcal{B}$. In this case, using the fact that g is partially differentiable in c_1 and c_2 ($W^{(q)}\in C^1((0,\infty))$), we get that $\frac{\partial g(c_1,c_2)}{\partial c_1}(c_1^*)=0$ and $\frac{\partial g(c_1,c_2)}{\partial c_2}(c_2^*)=0$. Hence, we obtain (2.13) and (i).

(b) The second possibility is when $c_1^*=0$. Then, we have that c_2^* minimizes function $g_0(c_2)=g(0,c_2)=\frac{V^{(q)}(c_2)-V^{(q)}(0)}{c_2-\beta}$. We get (ii) because $g_0'(c_2^*)=0$.

□

To start the optimisation reasoning, we need the following proposition and lemma.

Proposition 2.4.3. *Assume that $W^{(q)} \in C^1((0,\infty))$. For each $(c_1^*, c_2^*) \in C^*$, we have that*

$$\upsilon^{\kappa^r}_{c_1^*,c_2^*}(x) = \begin{cases} \frac{V^{(q)}(x)}{V^{(q)\prime}(c_2^*)}, & \text{for } x \le c_2^*, \\ (x - c_2^*) + \frac{V^{(q)}(c_2^*)}{V^{(q)\prime}(c_2^*)}, & \text{for } x > c_2^*. \end{cases}$$

Proof. From Proposition 2.4.2, it follows that:

(i) For $x \le c_2^*$,

$$\upsilon^{\kappa^r}_{c_1^*,c_2^*}(x) = (c_2^* - c_1^* - \beta)\frac{V^{(q)}(x)}{V^{(q)}(c_2^*) - V^{(q)}(c_1^*)} = \frac{V^{(q)}(x)}{V^{(q)\prime}(c_2^*)}.$$

(ii) For $x > c_2^*$,

$$\begin{aligned}\upsilon^{\kappa^r}_{c_1^*,c_2^*}(x) &= x - c_1^* - \beta + (c_2^* - c_1^* - \beta)\frac{V^{(q)}(c_1^*)}{V^{(q)}(c_2^*) - V^{(q)}(c_1^*)} \\ &= x - c_2^* + (c_2^* - c_1^* - \beta)\frac{V^{(q)}(c_2^*)}{V^{(q)}(c_2^*) - V^{(q)}(c_1^*)} \\ &= x - c_2^* + \frac{V^{(q)}(c_2^*)}{V^{(q)\prime}(c_2^*)}.\end{aligned}$$

□

For the function from Proposition 2.4.3 we will also use the notation $\upsilon^{\kappa^r}_{c_2^*}$, to underline that this function does not depends on c_1^*. It is also straightforward to see that this function's formula looks like the one for the barrier strategy in case of the spectrally negative Lévy processes (see Theorem 1.5.1, $V^{(q)}$ needs to be replaced by $W^{(q)}$).

Lemma 2.4.4. *Let $(c_1^*, c_2^*) \in C^*$ and $x \ge y \ge 0$. Then*

$$\upsilon^{\kappa^r}_{c_1^*,c_2^*}(x) - \upsilon^{\kappa^r}_{c_1^*,c_2^*}(y) \ge x - y - \beta$$

Proof. Note that $\upsilon^{\kappa^r}_{c_1^*,c_2^*}$ is an increasing function, and because of that, one can assume $x - y > \beta$. Consider the following possibilities.

(i) For $c_2^* \le y \le x$, one can obtain that

$$\upsilon^{\kappa^r}_{c_1^*,c_2^*}(x) - \upsilon^{\kappa^r}_{c_1^*,c_2^*}(y) = x - y > x - y - \beta.$$

(ii) For $y \le x \le c_2^*$,

$$\begin{aligned}\upsilon^{\kappa^r}_{c_1^*,c_2^*}(x) - \upsilon^{\kappa^r}_{c_1^*,c_2^*}(y) =& \frac{(c_2^* - c_1^* - \beta)(V^{(q)}(x) - V^{(q)}(y))}{V^{(q)}(c_2^*) - V^{(q)}(c_1^*)} \\ \ge& \frac{(x - y - \beta)(V^{(q)}(x) - V^{(q)}(y))}{V^{(q)}(x) - V^{(q)}(y)} = x - y - \beta.\end{aligned}$$

The above inequality follows from fact that $(c_1^*, c_2^*) \in C^*$, so (c_1^*, c_2^*) minimizes function

$$g(c_1, c_2) = \frac{V^{(q)}(c_2) - V^{(q)}(c_1)}{c_2 - c_1 - \beta}.$$

(iii) For $y \leq c_2^* \leq x$,

$$\begin{aligned} v^{\kappa^r}_{c_1^*,c_2^*}(x) - v^{\kappa^r}_{c_1^*,c_2^*}(y) &= x - c_1^* - \beta + (c_2^* - c_1^* - \beta)\left(\frac{V^{(q)}(c_1^*) - V^{(q)}(y)}{V^{(q)}(c_2^*) - V^{(q)}(c_1^*)}\right) \\ &= x - c_2^* + (c_2^* - c_1^* - \beta)\left(1 + \frac{V^{(q)}(c_1^*) - V^{(q)}(y)}{V^{(q)}(c_2^*) - V^{(q)}(c_1^*)}\right) \\ &= x - c_2^* + (c_2^* - c_1^* - \beta)\left(\frac{V^{(q)}(c_2^*) - V^{(q)}(y)}{V^{(q)}(c_2^*) - V^{(q)}(c_1^*)}\right) \\ &\geq x - y - \beta. \end{aligned}$$

The last inequality follows from point (ii) with $x = c_2^*$.

□

2.4.4 Optimality

We will start this section by analysing the refracted process R. This process is not spatially homogeneous and thus is not a Lévy process. However, roughly speaking, we can see that this process is similar to the Lévy process. Especially we observed that above zero, it behaves like process Y, and below zero, it behaves like process X. We will show that this process belongs to the family of Feller processes. We recall Section 1.3 for a brief introduction to the Markov processes and the Feller semigroup.

Fact 2.4.5. *Refracted process R is a Feller process, and its infinitesimal generator is of the form*

$$\begin{aligned} \Gamma f(x) =& (\gamma - \delta \mathbf{1}_{\{x>0\}})f'(x) + \frac{1}{2}\sigma^2 f''(x) \\ &+ \int_{0+}^{\infty} \left(f(x-z) - f(x) + f'(x)z\mathbf{1}_{\{0<z<1\}}\right)\Pi(dz), \end{aligned} \tag{2.15}$$

where $x \in \mathbb{R}$ and f is a function on $\mathbb{R}$ such that $\Gamma f(x)$ is well defined.

Proof. Recall that $C_0(\mathbb{R})$ denotes the space of continuous functions vanishing at infinity. For $q > 0$, $x \in \mathbb{R}$ and a function $f \in C_0(\mathbb{R})$ define $P_R^{(q)} f := \mathbb{E}_x\left[\int_0^\infty e^{-qt} f(R_t)dt\right]$. Recall from Theorem 1.3.13 that it is sufficient to verify the following conditions:

1. for all $q, p > 0$, $P_R^{(q)} - P_R^{(p)} = (p-q)P_R^{(q)}P_R^{(p)}$,
2. for all $q > 0$, $\left\|qP_R^{(q)}1\right\| \leq 1$,
3. for all $q > 0$, $P_R^{(q)}$ is a map from $C_0(\mathbb{R})$ to $C_0(\mathbb{R})$,
4. for all $f \in C_0(\mathbb{R})$, $\lim_{q\to\infty}\left\|qP_R^{(q)}f - f\right\| = 0$.

Let us note that $C_0(\mathbb{R})$ is a Banach space when equipped with the uniform norm $\|f\| = \sup_{x\in\mathbb{R}}|f(x)|$. Because the process R is a strong Markov process (for details, see Kyprianou and Loeffen [37]), one can observe that condition (1) is automatically fulfilled. Condition (2) is obvious. One can even prove a stronger result, namely that for every $q > 0$ and $f \in C_0(\mathbb{R})$ we have that $\left\| qP_R^{(q)} f \right\| \leq \|f\|$. To prove (3) and (4), the reasoning is similar as in Noba and Yano [49], and Noba [48] except that we need to use fluctuation identities obtained in Kyprianou and Loeffen [37]. However, for the consistency of the proof, let us present this reasoning.

At first, we will prove (3). Fix $f \in C_0(\mathbb{R})$ and $q > 0$. Moreover, let $\epsilon > 0$ and $x \in \mathbb{R}$. Let us start with the right-continuity. As R has no positive jumps, we have

$$\begin{aligned}\left|P_R^{(q)} f(x+\epsilon) - P_R^{(q)} f(x)\right| \leq& \left|P_R^{(q)} f(x+\epsilon) - \mathbb{E}_x\left(e^{-q\kappa_{x+\epsilon}^+}\mathbf{1}_{\{\kappa_{x+\epsilon}^+<\infty\}}\right) P_R^{(q)} f(x+\epsilon)\right| \\ &+ \left|\mathbb{E}_x\left(\int_0^{\kappa_{x+\epsilon}^+} e^{-qt} f(R_t)dt\right)\right| \\ \leq& \left|P_R^{(q)} f(x+\epsilon)\right|\left(1-\mathbb{E}_x\left(e^{-q\kappa_{x+\epsilon}^+}\mathbf{1}_{\{\kappa_{x+\epsilon}^+<\infty\}}\right)\right) \\ &+ \|f\|\,\mathbb{E}_x\left(\int_0^{\kappa_{x+\epsilon}^+} e^{-qt}dt\right) \\ \leq& \frac{2}{q}\|f\|\left(1-\mathbb{E}_x\left(e^{-q\kappa_{x+\epsilon}^+}\mathbf{1}_{\{\kappa_{x+\epsilon}^+<\infty\}}\right)\right) \stackrel{\epsilon\to 0}{\longrightarrow} 0,\end{aligned}$$

as from (2.3) we have

$$\mathbb{E}_x\left(e^{-q\kappa_{x+\epsilon}^+}\mathbf{1}_{\{\kappa_{x+\epsilon}^+<\infty\}}\right) \stackrel{\epsilon\to 0}{\longrightarrow} 1.$$

Thus, we proved right-continuity. Next, we will prove left-continuity.

$$\begin{aligned}\left|P_R^{(q)} f(x-\epsilon) - P_R^{(q)} f(x)\right| \leq& \left|\mathbb{E}_{x-\epsilon}\left(e^{-q\kappa_x^+}\mathbf{1}_{\{\kappa_x^+<\infty\}}\right) P_R^{(q)} f(x) - P_R^{(q)} f(x)\right| \\ &+ \left|\mathbb{E}_{x-\epsilon}\left(\int_0^{\kappa_x^+} e^{-qt} f(R_t)dt\right)\right|.\end{aligned}$$

The idea of the remaining proof is the same as for the right-continuity; thus, let us proceed to the vanishing at infinity. At first, let us note that for all $x \in (0,+\infty)$ we have

$$\lim_{y\to\infty}\mathbb{E}_y\left[e^{-q\kappa_x^-}\mathbf{1}\{\kappa_x^- < \infty\}\right] = \lim_{y\to\infty}\mathbb{E}_y\left[e^{-q\tau_x^-}\mathbf{1}\{\tau_x^- < \infty\}\right] = 0,$$

where $\tau_x^- = \inf\{t > 0 : X_t < x\}$. Since, as $f \in C_0(\mathbb{R})$ we have that for all $\epsilon > 0$ there exists $\delta_\epsilon \in (0,\infty)$ such that $\sup_{x\in(\delta_\epsilon,+\infty)} |f(x)| < \epsilon$. Then,

$$\begin{aligned}\lim_{x\to\infty}\left|P_R^{(q)} f(x)\right| \leq& \lim_{x\to\infty}\left(\mathbb{E}_x\left[\int_0^{\tau_{\delta_\epsilon}^-} e^{-qt}\,|f(X_t)|\,dt\right] + \mathbb{E}_x\left[\int_{\kappa_{\delta_\epsilon}^-}^{\infty} e^{-qt}\,\|f\|\,dt\right]\right) \\ \leq& \frac{\epsilon}{q} + \lim_{x\to\infty}\mathbb{E}_x\left[e^{-q\kappa_{\delta_\epsilon}^-}\mathbf{1}_{\{\kappa_{\delta_\epsilon}^-<\infty\}}\right] = \frac{\epsilon}{q}.\end{aligned}$$

Thus, we got that $\lim_{x\to\infty}\left|P_R^{(q)} f(x)\right| = 0$. Similar argument led to $\lim_{x\to-\infty}\left|P_R^{(q)} f(x)\right| = 0$. We now proceed to (4). Let us fix $f \in C_0(\mathbb{R})$. At first, we will prove point-wise convergence. Fix $x \in \mathbb{R}$. For all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$|x-y| < \delta_\epsilon \Rightarrow |f(x) - f(y)| < \epsilon, \quad y \in \mathbb{R}.$$

We define

$$T_{\delta_\epsilon} = \inf\{t > 0 : |R_t - x| \geq \delta_\epsilon\}.$$

Then

$$\begin{aligned}\left|qP_R^{(q)} f(x) - f(x)\right| &\leq q\mathbb{E}_x\left[\int_0^{T_{\delta_\epsilon}} e^{-qt}|f(R_t) - f(x)|dt\right] + q\mathbb{E}_x\left[\int_{T_{\delta_\epsilon}}^{\infty} e^{-qt}|f(R_t) - f(x)|dt\right] \\ &\leq \epsilon\left(1 - \mathbb{E}_x\left[e^{-qT_{\delta_\epsilon}}\mathbf{1}_{\{T_{\delta_\epsilon}<\infty\}}\right]\right) + 2\|f\|\,\mathbb{E}_x\left[e^{-qT_{\delta_\epsilon}}\mathbf{1}_{\{T_{\delta_\epsilon}<\infty\}}\right].\end{aligned}$$

Thus, by the dominated convergence theorem, we have

$$\limsup_{q\to\infty}\left|qP_R^{(q)} f(x) - f(x)\right| \leq \epsilon,$$

and so we have $\lim_{q\to\infty}\left|qP_R^{(q)} f(x) - f(x)\right| = 0$. Thus, we established the point-wise convergence. Now, one can prove that for all $q > 0$ set $P_R^{(q)}C_0(\mathbb{R})$ is dense in $C_0(\mathbb{R})$. At first, observe that the range of $P_R^{(q)}C_0(\mathbb{R})$ does not depend on q. To prove that fix $q_1, q_2 > 0$. Then, fix $f \in P_R^{(q_1)}$. We know that there exists $g \in C_0(\mathbb{R})$ such that $f = P_R^{(q_1)}g$. From (1), we have that

$$f = P_R^{(q_1)}g = P_R^{(q_2)}g + (q_2 - q_1)P_R^{(q_1)}P_R^{(q_2)}g = P_R^{(q_2)}\left(g + (q_2 - q_1)P_R^{(q_1)}g\right) \in P_R^{(q_2)}C_0(\mathbb{R}).$$

Thus $P_R^{(q_1)}C_0(\mathbb{R}) \subset P_R^{(q_2)}C_0(\mathbb{R})$. After reversing q_1 and q_2 one can also obtain that $P_R^{(q_2)}C_0(\mathbb{R}) \subset P_R^{(q_1)}C_0(\mathbb{R})$. Thus let us set $\mathcal{P}_R = P_R^{(q)}C_0(\mathbb{R})$ for some $q > 0$. We have already checked that $\mathcal{P}_R$ is dense in $C_0(\mathbb{R})$ with respect to the point-wise convergence of uniformly bounded sequences. Now we will use standard reasoning (see, e.g. Chapter 17 from Kallenberg [31]). Suppose that $\mathcal{P}_R$ is not dense with respect to uniform convergence. Then, by the Hahn-Banach theorem, there exists a bounded linear map $\varphi : C_0(\mathbb{R}) \to \mathbb{R}$ that vanishes on $\mathcal{P}_R$ but not on all $C_0(\mathbb{R})$. Let $f_0 \in C_0(\mathbb{R})$ such that $\varphi(f) \neq 0$. By the Riesz-Markov theorem, there exists a finite signed measure μ such that $\varphi(f) = \int_{\mathbb{R}} f d\mu$ for all $f \in C_0(\mathbb{R})$. Let $(g_n)_{n\in\mathbb{R}}$ be a uniformly bounded sequence in $\mathcal{P}_R$ with $g_n \to f_0$ point-wise. Then one can observe that

$$0 = \varphi(g_n) = \int_{\mathbb{R}} g_n d\mu \to \int_{\mathbb{R}} f d\mu,$$

by the bounded convergence theorem for finite signed measures. Thus, $\varphi(f) = 0$ by contradiction. Thus, $\mathcal{P}_R$ is dense in $C_0(\mathbb{R})$ with respect to uniform convergence.
Let $f = P_R^{(1)}g$ for some $g \in C_0(\mathbb{R})$. Using condition (1), one can get that

$$\left\|f - qP_R^{(q)}f\right\| = \left\|P_R^{(q)}g - (1-q)P_R^{(q)}f - qP_R^{(q)}f\right\| = \left\|P_R^{(q)}g - P_R^{(q)}f\right\| \leq \frac{1}{q}\|g - f\| \to 0.$$

Since $\mathcal{P}_R$ is dense, we got a conclusion. The form (2.15) of the generator follows from Theorem 1.3.15 and the construction of the refracted process with $l(x) = \gamma - \delta\mathbf{1}_{\{x>0\}}$, $q(x) = \sigma^2$ and $\Pi(x,\cdot) = \Pi(\cdot)$. □

For the remainder of the section, we will focus on verifying the optimality of the impulse strategy at the threshold level (c_1^*, c_2^*). Standard Markovian arguments lead the proof to show that the impulse strategy fulfils the following Verification Lemma.

Lemma 2.4.6 (Verification Lemma). *Suppose $\hat{\pi}$ is an admissible dividend strategy such that $v_{\hat{\pi}}$ is sufficiently smooth on $\mathbb{R}$ (i.e. its first or second derivative (for X of bounded or unbounded variations, respectively) has at most a finite number of single discontinuities), satisfies*

$$(\Gamma - q)v_{\hat{\pi}}(x) \leq 0, \quad \text{for almost all } x \in \mathbb{R} \tag{2.16}$$

$$v_{\hat{\pi}}(x) - v_{\hat{\pi}}(y) \geq x - y - \beta, \quad \text{for} \quad x \geq y. \tag{2.17}$$

Then, $v_{\hat{\pi}}(x) = v_*(x)$ for $x \in \mathbb{R}$ and hence $\hat{\pi} = \pi_*$ is an optimal strategy.

Proof. By the definition of v_* as a supremum, it follows that $v_{\hat{\pi}}(x) \leq v_*(x)$ for all $x \in \mathbb{R}$. We write $h := v_{\hat{\pi}}$ and show that $h(x) \geq v_\pi(x)$ for all $\pi \in \mathcal{A}$ for all $x \in \mathbb{R}$.

Fix $\pi \in \mathcal{A}$ and let $\tilde{U}^\pi$ and $\tilde{L}^\pi$ be a right-continuous modification of U^π and L^π. Let $(T_n)_{n\in\mathbb{N}}$ be the sequence of stopping times defined by

$$T_n := \inf\{t > 0 : |\tilde{U}^\pi_t| > n\} \wedge \kappa^r.$$

Note that we can still use κ^r, as the Parisian ruin time for U^π and $\tilde{U}^\pi$ is the same due to the fact that both processes can up-cross 0 only by the continuous passing. Because $\tilde{U}^\pi$ is a semimartingale and h is sufficiently smooth on $\mathbb{R}$, we will use to the stopped process $(h(\tilde{U}^\pi_{t\wedge T_n}); t \geq 0)$ the Bouleau and Yor [8] formula for bounded variation processes (see also Theorem 1.6.20 quoted from Protter [51]) and the Extant Second Derivative Meyer- Itô's formula (see Theorem 1.6.17 quoted from Protter [51]) for unbounded variation case, and deduce that under $\mathbb{P}_x$

$$h(\tilde{U}^\pi_{t\wedge T_n}) - h(\tilde{U}^\pi_0) = \int_{0+}^{t\wedge T_n} h'(\tilde{U}^\pi_{s-})d\tilde{U}^\pi_s + \frac{1}{2}\int_0^t h''(\tilde{U}^\pi_{s-})d[\tilde{U}^\pi, \tilde{U}^\pi]^c_s + \sum_{s\leq t\wedge T_n}\left[\Delta h(\tilde{U}^\pi_s) - h'(\tilde{U}^\pi_{s-})\Delta\tilde{U}^\pi_s\right], \tag{2.18}$$

where we use the following notation: $\Delta\zeta(s) := \zeta(s) - \zeta(s-)$ and $\Delta h(\zeta(s)) := h(\zeta(s)) - h(\zeta(s-))$ for any process ζ with left-hand limits. If X is of bounded variation, the last integral should be treated as missing. Instead, there should be integral

$$-\frac{1}{2}\int_{-\infty}^{\infty} f(a)d_aL^a_{t\wedge T_n},$$

where L^a_t is a local time of $\tilde{U}^\pi$. The above integral is zero due to the fact that $[\tilde{U}^\pi, \tilde{U}^\pi]^c_s = 0$. From the definition of U^π one can get that

$$\frac{1}{2}\int_0^{t\wedge T_n} h''(\tilde{U}^\pi_{s-})d[\tilde{U}^\pi, \tilde{U}^\pi]^c_s = \frac{1}{2}\int_0^{t\wedge T_n} h''(\tilde{U}^\pi_{s-})d[X, X]^c_s = \frac{\sigma^2}{2}\int_0^{t\wedge T_n} h''(\tilde{U}^\pi_{s-})ds.$$

Next, using above and stochastic integration by parts (for the stopped process $(e^{-q(t\wedge T_n)} : t \geq 0)$), equation (2.18) can be written ($\mathbb{P}_x$-a.s.) as

$$\begin{aligned} e^{-q(t\wedge T_n)}h(\tilde{U}^\pi_{t\wedge T_n}) - h(\tilde{U}^\pi_0) = &-q\int_{0+}^{t\wedge T_n} e^{-qs}h(\tilde{U}^\pi_{s-})\mathrm{d}s + \frac{\sigma^2}{2}\int_0^{t\wedge T_n} e^{-qs}h''(\tilde{U}^\pi_{s-})ds \\ &+ \int_{0+}^{t\wedge T_n} e^{-qs}h'(\tilde{U}^\pi_{s-})d\tilde{U}^\pi_s + \sum_{s\leq t\wedge T_n} e^{-qs}\left[\Delta h(\tilde{U}^\pi_s) - h'(\tilde{U}^\pi_{s-})\Delta\tilde{U}^\pi_s\right]. \end{aligned} \tag{2.19}$$

From the definition of $\tilde{U}^\pi$ and the fact that $\tilde{L}^\pi$ is a pure-jump process, we have that

$$\int_{0+}^{t\wedge T_n} e^{-qs}h'(\tilde{U}^\pi_{s-})d\tilde{U}^\pi_s = \int_{0+}^{t\wedge T_n} e^{-qs}h'(\tilde{U}^\pi_{s-})dX_s - \delta\int_{0+}^{t\wedge T_n} e^{-qs}h'(\tilde{U}^\pi_{s-})\mathbf{1}_{\{\tilde{U}^\pi_{s-}>0\}}ds - \sum_{s\le t\wedge T_n} e^{-qs}h'(\tilde{U}^\pi_{s-})\Delta\tilde{L}^\pi_s.$$

Thus, equation (2.19) can be simplify to

$$\begin{aligned} e^{-q(t\wedge T_n)}h(\tilde{U}^\pi_{t\wedge T_n}) - h(\tilde{U}^\pi_0) =& -q\int_{0+}^{t\wedge T_n} e^{-qs}h(U^\pi_{s-})\mathrm{d}s + \frac{\sigma^2}{2}\int_0^{t\wedge T_n} h''(\tilde{U}^\pi_{s-})ds \\ &+ \int_{0+}^{t\wedge T_n} e^{-qs}h'(\tilde{U}^\pi_{s-})dX_s - \delta\int_{0+}^{t\wedge T_n} e^{-qs}h'(\tilde{U}^\pi_{s-})\mathbf{1}_{\{\tilde{U}^\pi_{s-}>0\}}ds \\ &+ \sum_{s\le t\wedge T_n} e^{-qs}\left[\Delta h(\tilde{U}^\pi_s) - h'(\tilde{U}^\pi_{s-})\Delta X_s\right], \end{aligned} \tag{2.20}$$

as $\Delta\tilde{U}^\pi_s = \Delta X_s - \Delta\tilde{L}^\pi_s$. Now, observe that

$$\begin{aligned} \sum_{s\le t\wedge T_n} e^{-qs}\left[\Delta h(\tilde{U}^\pi_s) - h'(\tilde{U}^\pi_{s-})\Delta X_s\right] =& \sum_{s\le t\wedge T_n} e^{-qs}\left[\Delta h(\tilde{U}^\pi_{s-}+\Delta X_s) - h'(\tilde{U}^\pi_{s-})\Delta X_s\right] \\ &- \sum_{s\le t\wedge T_n} e^{-qs}[h(X_s - \tilde{L}^\pi_{s-} - \int_0^s \mathbf{1}_{\{\tilde{U}^\pi_t>0\}}dt) - h(X_s - \tilde{L}^\pi_{s-} - \Delta\tilde{L}^\pi_s - \int_0^s \mathbf{1}_{\{\tilde{U}^\pi_t>0\}}dt)], \end{aligned}$$

since

$$\begin{aligned} &\Delta h(\tilde{U}^\pi_s) = h(\tilde{U}^\pi_s) - h(\tilde{U}^\pi_{s-}), \\ &\Delta h(\tilde{U}^\pi_{s-}+\Delta X_s) = h(\tilde{U}^\pi_{s-} + X_s - X_{s-}) - h(\tilde{U}^\pi_{s-}), \\ &h(\tilde{U}^\pi_{s-} + X_s - X_{s-}) = h(X_s - \tilde{L}^\pi_{s-} - \int_0^s \mathbf{1}_{\{\tilde{U}^\pi_t>0\}}dt), \\ &h(\tilde{U}^\pi_s) = h(X_s - \tilde{L}^\pi_{s-} - \Delta\tilde{L}^\pi_s - \int_0^s \mathbf{1}_{\{\tilde{U}^\pi_t>0\}}dt). \end{aligned}$$

Using (2.17) we get that

$$h(X_s - \tilde{L}^\pi_{s-} - \int_0^s \mathbf{1}_{\{\tilde{U}^\pi_t>0\}}dt) - h(X_s - \tilde{L}^\pi_{s-} - \Delta\tilde{L}^\pi_s - \int_0^s \mathbf{1}_{\{\tilde{U}^\pi_t>0\}}dt) \ge \Delta\tilde{L}^\pi_s - \beta.$$

Thus, using (2.20) we have the following inequality

$$\begin{aligned} e^{-q(t\wedge T_n)}h(\tilde{U}^\pi_{t\wedge T_n}) - h(\tilde{U}^\pi_0) \le& -q\int_{0+}^{t\wedge T_n} e^{-qs}h(\tilde{U}^\pi_{s-})\mathrm{d}s + \frac{\sigma^2}{2}\int_0^{t\wedge T_n} h''(\tilde{U}^\pi_{s-})ds \\ &+ \int_{0+}^{t\wedge T_n} e^{-qs}h'(\tilde{U}^\pi_{s-})dX_s - \delta\int_{0+}^{t\wedge T_n} e^{-qs}h'(\tilde{U}^\pi_{s-})\mathbf{1}_{\{\tilde{U}^\pi_{s-}>0\}}ds \\ &+ \sum_{s\le t\wedge T_n} e^{-qs}\left[\Delta h(\tilde{U}^\pi_{s-}+\Delta X_s) - h'(\tilde{U}^\pi_{s-})\Delta X_s\right] \\ &- \sum_{s\le t\wedge T_n} e^{-qs}\left[\Delta\tilde{L}^\pi_s - \beta\right]. \end{aligned} \tag{2.21}$$

Rewriting the above leads to

$$\begin{aligned}
e^{-q(t\wedge T_n)}h(\tilde{U}^\pi_{t\wedge T_n}) - h(\tilde{U}^\pi_0) \le & \int_{0+}^{t\wedge T_n} e^{-qs}(\Gamma - q)h(\tilde{U}^\pi_{s-})ds - \sum_{s\le t\wedge T_n} e^{-qs}\left[\Delta \tilde{L}^\pi_s - \beta\right] \\
& + \Big\{\int_{0+}^{t\wedge T_n} e^{-qs}h^{'}(\tilde{U}^\pi_{s-})d(X_t - \gamma s - \sum_{u\le s}\Delta X_u \mathbf{1}_{\{|\Delta X_u|\ge 1\}})\Big\} \\
& + \Big\{\sum_{s\le t\wedge T_n} e^{-qs}(\Delta h(\tilde{U}^\pi_{s-} + \Delta X_s) - h^{'}(\tilde{U}^\pi_{s-})\Delta X_s \mathbf{1}_{\{\Delta X_s<1\}}) \\
& - \int_{0+}^{t\wedge T_n}\int_{0+}^{+\infty} e^{-qs}(h(\tilde{U}^\pi_{s-} - y) - h(\tilde{U}^\pi_{s-}) + h^{'}(\tilde{U}^\pi_{s-})y\mathbf{1}_{\{0<y<1\}})\Pi(dy)ds\Big\}.
\end{aligned}$$

Let us note that the first bracket contains the martingale part of X. The second part is a zero-mean martingale from the compensation formula for the Poisson random measure (see Theorem 4.4 in Kyprianou [36]). Thus, we can simplify the above to

$$e^{-q(t\wedge T_n)}h(\tilde{U}^\pi_{t\wedge T_n}) - h(\tilde{U}^\pi_0) \le \int_{0+}^{t\wedge T_n} e^{-qs}(\Gamma - q)h(\tilde{U}^\pi_{s-})ds - \sum_{s\le t\wedge T_n} e^{-qs}\left[\Delta \tilde{L}^\pi_s - \beta\right] + M_{t\wedge T_n}, \quad (2.22)$$

where $M = \{M_t : t \ge 0\}$ is a zero-mean (local) $\mathbb{P}_x$-martingale. Hence, using the assumption (2.16) we obtain the following

$$h(\tilde{U}^\pi_0) \ge \int_{0+}^{t\wedge T_n} e^{-qs}\mathrm{d}\left(\tilde{L}^\pi_s - \beta\sum_{z\le s}\mathbf{1}_{\{\Delta \tilde{L}^\pi_z>0\}}\right) - M_{t\wedge T_n} + e^{-q(t\wedge T_n)}h(\tilde{U}^\pi_{t\wedge T_n}). \quad (2.23)$$

To see it, it is sufficient to show that for any $t > 0$

$$\int_{0+}^{t} e^{-qs}(\Gamma - q)h(\tilde{U}^\pi_{s-})ds \le 0 \quad (2.24)$$

almost surely. We need to consider two cases separately, namely, when $\sigma = 0$ and $\sigma > 0$. If $\sigma = 0$, then process X is a quadratic pure jump semimartingale (see e.g. a paragraph after Definition 1.6.8, which was quoted from Protter [51]) and thus (2.24) automatically holds. We left with $\sigma > 0$. One can prove it using the occupation formula for the semimartingale local time. We can follow the same argument as in Lemma 6 from Loeffen [43]. Namely, by the assumption we have that $A = \{x \in \mathbb{R} : (\Gamma - q)h(x) > 0\}$ is of Lebesgue measure 0. Set also $B = \{s \in (0, t] : \tilde{U}^\pi_{s-} \in A\}$. Then a.s.

$$\int_0^t \mathbf{1}_{\{s\in B\}}\sigma^2 ds = \int_0^t \mathbf{1}_{\{s\in B\}} d[\tilde{U}^\pi, \tilde{U}^\pi]^c_s = \int_{-\infty}^{\infty} L^a_t \mathbf{1}_{\{a\in A\}} da,$$

where L^a_t is a local time of $\tilde{U}^\pi$. Given that $Leb(A) = 0$ hence $Leb(B) = 0$ (as $\sigma > 0$), where $Leb(.)$ is a Lebesgue measure.

Now, taking expectations in (2.23), using the fact that $(M_{t\wedge T_n} : t \ge 0)$ is a zero-mean $\mathbb{P}_x$-martingale and $h \geq 0$, letting t and n go to infinity ($T_n \xrightarrow{n\uparrow\infty} \kappa^r$ $\mathbb{P}_x$-a.s.), the dominated convergence gives

$$\begin{aligned} h(\tilde{U}_0^\pi) \geq & \lim_{t,n\uparrow\infty} \mathbb{E}_x\left[\int_{0+}^{t\wedge T_n} e^{-qs}\mathrm{d}\left(\tilde{L}_s^\pi - \beta\sum_{0<z\leq s}\mathbf{1}_{\{\Delta\tilde{L}_z^\pi>0\}}\right) - M_{t\wedge T_n} + e^{-q(t\wedge T_n)}h(\tilde{U}_{t\wedge T_n}^\pi)\right] \\ \geq & \mathbb{E}_x\left[\int_{0+}^{\kappa^r} e^{-qs}\mathrm{d}\left(\tilde{L}_s^\pi - \beta\sum_{0<z\leq s}\mathbf{1}_{\{\Delta\tilde{L}_z^\pi>0\}}\right) + \lim_{t,n\uparrow\infty} e^{-q(t\wedge T_n)}h(\tilde{U}_{t\wedge T_n}^\pi)\right] \\ \geq & \mathbb{E}_x\left[\int_{0+}^{\kappa^r} e^{-qs}\mathrm{d}\left(\tilde{L}_s^\pi - \beta\sum_{0<z\leq s}\mathbf{1}_{\{\Delta\tilde{L}_z^\pi>0\}}\right)\right]. \end{aligned}$$

If $L_{0+}^\pi = 0$ then above is equivalent to

$$h(x) \geq \mathbb{E}_x\left[\int_0^{\kappa^r} e^{-qs}\mathrm{d}\left(L_s^\pi - \beta\sum_{0\leq z\leq s}\mathbf{1}_{\{\Delta L_z^\pi>0\}}\right)\right] = v_\pi(x).$$

Thus, let us assume that $L_{0+}^\pi > 0$. Then

$$\mathbb{E}_x\left[\int_{0+}^{\kappa^r} e^{-qs}\mathrm{d}\left(\tilde{L}_s^\pi - \beta\sum_{0<z\leq s}\mathbf{1}_{\{\Delta\tilde{L}_z^\pi>0\}}\right)\right] = \mathbb{E}_x\left[\int_0^{\kappa^r} e^{-qs}\mathrm{d}\left(L_s^\pi - \beta\sum_{0\leq z\leq s}\mathbf{1}_{\{\Delta L_z^\pi>0\}}\right)\right] - (L_{0+}^\pi - \beta),$$

and

$$h(\tilde{U}_0^\pi) = h(x - \tilde{L}_0^\pi) = h(x - L_{0+}^\pi).$$

Using inequality (2.17) one can get that

$$h(x) - h(x - L_{0+}^\pi) \geq L_{0+}^\pi - \beta.$$

Thus

$$h(\tilde{U}_0^\pi) \leq h(x) - (L_{0+}^\pi - \beta),$$

and as a result we get again that

$$h(x) \geq \mathbb{E}_x\left[\int_0^{\kappa^r} e^{-qs}\mathrm{d}\left(L_s^\pi - \beta\sum_{0\leq z\leq s}\mathbf{1}_{\{\Delta L_z^\pi>0\}}\right)\right] = v_\pi(x).$$

which completes the proof. □

Remark 2.4.7. *The lemmas presented below require some smoothness on the value function of a $(c_1; c_2)$ policy. In the view of Proposition 2.4.3, it means that some smoothness conditions on the scale function $V^{(q)}$ are required. We will call the scale function $V^{(q)}$ sufficiently smooth if $W^{(q)} \in C^1((0,\infty))$ when X is of bounded variation. From Theorem 2.9 of Kyprianou et al. [39], one can see that a necessary and sufficient condition for this is that the Lévy measure has no atoms. When X is of unbounded variation, we call the scale function $V^{(q)}$ sufficiently smooth if $W^{(q)} \in C^1((0,\infty))$ and $W^{(q)'}$ is absolutely continuous on $(0,\infty)$ with a density which is bounded on sets of the form $[1/n, n]$, $n \geq 1$. Moreover, in Theorem 2.6 of Kyprianou et al. [39], it is proved that $W^{(q)} \in C^2((0,\infty))$ if the Gaussian coefficient σ is strictly positive.*

To prove the optimality of our strategy, we will need the following technical lemma.

Lemma 2.4.8. *Let $V^{(q)}$ be sufficiently smooth. Moreover, let Γ_Y be an infinitesimal generator of the process Y. Then, for $z > 0$, the following holds*

$$(\Gamma_Y - q)\left(\int_0^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y+z)dy\right)(x) = W^{(q)\prime}(x+z). \tag{2.25}$$

Proof. We will break this lemma into two parts. Namely, when X is of bounded variation or unbounded variation. We will start with the unbounded variation part. In such case we have an assumption that $W^{(q)\prime}$ is absolutely continuous on $(0,\infty)$ (note that we are interested in $W^{(q)\prime}(x+z)$ for $z > 0$ and $x > 0$, thus we are safely separated from 0). From the proof of Lemma 4.5 from Egami and Yamazaki [21] one can get that

$$(\Gamma_Y - q)\left(\int_0^x \mathbb{W}^{(q)}(x-y)l(y)dy\right)(x) = l(x),$$

where l needs to be smooth enough to perform integration by parts, and here we use an assumption that $W^{(q)\prime}$ is absolutely continuous. In more detail, they showed that

$$q\int_0^x \mathbb{W}^{(q)}(x-y)l(y)dy = -l(M) + l(0)\mathbb{Z}^{(q)}(x) + \int_0^M l'(y)\mathbb{Z}^{(q)}(x-y)dy,$$

for any $M > x$. It is known from, e.g. Kyprianou [36], that (the same applies for X and its infinitesimal operator)

$$(\Gamma_Y - q)\mathbb{W}^{(q)}(x) = 0, \quad \text{for } x \in \mathbb{R},$$

and

$$(\Gamma_Y - q)\mathbb{Z}^{(q)}(x) = 0, \quad \text{for } x \in \mathbb{R}.$$

Thus, for $l(x) = W^{(q)\prime}(x+z)$ one can get that (for $f(x) = \left(\int_0^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y+z)dy\right)(x)$)

$$\begin{aligned}(\Gamma_Y - q)f(x) &= \frac{1}{q}(\Gamma_Y - q)\left(-W^{(q)\prime}(M+z) + \int_0^M W^{(q)\prime\prime}(y+z)\mathbb{Z}^{(q)}(x-y)dy\right)\\ &= -W^{(q)\prime}(M+z) + \frac{1}{q}\int_0^M W^{(q)\prime\prime}(y+z)(\Gamma_Y - q)\mathbb{Z}^{(q)}(x-y)dy\\ &= -W^{(q)\prime}(M+z) - \int_x^M W^{(q)\prime\prime}(y+z)dy = W^{(q)\prime}(x+z)\end{aligned}$$

For the bounded variation part, we cannot use the same result. However, we can do calculations by hand. In such a case, the infinitesimal generator of Y is equal to

$$\Gamma_Y f(x) = (\gamma - \delta)f'(x) + \int_{0+}^\infty (f(x-s) - f(x) + f'(x)s\mathbf{1}_{\{0<s<1\}})\Pi(ds).$$

for sufficiently smooth function f. Set $f(x) = \int_0^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y+z)dy$. Then

$$(\gamma - \delta)f'(x) = (\gamma - \delta)\int_0^x \mathbb{W}^{(q)\prime}(x-y)W^{(q)\prime}(y+z)dy + (\gamma - \delta)\mathbb{W}^{(q)}(0)W^{(q)\prime}(x+z)$$

and

$$\int_{0+}^{\infty}(f(x-s)-f(x)+f^{'}(x)s\mathbf{1}_{\{0<s<1\}})\Pi(ds)$$
$$=\int_{0+}^{\infty}\left[\int_0^{x-s}\mathbb{W}^{(q)}(x-s-y)W^{(q)\prime}(y+z)dy-\int_0^x\mathbb{W}^{(q)}(x-y)\mathbb{W}^{(q)'}(y+z)dy\right]$$
$$+\left[\int_0^x\mathbb{W}^{(q)'}(x-y)W^{(q)\prime}(y+z)dy+\mathbb{W}^{(q)}(0)\mathbb{W}^{(q)'}(x+z)\right]s\mathbf{1}_{\{0<s<1\}}\Pi(ds)$$

Thus, one can simplify $(\Gamma_Y-q)f(x)$ to

$$(\Gamma_Y-q)f(x)=\int_0^x(\Gamma_Y-q)\mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y+z)dy+W^{(q)\prime}(x+z)\mathbb{W}^{(q)}(0)\left((\gamma-\delta)+\int_{0+}^1 s\Pi(ds)\right)$$
$$-\int_0^{\infty}\int_{x-s}^x\mathbb{W}^{(q)}(x-s-y)W^{(q)\prime}(y+z)dy\Pi(ds).$$

Thus, the first integral is equal to 0. Next, $\mathbb{W}^{(q)}(0)=\frac{1}{(\gamma-\delta)+\int_{0+}^1 s\Pi(ds)}$, and hence the second expression is equal to $W^{(q)\prime}(x+z)$. The last integral is equal to zero as

$$\int_{x-s}^x\mathbb{W}^{(q)}(x-s-y)W^{(q)\prime}(y+z)dy=0,$$

due to the fact that $\mathbb{W}^{(q)}(x)=0$ for $x<0$. This ends the proof. □

Using above, one can prove the following lemma.

Lemma 2.4.9. *If $V^{(q)}$ is sufficiently smooth and fulfils*

$$(\Gamma-q)v^{\kappa^r}_{c_1^*,c_2^*}(x)\leq 0,\quad for\quad x>c_2^*, \tag{2.26}$$

then $v^{\kappa^r}_{c_1^*,c_2^*}=v_*$ for every $x\in\mathbb{R}$.

Proof. From Lemma 2.4.4, one can see that it is sufficient to prove that (2.16) holds. Thus, due to assumption (2.26), it suffices to prove that $(\Gamma-q)v^{\kappa^r}_{c_1^*,c_2^*}(x)\leq 0$, for almost all $x<c_2^*$. As one can observe from Proposition 2.4.3 (for $x\leq c_2^*$) we can use notation $(\Gamma-q)v^{\kappa^r}_{c_2^*}(x)\leq 0$. We will prove even more as we will need the following identity later, namely

$$(\Gamma-q)v^{\kappa^r}_{c_2^*}(x)=0,\quad \text{for } x\in(0,c_2^*), \tag{2.27}$$

and also

$$(\Gamma-q)v^{\kappa^r}_{c_2^*}(x)<0,\quad \text{for } x\in(-\infty,0).$$

Above, again from Proposition 2.4.3, can be simplify to

$$(\Gamma-q)V^{(q)}(x)=0,\quad \text{for } x\in(0,c_2^*), \tag{2.28}$$

and

$$(\Gamma-q)V^{(q)}(x)<0,\quad \text{for } x\in(-\infty,0). \tag{2.29}$$

Thus, we will prove the above. At first, one can observe that for $x \leq 0$ infinitesimal operator Γ is equal to the respective infinitesimal operator for the process X and we will call it Γ_X. The same is for $x > 0$ and the process Y. Recall again that

$$(\Gamma_X - q)W^{(q)}(x) = 0, \quad \text{for } x \in \mathbb{R},$$

and similarly

$$(\Gamma_Y - q)\mathbb{W}^{(q)}(x) = 0, \quad \text{for } x \in \mathbb{R}.$$

Therefore, one can check that

$$(\Gamma - q)W^{(q)}(x) = -\delta W^{(q)\prime}(x) < 0, \quad \text{for } x > 0. \tag{2.30}$$

Thus, for $x < 0$ we have that

$$\begin{aligned}
(\Gamma - q)V^{(q)}(x) &= \int_0^\infty (\Gamma - q)w^{(q)}(x; -z)\frac{z}{r}\mathbb{P}(X_r \in dr) \\
&= \int_{-x}^\infty (\Gamma - q)W^{(q)}(x+z)\frac{z}{r}\mathbb{P}(X_r \in dr) \\
&= \int_{-x}^\infty (-\delta W^{(q)\prime}(x+z))\frac{z}{r}\mathbb{P}(X_r \in dr) < 0,
\end{aligned}$$

where last equality follows from (2.30). To prove the part for $x > 0$ we will use Lemma 2.4.8. Then

$$\begin{aligned}
(\Gamma - q)V^{(q)}(x) &= \int_0^\infty (\Gamma_Y - q)w^{(q)}(x; -z)\frac{z}{r}\mathbb{P}(X_r \in dr) \\
&= \int_0^\infty (\Gamma_Y - q)W^{(q)}(x+z)\frac{z}{r}\mathbb{P}(X_r \in dr) \\
&\quad + \delta \int_0^\infty (\Gamma_Y - q)\left(\int_0^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y+z)dy\right)\frac{z}{r}\mathbb{P}(X_r \in dr) \\
&= \int_0^\infty (-\delta W^{(q)\prime}(x+z))\frac{z}{r}\mathbb{P}(X_r \in dr) + \delta \int_0^\infty W^{(q)\prime}(x+z)\frac{z}{r}\mathbb{P}(X_r \in dr) = 0.
\end{aligned}$$

This ends the proof. □

Theorem 2.4.10. *Suppose that $V^{(q)}$ is sufficiently smooth and that there exists $(c_1^*, c_2^*) \in C^*$ such that*

$$V^{(q)\prime}(x) \leq V^{(q)\prime}(y) \quad \text{for all} \quad c_2^* \leq x \leq y. \tag{2.31}$$

Then, the strategy $\pi_{c_1^*,c_2^*}$ is an optimal strategy for the impulse control problem.

Proof. From Lemma 2.4.9, one can see that it suffices to prove that (2.26) holds. This will be proven using ideas from Theorem 2 of Loeffen [43]. For $x > c_2^*$, Proposition 2.4.3 gives that we can use notation $v_{c_2^*}^{\kappa^r}$ instead of $v_{c_1^*,c_2^*}^{\kappa^r}$. At first, let us prove that

$$\lim_{y \uparrow x}(\Gamma - q)(v_{c_2^*}^{\kappa^r} - v_x^{\kappa^r})(y) \leq 0, \quad \text{for } x > c_2^*. \tag{2.32}$$

Assume that X is of unbounded variation and $x > c_2^*$ - the case for bounded variation is almost the same. By assumption on the smoothness of the scale function $V^{(q)}$, v_x and $v_{c_2^*}$ are twice differentiable on $(0,\infty)$, except for the possibility that $\lim_{y\uparrow x} v_x''(y) \neq \lim_{y\downarrow x} v_x''(y)$. One can get that

$$\begin{aligned}\lim_{y\uparrow x}(\Gamma-q)(v_{c_2^*}^{\kappa^r}-v_x^{\kappa^r})(y) =&(\gamma-\delta)(v_{c_2^*}'^{\kappa^r}-v_x'^{\kappa^r})(x)+\frac{\sigma^2}{2}\left(v_{c_2^*}''^{\kappa^r}(x)-\lim_{y\uparrow x}v_x''^{\kappa^r}(y)\right)-q(v_{c_2^*}^{\kappa^r}-v_x^{\kappa^r})(x)\\ &+\int_{(0,+\infty)}\Big[(v_{c_2^*}^{\kappa^r}-v_x^{\kappa^r})(x-z)-(v_{c_2^*}^{\kappa^r}-v_x^{\kappa^r})(x)\Big]\\ &+(v_{c_2^*}'^{\kappa^r}-v_x'^{\kappa^r})(x)z\mathbf{1}_{\{0<z<1\}}\Pi(dz).\end{aligned}$$

One can observe that:

(i) $\lim_{y\uparrow x} v_x''^{\kappa^r}(y) \geq 0 = v_{c_2^*}''^{\kappa^r}(x)$, where inequality is due to assumption (2.31).

(ii) We have that $(v_{c_2^*}'^{\kappa^r} - v_x'^{\kappa^r})(u) \geq 0$ for $u \in (-\infty, x]$. This holds because for $u \in (-\infty, c_2^*]$ we have that

$$\frac{V^{(q)\prime}(u)}{V^{(q)\prime}(c_2^*)} \geq \frac{V^{(q)\prime}(u)}{V^{(q)\prime}(x)},$$

due to assumption (2.31). Moreover, for $u \in (c_2^*, x]$ we have that

$$1-\frac{V^{(q)\prime}(u)}{V^{(q)\prime}(x)} \geq 0,$$

due to, again, assumption (2.31). Thus, we one can obtain that for $z \in (0,\infty)$

$$(v_{c_2^*}^{\kappa^r}-v_x^{\kappa^r})(x-z) \leq (v_{c_2^*}^{\kappa^r}-v_x^{\kappa^r})(x).$$

(iii) We have that

$$(v_{c_2^*}^{\kappa^r}-v_x^{\kappa^r})(x) \geq (v_{c_2^*}^{\kappa^r}-v_x^{\kappa^r})(x) \geq 0.$$

The first inequality follows from (ii), the second from assumption (2.31).

(iv) $v_{c_2^*}'^{\kappa^r}(x) = v_x'^{\kappa^r}(x) = 1$.

Collecting all the above, we get (2.32). Using this one can prove (2.26) by contradiction. Namely, let us assume that for some $x > c_2^*$ we have that $(\Gamma - q)v_{c_1^*,c_2^*}^{\kappa^r}(x) > 0$. Then by (2.32) one can get that $\lim_{y\uparrow x}(\Gamma - q)v_x^{\kappa^r}(y) > 0$, but this contradicts (2.27). The theorem follows by the lemma 2.4.9. □

2.4.5 Examples

In this part, we will present the results concerning the numerical calculations of the optimal impulse policy (c_1^*, c_2^*). From Proposition 2.4.1, we know that when C^* is not an empty set, then (c_1^*, c_2^*) needs to satisfy one of the possibilities listed there. Such an observation will define the way of constructing numerical calculations. First, however, one needs to know how to calculate the Parisian refracted scale function to start the computations. Therefore, we will find an analytical representation for $w^{(q)}$ and $V^{(q)}$ for the linear Brownian motion and the Crámer-Lundberg process with the exponential claims. Moreover, we will prove that there is a unique (c_1, c_2) policy for these two processes, which is optimal for the impulse control problem.

Linear Brownian motion

Let us assume that the process X is a linear Brownian motion which can be represented as

$$X_t = \mu t + \sigma B_t, \qquad \text{for } t \geq 0,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $B = \{B_t : t \geq 0\}$ is the standard Brownian motion. Fix $q > 0$ and $\delta > 0$. Let us recall from Section 1.4.2 that

$$W^{(q)}(x) = \frac{2}{\sigma^2 \rho}\left(e^{\rho_2 x} - e^{-\rho_1 x}\right),$$

where

$$\rho_1 = \frac{\sqrt{\mu^2 + 2q\sigma^2} + \mu}{\sigma^2}, \ \rho_2 = \frac{\sqrt{\mu^2 + 2q\sigma^2} - \mu}{\sigma^2}, \ \rho = \rho_1 + \rho_2 = \frac{2\sqrt{\mu^2 + 2q\sigma^2}}{\sigma^2}.$$

For process Y, we will use Y superscript for the above parameters (i.e. ρ_1^Y, ρ_2^Y and ρ^Y). Our first step is to present the formula for $w^{(q)}$.

Proposition 2.4.11. *For the linear Brownian motion, the function $w^{(q)}$ is of the following form*

$$w^{(q)}(x; -z) = \frac{\sigma^2}{2} W^{(q)\prime}(z) \mathbb{W}^{(q)}(x) + \frac{W^{(q)}(z)}{\rho^Y}\left(\rho_1^Y e^{\rho_2^Y x} + \rho_2^Y e^{-\rho_1^Y x}\right).$$

Proof. The proof contains simple calculations, which involve the following relations between parameters of $W^{(q)}$ and $\mathbb{W}^{(q)}$

$$\frac{\rho_2}{\rho_2 - \rho_2^Y} - \frac{\rho_2}{\rho_1^Y + \rho_2} = -\frac{\sigma^2 \rho^Y}{2\delta}, \ \frac{\rho_1}{\rho_2^Y + \rho_1} + \frac{\rho_1}{\rho_1^Y - \rho_1} = -\frac{\sigma^2 \rho^Y}{2\delta}. \tag{2.33}$$

□

Now, we will consider the formula for the function $V^{(q)}$.

Proposition 2.4.12. *For the linear Brownian motion, the function $V^{(q)}$ is of the following form. For $x \geq 0$*

$$V^{(q)}(x) = \frac{\sigma^2}{2} \mathbb{W}^{(q)}(x) \left[\frac{2}{\sqrt{2\pi\sigma^2 r}} e^{\frac{-r\mu^2}{2\sigma^2}} + \rho_2 e^{qr} - \rho e^{qr} \Phi\left(\frac{-r\sqrt{\mu^2 + 2q\sigma^2}}{\sigma\sqrt{r}}\right)\right] + \frac{e^{qr}}{\rho^Y}\left(\rho_1^Y e^{\rho_2^Y x} + \rho_2^Y e^{-\rho_1^Y x}\right),$$

and for $x < 0$

$$V^{(q)}(x) = e^{qr}\left(e^{\rho_2 x} \Phi\left(\frac{x + r\sqrt{\mu^2 + 2q\sigma^2}}{\sigma\sqrt{r}}\right) + e^{-\rho_1 x} \Phi\left(\frac{x - r\sqrt{\mu^2 + 2q\sigma^2}}{\sigma\sqrt{r}}\right)\right),$$

where Φ is the cumulative distribution function of the standard normal variable.

Proof. We will separate our proof into two parts. Assume that $x \geq 0$. Using the formula for the $w^{(q)}$ from the last proposition and equation (2.5), one can get

$$V^{(q)}(x) = \frac{\sigma^2}{2}\mathbb{W}^{(q)}(x)\int_0^\infty W^{(q)\prime}(z)\frac{z}{r}\mathbb{P}(X_r \in dz) + \frac{e^{qr}}{\rho^Y}\left(\rho_1^Y e^{\rho_2^Y x} + \rho_2^Y e^{-\rho_1^Y x}\right).$$

Hence, one needs to calculate integral $\int_0^\infty W^{(q)\prime}(z)\frac{z}{r}\mathbb{P}(X_r \in dz)$. This simply but long calculation is sufficient to end the proof of this part. Now, fix $x < 0$. Then

$$V^{(q)}(x) = \int_{-x}^\infty \frac{2}{\sigma^2\rho}\left(e^{\rho_2(x+z)} - e^{-\rho_1(x+z)}\right)\frac{z}{r}\frac{1}{\sqrt{2\pi\sigma^2 r}}e^{\frac{-(z-\mu r)^2}{2\sigma^2 r}}dz.$$

Therefore, again after some calculations, one can also obtain this part. □

Proposition 2.4.13. *For any $q > 0$ and $z > 0$, there exists a constant $a_R^* \geq 0$ such that the function $w^{(q)\prime}(x; -z)$ is decreasing on $(0, a_R^*)$ and is increasing on (a_R^*, ∞). This also implies the same for $V^{(q)\prime}(x)$.*

Proof. To prove the proposition, we will examine the second derivative with respect to x of $w^{(q)\prime}(x; -z)$. Indeed, using Proposition 2.4.11 and the formula for the scale function $\mathbb{W}^{(q)}$, one can get

$$\begin{aligned} w^{(q)\prime\prime}(x; -z) &= \frac{(\rho_2^Y)^2}{\rho^Y}e^{\rho_2^Y x}\left(W^{(q)\prime}(z) + \rho_1^Y W^{(q)}(z)\right) - \frac{(\rho_1^Y)^2}{\rho^Y}e^{-\rho_1^Y x}\left(W^{(q)\prime}(z) - \rho_2^Y W^{(q)}(z)\right) \\ &= \frac{(\rho_2^Y)^2}{\rho^Y}e^{\rho_2^Y x}A - \frac{(\rho_1^Y)^2}{\rho^Y}e^{-\rho_1^Y x}B, \end{aligned}$$

where $\rho_1^Y, \rho_2^Y > 0$. The constant A is strictly positive because the scale function $W^{(q)}$ is increasing and strictly positive on the whole positive half-line. Now, if $B < 0$, then function $w^{(q)\prime\prime}(x; -z)$ is positive for all $x, z > 0$ and hence $a_R^* = 0$. If $B > 0$, then $w^{(q)\prime\prime}(x; -z)$ is an increasing and unbounded function of x as a sum of two increasing exponential functions. This completes the proof for $w^{(q)}$. For $V^{(q)\prime}(x)$, we get the result directly from its definition. □

Theorem 2.4.14. *For the linear Brownian motion model, there is a unique $(c_1; c_2)$ policy which is optimal for the impulse control problem.*

Proof. The proof of the theorem follows directly from the Proposition 2.4.13 together with the Lemma 2.4.10. □

Now, we begin the numerical examples of (c_1^*, c_2^*) pairs with the respect to three parameters β, r and δ. Let us start with the parameter β and consider the following parameters

$$\mu = 0.5, \quad \sigma = 0.75, \quad r = 3, \quad \delta = 0.05, \quad q = 0.05.$$

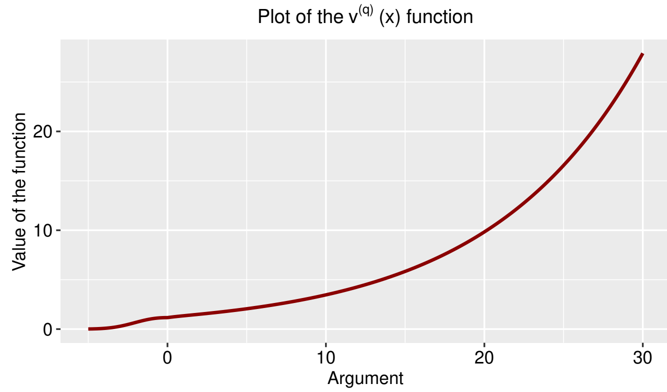

Figure 2.1: Plot of the $V^{(q)}$ function for the linear Brownian motion

Note that, from Figure 2.1, the shape of this function is similar to the classic scale function for linear Brownian motion. In Figure 2.2, we consider two plots. On the first plot, we show optimal points (c_1^*, c_2^*) imposed on the graph of $V^{(q)\prime}$. On the second plot, we show (c_1^*, c_2^*) points alone. We draw both plots for a varying parameter of $\beta \in (0, 1]$.

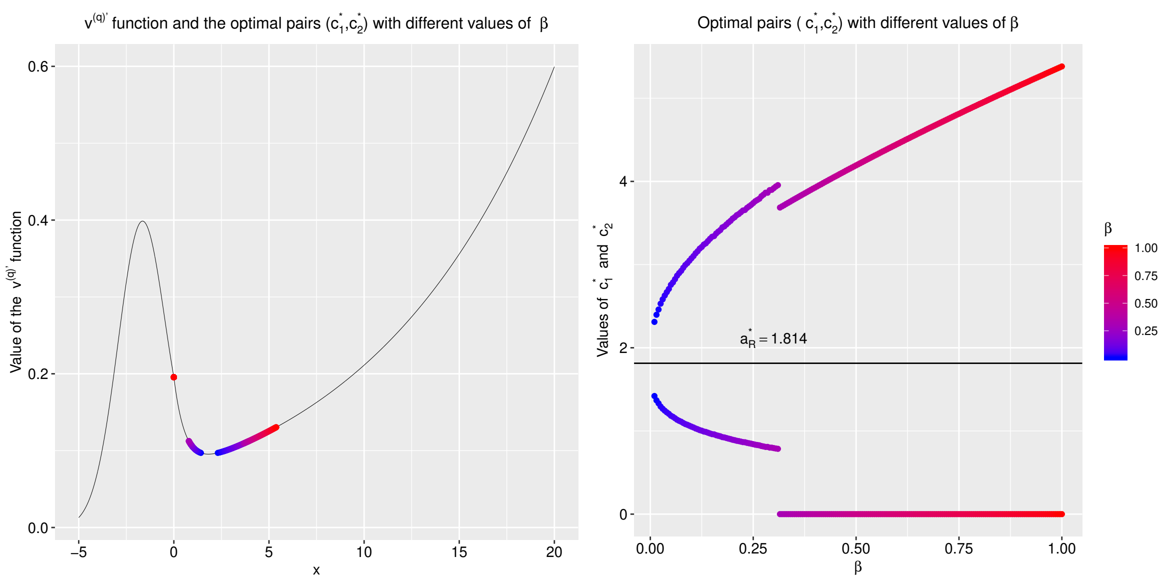

Figure 2.2: Plots of the $V^{(q)\prime}$ function for linear Brownian motion and the optimal pairs (c_1^*, c_2^*) in case of changing cost of the transactions.

In the second plot, one can see that c_1^*'s are below point a_R^*, c_2^*'s one can find above this level, and for fixed β pair (c_1^*, c_2^*) is of the same colour. One can see that for β big enough optimal pair is not belonging to the set $\mathcal{B}$ from the Proposition 2.4.2. One can think that if the cost of transactions is too restrictive, then an optimal behaviour is to pay everything we can. However, c_2^* must be big enough to have some profit on the dividend payment. Moreover, one can be interested in the sensitivity of optimal points concerning other parameters. Thus, now we will consider parameter $r \in (0, 3]$. We will stay with the same set of the parameters as before, except r.

In Figure 2.3 one can see two cases, one for $\beta = 0.05$ and second for relative high cost of transaction, namely $\beta = 0.45$

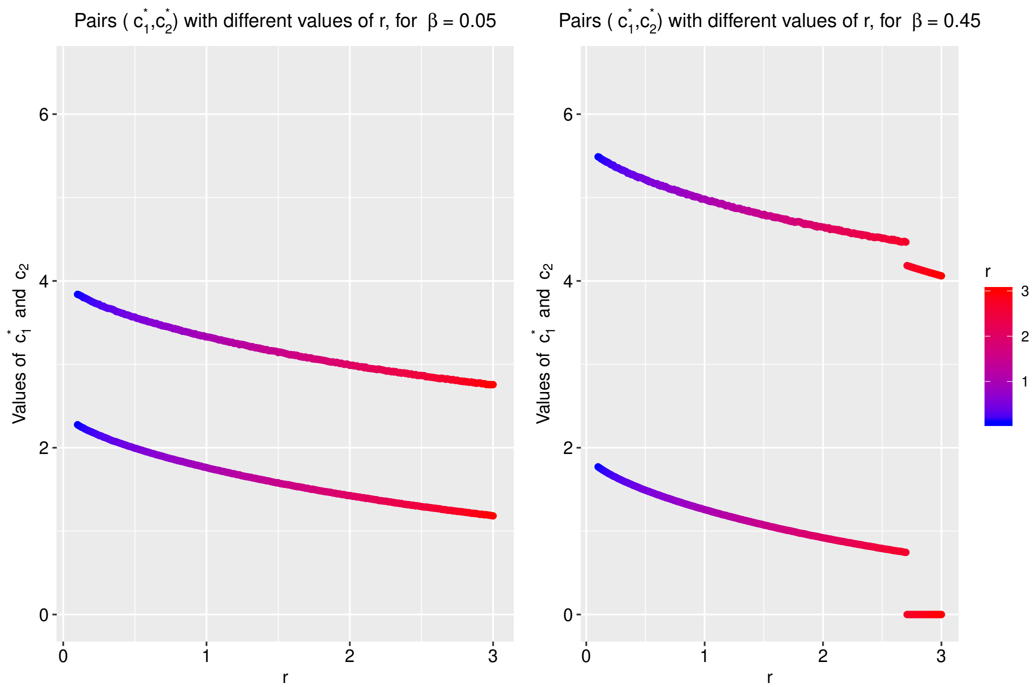

Figure 2.3: Plots of the optimal pairs (c_1^*, c_2^*) in case of changing a parameter r for two different values of the parameter β.

Again, for the fixed level r, (c_1^*, c_2^*) is of the same colour as r, where c_1^* is on the bottom curve and c_2^* is on the top one. In both cases, one can see that if we increase parameter r, then (c_1^*, c_2^*) will decrease in both coordinates. The explanation is quite simple, namely, for more significant r process is safer in terms of possible ruin. Thus, one can lower both barriers to have more frequent dividends payments. At the end of this part, we will consider sensitivity concerning parameter $\delta \in (0, 0.25)$. The rest of the parameters stay the same as before, i.e.

$$\mu = 0.5, \quad \sigma = 0.75, \quad r = 3, \quad q = 0.05.$$

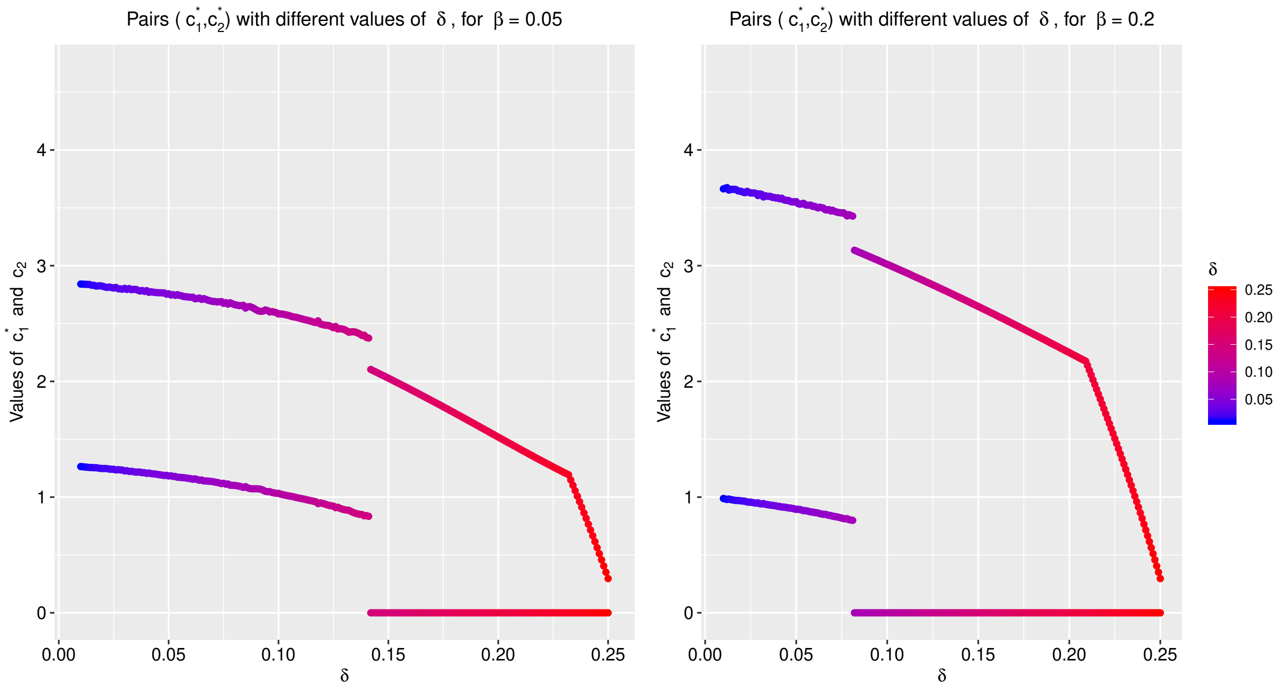

Figure 2.4: Plots of the optimal pairs (c_1^*, c_2^*) in case of changing a parameter δ for two different values of the parameter β.

From Figure 2.4, one can, again, observe that changing parameter β leads to bigger distance between c_1^* and c_2^*. Moreover, increasing δ implies lowering both levels of c_1^* and c_2^*. This time it is because with increasing δ, we decrease the overall drift of the process. Thus, it is harder to reach higher values.

Cramér-Lundberg Process with exponential claims

In the second example, we will consider as the process X the Cramér-Lundberg process with exponential claims which can be represented as

$$X_t = pt - \sum_{i=1}^{N_t} U_i, \qquad \text{for } t \geq 0,$$

where $p > 0$, $\{U_i\}_{i=1}^{\infty}$ is an *i.i.d.* sequence of exponential random variables with the parameter $\mu > 0$, and $N = \{N_t : t \geq 0\}$ is a homogeneous Poisson process with intensity $\lambda > 0$. We also assume that the Poisson process and the exponential random variables are mutually independent. For this process, let us recall from Section 1.4.2 that the scale function is of the following form

$$W^{(q)}(x) = \frac{1}{p}\Big(A^+ e^{q^+ x} - A^- e^{q^- x}\Big),$$

with

$$q^{\pm} = \frac{q + \lambda - \mu p \pm \sqrt{(q + \lambda - \mu p)^2 + 4pq\mu}}{2p}, \quad \text{and} \quad A^{\pm} = \frac{\mu + q^{\pm}}{q^+ - q^-}.$$

The respective parameters for the process Y are denoted by subscript Y (i.e. $q_Y^{\pm}, A_Y^{\pm}$).

Proposition 2.4.15. *For the Crámer-Lundberg process with exponential claims and $z > 0$, we have that*

$$w^{(q)}(x;-z) = (p-\delta)\mathbb{W}^{(q)}(x)W^{(q)}(z) - \frac{1}{\mu\lambda}\left[(q+\lambda)W^{(q)}(z) - pW^{(q)\prime}(z)\right] \cdot \left[(q+\lambda)\mathbb{W}^{(q)}(x) - (p-\delta)\mathbb{W}^{(q)\prime}(x)\right].$$

Proof. To obtain such a representation, we need to use the following relations between the parameters of the scale functions

$$\frac{A_Y^+ q^+}{q^+ - q_Y^+} - \frac{A_Y^- q^+}{q^+ - q_Y^-} = -\frac{p-\delta}{\delta}, \qquad \frac{A_Y^- q^-}{q^- - q_Y^-} - \frac{A_Y^+ q^-}{q^- - q_Y^+} = \frac{p-\delta}{\delta},$$

$$\frac{q^+}{q^+ - q_Y^+} = -\frac{p-\delta}{\delta}\cdot\frac{q^+ - q_Y^-}{q^+ + \mu}, \qquad \frac{q^-}{q^- - q_Y^+} = -\frac{p-\delta}{\delta}\cdot\frac{q^- - q_Y^-}{q^- + \mu}.$$

□

We will obtain the formula for a Parisian refracted scale function, divided into three parts.

Proposition 2.4.16. *For the Crámer-Lundberg process with exponential claims, the function $V^{(q)}$ is of the following form.*

(i) For $x > 0$,

$$V^{(q)}(x) = e^{qr}(p-\delta)\mathbb{W}^{(q)}(x) - \frac{1}{\mu\lambda}\left[(q+\lambda)\mathbb{W}^{(q)}(x) - (p-\delta)\mathbb{W}^{(q)\prime}(x)\right]\left[(q+\lambda)e^{qr} - pC\right],$$

$$C := e^{-\lambda r}\left[pW^{(q)\prime}(pr) + C^+ - C^- + \frac{e^{-\mu pr}}{pr}\sum_{m=1}^{\infty}\frac{(p\lambda\mu r^2)^m}{(m-1)!m!}\right],$$

$$C^\pm := A^\mp q^\pm e^{q^\pm pr}\sum_{m=1}^{\infty}\frac{pr(q^\mp+\mu))^{(m-1)}}{(m-1)!m!}\gamma(m,(q^\pm+\mu)pr)[pr(q^\pm+1)-m],$$

where $\gamma(x,a) = \int_0^x e^{-t}t^{a-1}dt$ is an incomplete gamma function.

(ii) For $x \in [-pr, 0]$,

$$V^{(q)}(x) = e^{-\lambda r}\left[pW^{(q)}(x+pr) + K^+(x) - K^-(x)\right],$$

with

$$K^\pm(x) := A^\mp e^{q^\pm(x+pr)}\sum_{m=1}^{\infty}\frac{(pr(q^\mp+\mu))^{m-1}}{(m-1)!m!}\gamma(m,(pr+x)(q^\pm+\mu))\left[pr(q^\pm+\mu)-m\right].$$

(iii) For $x < -pr$, we have $V^{(q)}(x) = 0$.

Proof. We will divide this proof into three parts.

(i) For $x > 0$ we use the formula from the Proposition 2.4.15 and equation (2.5)

$$V^{(q)}(x) = (p-\delta)\mathbb{W}^{(q)}(x)e^{qr} - \frac{1}{\mu\lambda}\Big[(q+\lambda)\mathbb{W}^{(q)}(x) - (p-\delta)\mathbb{W}^{(q)\prime}(x)\Big] \cdot \Big[(q+\lambda)e^{qr} - p\int_0^\infty W^{(q)\prime}(z)\frac{z}{r}\mathbb{P}(X_r \in dz)\Big].$$

From Lkabous *et al.* [45] one can obtain the following

$$\mathbb{P}\left(\sum_{i=1}^{N_r} U_i \in dy\right) = e^{-\lambda r}\left(\delta_0(dy) + e^{-\mu y}\sum_{m=1}^{\infty}\frac{(\mu\lambda r)^m}{(m-1)!m!}y^{m-1}dy\right). \tag{2.34}$$

With the use of (2.34), it turns out that $\int_0^\infty W^{(q)\prime}(z)\frac{z}{r}\mathbb{P}(X_r \in dz) = C$. Putting all the pieces together, one gets the postulated formula for $V^{(q)}$ for $x > 0$.

(ii) Now, fix $x \in [-pr, 0]$. In this case,

$$V^{(q)}(x) = \int_0^\infty W^{(q)}(x+z)\frac{z}{r}\mathbb{P}(X_r \in dz).$$

The random variable X_r can achieve at most value pr with the probability one. Moreover, we know that $W^{(q)}(x+z) > 0$ iff $x + z \geq 0$. Therefore,

$$V^{(q)}(x) = \int_{-x}^{pr} W^{(q)}(x+z)\frac{z}{r}\mathbb{P}(X_r \in dz).$$

Using this observation, the rest of the proof involves simple but long calculations; thus, we omit this.

(iii) Fix $x < -pr$. As we state in the previous case, when $x < -pr$, then $x + z < 0$. Therefore, $W^{(q)}(x+z) = 0$ and $V^{(q)}(x) = 0$.

□

Proposition 2.4.17. *Fix $q > 0$ and $z > 0$. There exists a constant $a_R^* \geq 0$ such that the function $w^{(q)\prime}(x; -z)$ is decreasing on $(0, a_R^*)$ and is increasing on (a_R^*, ∞). This also implies the same for $V^{(q)\prime}(x)$.*

Proof. Firstly, let us note that $\frac{q+\lambda}{p-\delta} - q_Y^+ = q_Y^- + \mu$ and $\frac{q+\lambda}{p} - q^+ = q^- + \mu$. From this observation and Proposition 2.4.15, one can obtain

$$(q+\lambda)\mathbb{W}^{(q)}(x) - (p-\delta)\mathbb{W}^{(q)\prime}(x) = \frac{\mu\lambda\left[e^{q_Y^+ x} - e^{q_Y^- x}\right]}{(p-\delta)(q_Y^+ - q_Y^-)},$$

$$(q+\lambda)W^{(q)}(z) - pW^{(q)\prime}(z) = \frac{\mu\lambda\left[e^{q^+ z} - e^{q^- z}\right]}{p(q^+ - q^-)}.$$

So,

$$w^{(q)}(x; -z) = (p-\delta)\mathbb{W}^{(q)}(x)W^{(q)}(z) - \frac{\mu\lambda\left[e^{q^+ z} - e^{q^- z}\right]\left[e^{q_Y^+ x} - e^{q_Y^- x}\right]}{p(p-\delta)(q^+ - q^-)(q_Y^+ - q_Y^-)}.$$

Thus, one can also get the more explicit form $w^{(q)}(x;-z) = D^+e^{q_Y^+ x} - D^- e^{q_Y^- x}$, where

$$D^{\pm} = A_Y^{\pm} W^{(q)}(z) - \frac{\mu\lambda\left(e^{q^+ z} - e^{q^- z}\right)}{p(p-\delta)(q^+ - q^-)(q_Y^+ - q_Y^-)}.$$

Next, $q_Y^+ > 0$, $q_Y^- < 0$ and $\lim_{x\to+\infty} e^{q_Y^+ x} = +\infty$, $\lim_{x\to+\infty} e^{q_Y^- x} = 0$. Then, from $\lim_{x\to+\infty} w^{(q)}(x;-z) = +\infty$, one can get that $D^+ > 0$. Moreover, we are interested in the sign of

$$w^{(q)''}(x;-z) = D^+(q_Y^+)^2 e^{q_Y^+ x} - D^-(q_Y^-)^2 e^{q_Y^- x}.$$

If $D^- < 0$, then $w^{(q)''}(x;-z)$ is positive on the whole positive half-line. In such a case, $a_R^* = 0$. If $D^- > 0$, then one can see that $w^{(q)''}(x;-z)$ is an increasing and unbounded function. This ends the proof. □

Theorem 2.4.18. *For the Cramér-Lundberg process with exponential claims, there is a unique $(c_1;c_2)$ policy which is optimal for the impulse control problem.*

Proof. The proof of the theorem follows directly from Proposition 2.4.17 together with Lemma 2.4.10. □

Using the above results, one can plot the picture of the $V^{(q)}$ and $V^{(q)\prime}$ for this process. Namely, let us set

$$p = 3, \quad \lambda = 2, \quad \mu = 1, \quad r = 2, \quad q = 0.05, \quad \delta = 0.25.$$

Note, that we set such parameters that $p > \frac{\lambda}{\mu}$. Moreover, we know that $V^{(q)}(x) = 0$ if $x < -pr$; therefore, we will consider x $\geq -pr$.

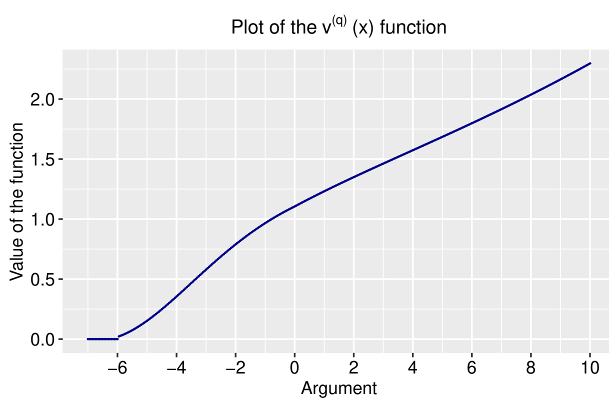

Figure 2.5: Plot of the $V^{(q)}$ for the Cramér-Lundberg process with exponential claims

From Figure 2.5, as in the linear Brownian motion setting, one can also see a similar shape of the Parisian scale function with the shape of a classical scale function. However, even if this is not directly clear from Figure 2.5, $V^{(q)}$ is not a continuous function at $x = -pr$.
Now, we will proceed to sensitive analyse of three parameters β, r and δ, as it was a case for the linear Brownian motion. Let us start with the parameter $\beta \in (0, 1.5]$.

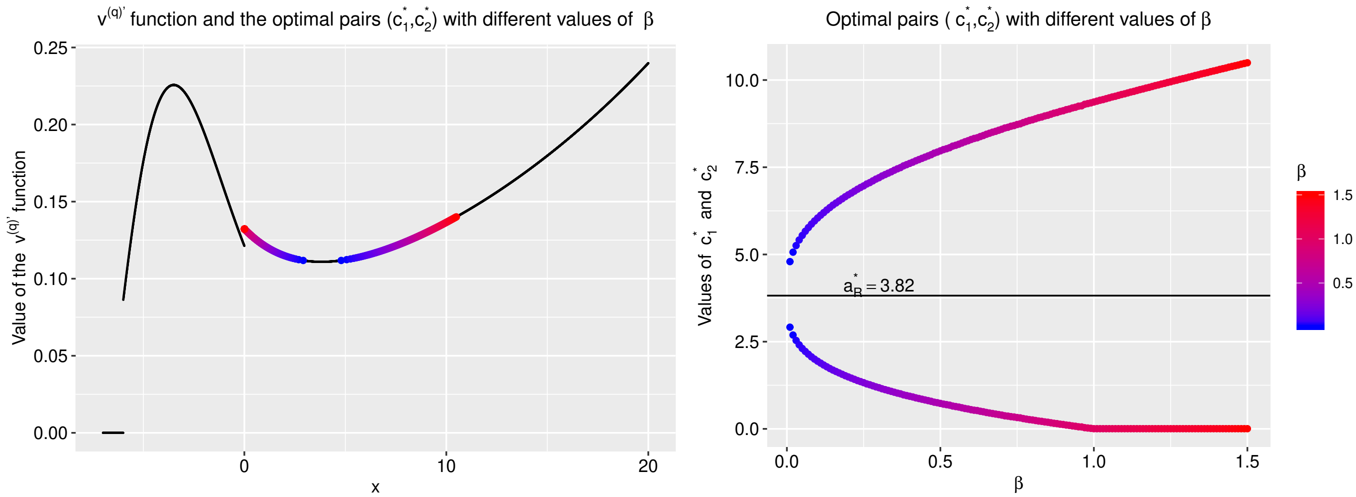

Figure 2.6: Plot of the $V^{(q)\prime}$ for the Cramér-Lundberg process with exponential claims and the optimal pairs (c_1^*, c_2^*) in case of changing cost of the transactions.

From Figure 2.6, one can see that $V^{(q)\prime}$ is not a continuous function at 0. Moreover, from the plot on the right, one can see that increasing parameter β causes the increasing distance between c_1^* and c_2^*. Furthermore, again one can see that for β big enough, optimal pair (c_1^*, c_2^*) does not belong to the set $\mathcal{B}$. Let us proceed to Figure 2.7, where one can see sensitivity for the parameter $r \in (0, 3]$.

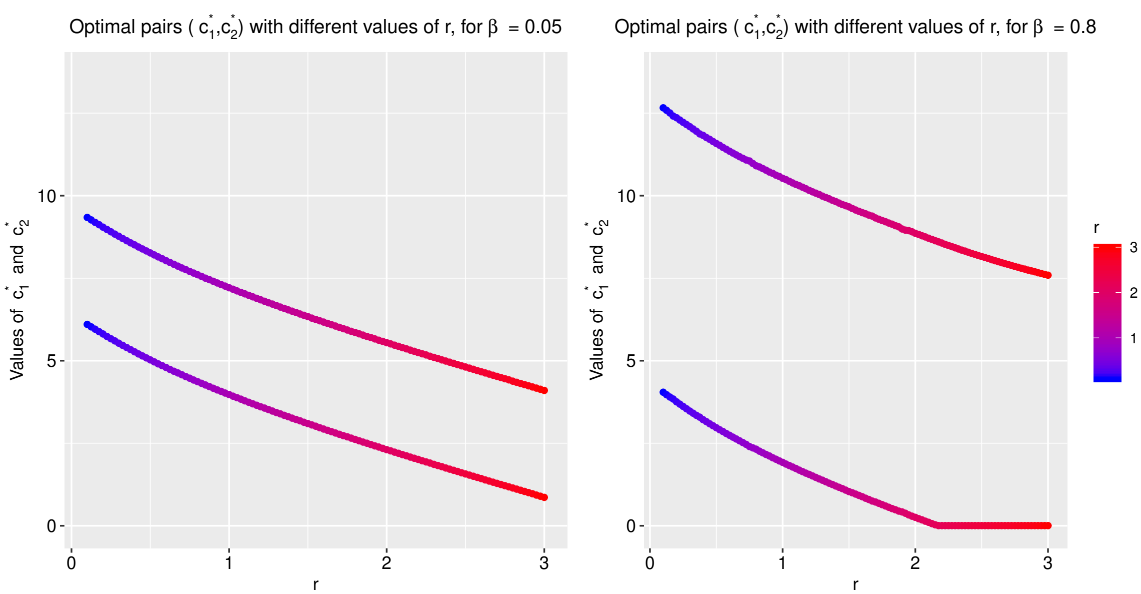

Figure 2.7: Plot of the optimal pairs (c_1^*, c_2^*) in case of changing a parameter r for two different values of the parameter β

From this picture, one can see that if one would increase the parameter r then optimal points

(c_1^*, c_2^*) are both lowering their levels. This is due to being safer from ruin, and thus one can stick closer to level 0. At last, let us consider sensitivity for the parameter $\delta \in [0.1, 0.5]$.

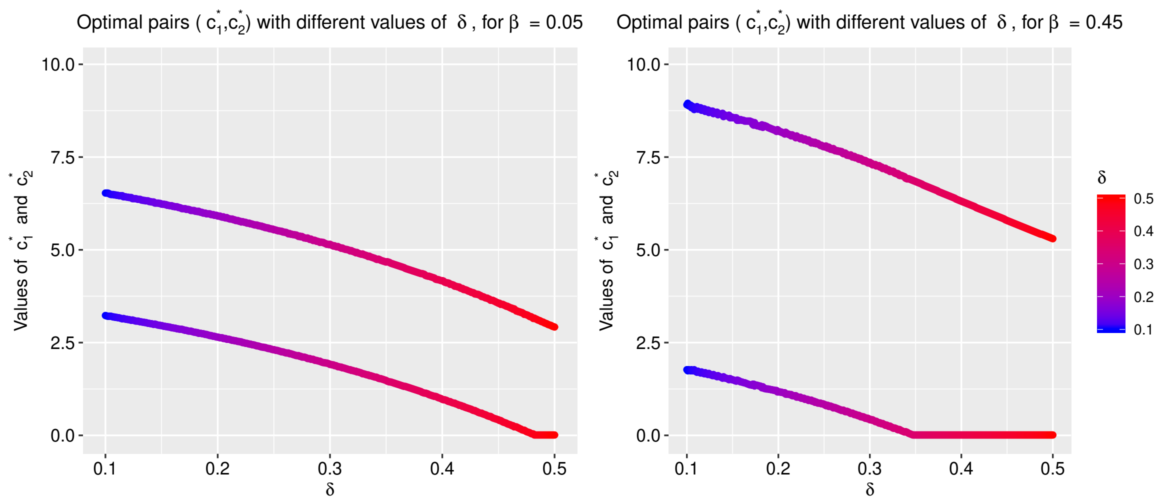

Figure 2.8: Plot of the optimal pairs (c_1^*, c_2^*) in case of changing a parameter δ for two different values of the parameter β

In Figure 2.8, one can see that with increasing the parameter δ optimal points (c_1^*, c_2^*) are decreasing in both co-ordinates. This is consistent with the previous conclusions on δ.

2.5 Comments

In this chapter, we solved the problem of optimal dividend payments with the refraction of the controlled risk process. In our model, we proposed Parisian ruin time as a definition of bankruptcy, and we assumed that every dividend payment is associated with a fixed transaction cost. Even though this setting was motivated by being close to reality, and we firmly believe that with success, there is still a big room for future developments. Namely, we interpreted the refracted model as a cash injection done by shareholders and dividend as a payout for them. However, we did not connect these two interpretations. It would be worthy to see some penalty function for being in the red zone which should be associated with the value function of the dividend payments. In such a case, finding optimal pair (c_1^*, c_2^*) would consider the risk of ruin and stay in the red zone. One possibility is to remove cash injected, through the refracted model, from the present value of the value function. Another idea could be to have a more sophisticated discounting structure, which would take care of how long we stayed in the red zone. Without these improvements, one can see that increasing parameters δ and r will lead to the same effect of decreasing (c_1^*, c_2^*) in both coordinates, which could be misleading. Namely, increasing the value of parameter r can be seen as a decreasing probability of ruin; however, increasing δ means we lower our potential drift of the process. Furthermore, one can see that a lot more can be done in the case of numerical computation. In this thesis, we focused on giving some intuitions behind the model. However, for example, one could be interested in some semi-explicit results on the behaviour of the optimal points concerning model parameters.

Chapter 3

Markov additive processes & ω-killing

This chapter intends to generalise the classical model, but in a different direction than we have done in Chapter 2. We saw that the refracted process changes its structure depending on its position. However, there are phenomena whose structure depends on the situation of the environment in which they are considered. Let us assume that we can distinguish certain states of the environment. For example, the state can be the season or the situation on the stock market (prosperity or fall). Due to the stationary increments, Lévy processes are not a good tool for studying such problems. However, looking at the short period, if the environment remains in its current state, it may turn out that the use of the Lévy process is justified in this period. This intuitive view guides the use of the Markov additive processes. It is a class of two-dimensional stochastic processes, where the first component is responsible for the position of the process and the second for the environmental condition.

On the other hand, when it comes to a different view of the generalisation of classical ruin, we can note that the main disadvantage of Parisian ruin time is the lack of control over how far the process can drop below zero. It may happen that after time r, the process will move so far that, from the point of view of a realistic approach, the phenomenon should have been bankrupt much earlier. By taking a different approach, we might want to impose specific penalties that will cause the process to go bankrupt when the accumulated size is exceeded. In particular, this approach is covered by the so-called Omega model, in which the idea of ruin is to count penalties using a certain function ω, depending on the position of the stochastic process. Moreover, it is assumed that if the process crosses the fixed level, it is called bankrupt immediately. This concept first appeared in the work of Albrecher *et al.* [1]. The authors considered the Wiener process for which they calculated the optimal form of the barrier strategy in the problem of optimal dividend payments until Omega type of ruin. In this model, the authors excluded the possibility of continuous dividend payments. Instead, they assumed that the waiting time for the possible next dividend payment is random, with an exponential distribution. The authors defined ω function such that $\omega(x) = 0$ for $x > 0$ and $\omega(x) \geq 0$ for $x \leq 0$ and level of immediately ruin was $-d < 0$. In this model probability of bankruptcy in infinitesimal period dt was $\omega(x)dt$.

Next, a similar model was considered in the work of Gerber *et al.* [23]. The authors also considered the Omega model with the Brownian motion, but they aimed to get the form of the probability of bankruptcy. In particular, they obtained a semi-explicit form of this probability in terms of Airy functions.

There is a need to solve some specific exit problems to approach the Omega model for spectrally negative Lévy processes. This was done in work Li and Palmowski [41], where the authors obtained the results in terms of the new scale functions. These results naturally introduce the concept of ω-killed types of exit problems. It turns out that the obtained results have a wide range of applications, including the consideration of more advanced concepts of the structure of the interest rates. Solving more general issues gave answers to the Omega model and other essential applications. In this chapter, our aim is similar. At first, we will produce representations for ω-killed types of exit problems for Markov additive processes. Next, we will use these results for a few chosen applications, including calculation of the representation of the value function for the dividend problem in the Omega model.

We will start this chapter with the definition of the Markov additive processes. We will show that also, in the case of this class of processes, there exist scale matrices which are a generalisation of the scale functions from the spectrally negative Lévy processes case. In terms of these matrices, we will show results for exit problems obtained in the literature. Then, in Section 3.1.3, as an example of a Markov additive process, we will consider Markov modulated Brownian motion, for which we will show explicit formulas for scale matrix for some limited case. Next, we will define the ω function used in the concept of ω-killing. Finally, to make some intuitions, we will show in Section 3.2.3 that the probability of bankruptcy in the Omega model for the Crámer-Lundberg process with exponential claims is a linear function of the probability of classical ruin time. Finally, in Section 3.3, we will show the main results of this chapter. Namely, we will consider some exit problems for Markov additive processes and ω-killing. Then, in Section 3.4, we will use obtained results to deliver the value function for the dividend problem in the Omega model framework. Furthermore, in Section 3.5, we will show examples for some specific choices of ω functions.

3.1 Markov additive processes

3.1.1 Definition

A Markov additive process (MAP in shorthand) is defined as follows. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtrated probability space, with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ which satisfies usual conditions. We say that a bivariate process (X, J) is a MAP if, given $\{J_t = i\}$ for $i \in E$, the vector $(X_{t+s} - X_t, J_{t+s})$ is independent of $\mathcal{F}_t$ and has the same law as $(X_s - X_0, J_s)$ given $\{J_0 = i\}$ for all $s, t \geq 0$ and $i \in E$, where E is finite state space and $|E| = N \in \mathbb{N}^+$. Usually, X is called an additive component, and J is a background process representing the environment. Moreover, one can find the following representation of every MAP of importance. It is straightforward from the definition that J forms a Markov chain. Furthermore, one can observe that the process X evolves as some Lévy process X^i when J is in state i. In addition, when J transits to state $j \neq i$, the process X jumps according to the distribution of the random variable U_{ij}, where $i, j \in E$. All these components are assumed to be independent. The above structure explains why the other name for MAP is "Markov-modulated Lévy process". The following picture can summarise this representation.

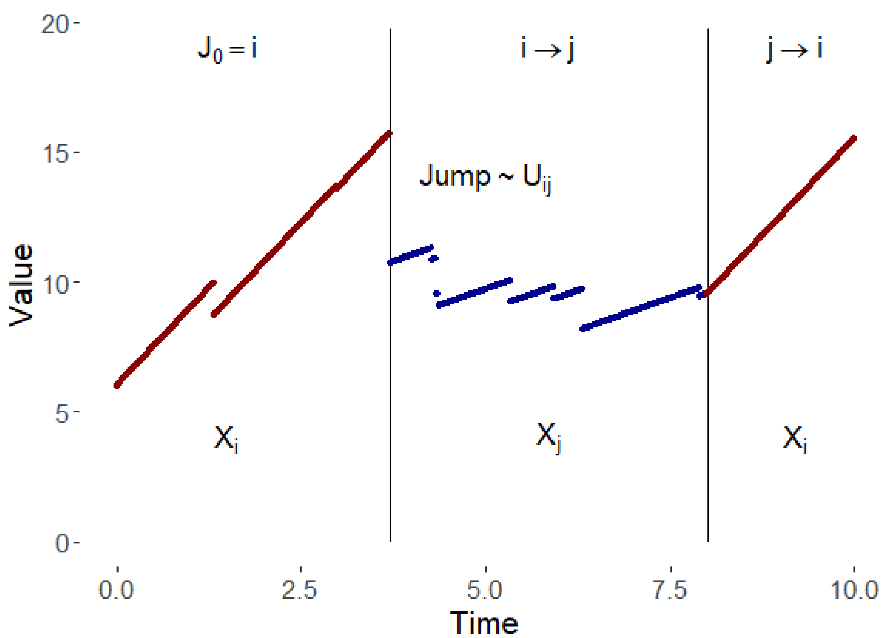

Figure 3.1: An example of one approximated sample path of the MAP

In particular, when J lives in a single state, X reduces to a Lévy process. This chapter assumes that the process X has no positive jumps. Thus X^i is a spectrally negative Lévy process and $U_{ij} \leq 0$ a.s. (for every $i, j \in E$). Furthermore, we exclude the case when X has monotone paths. We further assume that J is an irreducible Markov chain, with $\boldsymbol{Q} = (q_{ij})_{i,j \in E}$ being its transition probability matrix and $\boldsymbol{\pi}$ being its unique stationary vector. Throughout this chapter, the law of (X, J) such that $X_0 = x$ and $J_0 = i$ is denoted by $\mathbb{P}_{x,i}$ and its expectation by $\mathbb{E}_{x,i}$. We will also use equivalently $\mathbb{E}_x[\cdot|J_0 = i]$ for $\mathbb{E}_{x,i}[\cdot]$ to emphasis the starting state. When $x = 0$, we will write $\mathbb{P}(\cdot|J_0 = i)$ and $\mathbb{E}[\cdot|J_0 = i]$ or $\mathbb{P}_i(\cdot)$ and $\mathbb{E}_i[\cdot]$ respectively. For a stopping time κ, the notation

$\mathbb{E}_x[\cdot, J_\kappa|J_0]$ is used to denote a $N \times N$ matrix whose (i,j) entry equals to $\mathbb{E}_x[\cdot, J_\kappa = j|J_0 = i]$. One can find some of the first foundations of Markov additive processes in the Çinlar [11]. Another source is Chapter XI from Asmussen [2]. Especially one can find some information about the different settings, e.g., infinity state space or discrete-time.

3.1.2 Exit problems and properties of the MAPs

As we have seen before, the Laplace exponent was crucial when working with exit problems for the spectrally negative Lévy process. Namely, it served as a tool to define scale functions. Also, in the case of the MAPs, there exists a similar concept. Let us quote the following proposition from Asmussen [2], however, with the use of slightly different proof from Ivanovs [27].

Proposition 3.1.1. *Let $\mathbf{F}(\alpha)$ be the matrix analogue of the Laplace exponent of the spectrally negative Lévy process, which satisfies for $i, j \in E$*

$$\mathbb{E}\left[e^{\alpha X_t}, J_t = j|J_0 = i\right] = \left(e^{\mathbf{F}(\alpha)t}\right)_{i,j}, \quad \textit{for } \alpha \geq 0.$$

Then it has an explicit representation,

$$\mathbf{F}(\alpha) = diag(\psi_1(\alpha), ..., \psi_N(\alpha)) + \mathbf{Q} \circ \mathbb{E}(e^{\alpha U_{ij}})_{i,j\in E},$$

where $\psi_i(\cdot)$ is the Laplace exponent of the Lévy process X^i (i.e., $\mathbb{E}(e^{\alpha X_t^i}) = e^{\psi_i(\alpha)t}$), and $\boldsymbol{A} \circ \boldsymbol{B} = (a_{ij}b_{ij})$ stands for entry-wise (Hadamard) matrix product.

Proof. The overall idea is straightforward. Namely, one needs to build the system of the linear differential equations that involve $\mathbb{E}\left[e^{\alpha X_t}, J_t = j|J_0 = i\right]$ and its derivative for $i, j \in E$ and then solve it to get the result. The difficulty is related to the first part. Therefore, let $h > 0$, which will, eventually, goes to zero. Set $i, j \in E$. Let us recall that if $i \neq j$[1]

$$\mathbb{P}_i(J_h = j) = q_{ij}h + o(h),$$

and if $i = j$ then

$$\mathbb{P}_i(J_h = i) = 1 + q_{ii}h + o(h).$$

Therefore, up to $o(h)$ terms, we have the following. For $i \neq j$

$$\mathbb{E}_i\left[e^{\alpha X_h}, J_h = j\right] = \mathbb{E}_i\left[e^{\alpha(X_{T_1-} + U_{ij} + (X_h - X_{T_1}))}|J_h = j\right]\mathbb{P}_i(J_h = j) = \mathbb{E}_i[e^{\alpha U_{ij}}]q_{ij}h,$$

where $T_1 = \inf\{t > 0 : J_t = j\}$ and the last equations is due to $\mathbb{E}[e^{\alpha X_h^k}] = 1 + o(1)$ for every $k \in E$ and X^k is the Lévy process associated with the state k. For $i = j$ we have

$$\mathbb{E}_i\left[e^{\alpha X_h}, J_h = i\right] = \mathbb{E}_i\left[e^{\alpha X_h}|J_h = i\right]\mathbb{P}_i(J_h = i) = (1 + q_{ii}h)\mathbb{E}\left[e^{\alpha X_h^i}\right].$$

We know that $\mathbb{E}\left[e^{\alpha X_h^i}\right] = 1 + \psi_i(\alpha)h + o(h)$, thus

$$(1 + q_{ii}h)\mathbb{E}\left[e^{\alpha X_h^i}\right] = 1 + q_{ii}h + \psi_1(\alpha)h + o(h).$$

[1]The statement $f = o(g(h))$ means that $\lim_{h\to 0}\frac{f(h)}{g(h)} = 0$

Hence, we obtain up to $o(h)$ terms

$$\mathbb{E}_i\left[e^{\alpha X_h}, J_h = j\right] = 1_{\{i=j\}}(1 + q_{ii}h + \psi_i(\alpha)h))\mathbb{E}\left[e^{\alpha X_h^i}\right] + 1_{\{i\neq j\}}\mathbb{E}_i[e^{\alpha U_{ij}}]q_{ij}h,$$

and in the matrix form

$$\mathbb{E}\left[e^{\alpha X_h}, J_h|J_0\right] = \mathbf{I} + \boldsymbol{F}(\alpha)h,$$

where $\mathbf{I}$ denotes an identity matrix of appropriate size. The Markov property gives for $t > 0$

$$\mathbb{E}\left[e^{\alpha X_{t+h}}, J_{t+h}|J_0\right] = \mathbb{E}\left[e^{\alpha X_t}, J_t|J_0\right]\mathbb{E}\left[e^{\alpha X_h}, J_h|J_0\right],$$

thus

$$\frac{\partial}{\partial t}\mathbb{E}\left[e^{\alpha X_t}, J_t|J_0\right] = \mathbb{E}\left[e^{\alpha X_t}, J_t|J_0\right]\boldsymbol{F}(\alpha),$$

with $\mathbb{E}\left[e^{\alpha X_0}; J_0\right] = \mathbf{I}$. The above equation implies the result using the standard solution formula for the system of linear differential equations. □

Note that $\mathbf{F}(0)$ is the transition rate matrix of J, and hence a MAP is non-defective if and only if $\mathbf{F}(0)\vec{1} = \vec{0}$, where $\vec{0}$ and $\vec{1}$ denote the (column) vectors of 0s and 1s respectively (whereas the identity and the zero matrices are denoted by $\mathbf{I}$ and $\mathbf{0}$ respectively.)
In studying exit problems of spectrally negative MAPs, the so-called scale matrices play an essential role, which are defined analogously as the scale functions of spectrally negative Lévy processes. From Kyprianou and Palmowski [38], for $q \geq 0$, there exists a continuous, invertible matrix function $\boldsymbol{W}^{(q)} : [0,\infty) \to \mathbb{R}^{N\times N}$ such that for all $0 \leq x \leq a$,

$$\mathbb{E}_x\left[e^{-q\tau_a^+}, \tau_a^+ < \tau_0^-, J_{\tau_a^+}|J_0\right] = \boldsymbol{W}^{(q)}(x)\boldsymbol{W}^{(q)}(a)^{-1}. \tag{3.1}$$

Moreover, Ivanovs [27] and Ivanovs and Palmowski [29] showed that $\boldsymbol{W}^{(q)}$ can be characterized by

$$\widetilde{\mathbf{W}}^{(q)}(\alpha) = \left(\mathbf{F}(\alpha) - q\mathbf{I}\right)^{-1}, \quad \text{for large enough } \alpha, \tag{3.2}$$

where $\widetilde{f}(\alpha) = \int_0^\infty e^{-\alpha x} f(x)dx$ denotes the Laplace transform of the (matrix) function f. Furthermore, the domain of $\boldsymbol{W}^{(q)}$ can be extended to the negative half line by taking $\mathbf{W}^{(q)}(x) = \mathbf{0}$ for $x < 0$. The basis of the above transform lies on a probabilistic construction of the scale matrix $\mathbf{W}^{(q)}$, which involves the first hitting time at level x. Let $\mathbf{L}^q(x)$ denotes a matrix of expected occupation times at 0 up to the first passage time over x. In addition, the matrix $\mathbf{L}^q := \mathbf{L}^q(\infty)$ is the expected occupation density at 0. Then, $\mathbf{W}^{(q)}$ can be written as

$$\mathbf{W}^{(q)}(x) = e^{-\mathbf{\Lambda}^q x}\mathbf{L}^q(x),$$

with $\mathbf{\Lambda}^q$ being the transition rate matrix of the Markov chain $\{J_{\tau_x^+}\}_{x\geq 0}$. In other words, one has $\mathbb{P}(\tau_x^+ < \mathrm{e}_q, J_{\tau_x^+}) = e^{\mathbf{\Lambda}^q x}$ with e_q being an independent exponential random variable with rate $q > 0$. It is known that $\mathbf{L}^q$ has finite entries and is invertible unless the process is non-defective and $\boldsymbol{\pi}\mathbb{E}[X_1, J_1|J_0]\vec{1} = 0$ (see Ivanovs and Palmowski [29]). Hence, we have

$$\lim_{x\to\infty} e^{\mathbf{\Lambda}^q x}\mathbf{W}^{(q)}(x) = \lim_{x\to\infty}\mathbf{W}^{(q)}(x)e^{\mathbf{R}^q x} = \mathbf{L}^q, \tag{3.3}$$

where the matrix $\mathbf{R}^q := (\mathbf{L}^q)^{-1} \mathbf{\Lambda}^q \mathbf{L}^q$. Moreover, one can see that

$$\lim_{a\to\infty} \mathbf{W}^{(q)}(a)^{-1} = \mathbf{0},$$

since the expectation in (3.1) tends to $\mathbf{0}$ when $a \to \infty$. Therefore, from the above argument,

$$\lim_{x\to\infty} e^{\mathbf{\Lambda}^q x} = \lim_{x\to\infty} \mathbf{W}^{(q)}(x)^{-1}\mathbf{L}^q(x) = \mathbf{0}.$$

The second scale matrix $\mathbf{Z}^{(q)}$ is then defined through the $\mathbf{W}^{(q)}$ matrix function

$$\mathbf{Z}^{(q)}(x) = \mathbf{I} - \int_0^x \mathbf{W}^{(q)}(y)dy\,(\mathbf{F}(0) - q\mathbf{I}).$$

Note that $\mathbf{Z}^{(q)}(x)$ is continuous in x with $\mathbf{Z}^{(q)}(0) = \mathbf{I}$. Furthermore,

$$\lim_{x\to\infty} e^{\mathbf{\Lambda}^q x}\mathbf{Z}^{(q)}(x) = \int_0^\infty e^{\mathbf{\Lambda}^q z}dz\mathbf{L}^q\,(q\mathbf{I} - \mathbf{F}(0)).$$

Remark 3.1.2. *If the function is matrix-valued, it will always be bold in this chapter. It will also be the case for the constant matrices.*

Remark 3.1.3. *In the cases without exponential killing ($q = 0$), the upper subscript q will be omitted in the quantities mentioned above, which write as $\mathbf{W}(x), \mathbf{Z}(x), \mathbf{L}(x), \mathbf{\Lambda}$, etc.*

More details about the scale matrices, can be found in Ivanovs and Palmowski [29] and Ivanovs [28].

3.1.3 Markov modulated Brownian motion and its scale matrix

This part will consider a case when (X, J) is a Markov modulated Brownian motion (MMBM in shorthand). Some essential relations we will derive for later use in Section 3.5. Let X^i be the linear Brownian motion with variance $\sigma_i^2 > 0$ and drift μ_i for all $i \in E$. Further denote $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ as the (column) vectors of σ_i and μ_i, and $\Delta_{\boldsymbol{v}}$ as the diagonal matrix with $\boldsymbol{v}$ on the diagonal. Moreover, for every $i, j \in E$ we have that $U_{ij} = 0$. Therefore, the matrix Laplace exponent $\boldsymbol{F}(s)$ is given by

$$\boldsymbol{F}(s) = \frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 s^2 + \Delta_{\boldsymbol{\mu}} s + \boldsymbol{Q}.$$

Despite the case when $\kappa := \boldsymbol{\pi\mu} = 0$ and $q = 0$, Ivanovs [26] showed the representation of the q-scale matrix

$$\mathbf{W}^{(q)}(x) = \left(e^{-\mathbf{\Lambda}_q^+ x} - e^{\mathbf{\Lambda}_q^- x}\right)\Xi_q, \tag{3.4}$$

where $\Xi_q^{-1} = -\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2(\mathbf{\Lambda}_q^+ + \mathbf{\Lambda}_q^-)$ and $\mathbf{\Lambda}_q^\pm$ are the (unique) right solutions to the matrix integral equation $\boldsymbol{F}(\mp\mathbf{\Lambda}_q^\pm) = \mathbf{0}$, that is,

$$\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}(\mathbf{\Lambda}_q^\pm)^2 \mp \Delta_{\boldsymbol{\mu}}\mathbf{\Lambda}_q^\pm + \left(\boldsymbol{Q} - q\mathbf{I}\right) = \mathbf{0}. \tag{3.5}$$

In the next lemma, we present relations between $\mathbf{\Lambda}_q^+$ and $\mathbf{\Lambda}_q^-$.

Lemma 3.1.4. *For $q \geq 0$, we have*

$$\Delta_{\frac{2\boldsymbol{\mu}}{\boldsymbol{\sigma}^2}} = \boldsymbol{\Lambda}_q^+ - \boldsymbol{C}_q, \quad \boldsymbol{C}_q\boldsymbol{\Lambda}_q^+ = \Delta_{\frac{2}{\boldsymbol{\sigma}^2}}\Big[-\boldsymbol{Q} + q\boldsymbol{I}\Big], \tag{3.6}$$

and

$$\Delta_{\frac{2\boldsymbol{\mu}}{\boldsymbol{\sigma}^2}} = \boldsymbol{D}_q - \boldsymbol{\Lambda}_q^-, \quad \boldsymbol{D}_q\boldsymbol{\Lambda}_q^- = \Delta_{\frac{2}{\sigma^2}}\Big[-\boldsymbol{Q} + q\boldsymbol{I}\Big], \tag{3.7}$$

where

$$\boldsymbol{C}_q = (\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)\boldsymbol{\Lambda}_q^-(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)^{-1}, \quad \boldsymbol{D}_q = \Big(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-\Big)\boldsymbol{\Lambda}_q^+\Big(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-\Big)^{-1}.$$

Proof. Using equations (3.5) altogether, one can obtain

$$\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\Big((\boldsymbol{\Lambda}_q^+)^2 - (\boldsymbol{\Lambda}_q^-)^2\Big) = \Delta_{\boldsymbol{\mu}}\Big(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-\Big).$$

Hence, using $(\boldsymbol{\Lambda}_q^+)^2 - (\boldsymbol{\Lambda}_q^-)^2 = \boldsymbol{\Lambda}_q^+(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-) - (\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)\boldsymbol{\Lambda}_q^-$, we have

$$\Delta_{\frac{2\boldsymbol{\mu}}{\boldsymbol{\sigma}^2}} = \Big((\boldsymbol{\Lambda}_q^+)^2 - (\boldsymbol{\Lambda}_q^-)^2\Big)\Big(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-\Big)^{-1} = \boldsymbol{\Lambda}_q^+ - \boldsymbol{C}_q.$$

Now, the above relationship together with (3.5) gives that

$$\boldsymbol{C}_q\boldsymbol{\Lambda}_q^+ = \Delta_{\frac{2}{\boldsymbol{\sigma}^2}}\Big[-\boldsymbol{Q} + q\mathbf{I}\Big].$$

The remaining part of the proof can be done in a similar way by using

$$(\boldsymbol{\Lambda}_q^+)^2 - (\boldsymbol{\Lambda}_q^-)^2 = (\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)\boldsymbol{\Lambda}_q^+ - \boldsymbol{\Lambda}_q^-(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-).$$

□

In the special case of $q = 0$ we will write $\boldsymbol{\Lambda}^+$, $\boldsymbol{\Lambda}^-$, $\boldsymbol{C}$ and $\boldsymbol{D}$ for $\boldsymbol{\Lambda}_0^+$, $\boldsymbol{\Lambda}_0^-$, $\boldsymbol{C}_0$ and $\boldsymbol{D}_0$, respectively. Note that if (X, J) is the MMBM with single state (i.e., X is equal in law to the linear Brownian motion), we have, for $q \geq 0$,

$$\boldsymbol{\Lambda}_q^+ = -\rho_2, \quad \boldsymbol{\Lambda}_q^- = -\rho_1,$$

where $\rho_1 - \rho_2 = \frac{2\mu}{\sigma^2}$ and $\rho_1 + \rho_2 = \frac{2\sqrt{\mu^2+2q\sigma^2}}{\sigma^2}$. In general, for the MMBM, we can only calculate explicit analytical formulas for $\boldsymbol{W}^{(q)}(x)$, $\boldsymbol{\Lambda}_q^+$, and $\boldsymbol{\Lambda}_q^-$ for some special cases. For instance, consider the following parameters: $q > 0$,

$$\Delta_{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \Delta_{\boldsymbol{\mu}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{Q} = \begin{pmatrix} -q_{11} & q_{11} \\ q_{22} & -q_{22} \end{pmatrix}, \tag{3.8}$$

for σ_1, σ_2, q_{11}, $q_{22} > 0$. Then the matrix $\boldsymbol{F}(s) - q\mathbf{I}$ is of the form

$$\boldsymbol{F}(s) - q\mathbf{I} = \begin{pmatrix} \frac{\sigma_1^2}{2}s^2 - q_{11} - q & q_{11} \\ q_{22} & \frac{\sigma_2^2}{2}s^2 - q_{22} - q \end{pmatrix}.$$

Therefore, inversion of the Laplace transform (3.2) with respect to s gives

$$\boldsymbol{W}^{(q)}(x) = \begin{pmatrix} 2(q_{22}+q)-\alpha_2^2\sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11}+q)-\alpha_2^2\sigma_1^2 \end{pmatrix} \frac{e^{\alpha_2 x}-e^{-\alpha_2 x}}{(\alpha_1^2-\alpha_2^2)\alpha_2\sigma_1^2\sigma_2^2}$$
$$- \begin{pmatrix} 2(q_{22}+q)-\alpha_1^2\sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11}+q)-\alpha_1^2\sigma_1^2 \end{pmatrix} \frac{e^{\alpha_1 x}-e^{-\alpha_1 x}}{(\alpha_1^2-\alpha_2^2)\alpha_1\sigma_1^2\sigma_2^2}, \tag{3.9}$$

where

$$\alpha_1 = \frac{\sqrt{M_q+\sqrt{(M_q)^2-4\sigma_1^2\sigma_2^2K_q}}}{\sigma_1\sigma_2}, \quad \alpha_2 = \frac{\sqrt{M_q-\sqrt{(M_q)^2-4\sigma_1^2\sigma_2^2K_q}}}{\sigma_1\sigma_2},$$
$$M_q = \sigma_1^2(q_{22}+q)+\sigma_2^2(q_{11}+q), \quad K_q = (q_{11}+q_{22}+q)q.$$

It is straightforward that

$$\boldsymbol{W}^{(q)\prime}(x) = \begin{pmatrix} 2(q_{22}+q)-\alpha_2^2\sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11}+q)-\alpha_2^2\sigma_1^2 \end{pmatrix} \frac{e^{\alpha_2 x}+e^{-\alpha_2 x}}{(\alpha_1^2-\alpha_2^2)\sigma_1^2\sigma_2^2}$$
$$- \begin{pmatrix} 2(q_{22}+q)-\alpha_1^2\sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11}+q)-\alpha_1^2\sigma_1^2 \end{pmatrix} \frac{e^{\alpha_1 x}+e^{-\alpha_1 x}}{(\alpha_1^2-\alpha_2^2)\sigma_1^2\sigma_2^2}.$$

Likewise, one can be interested in the formulas of $\boldsymbol{\Lambda}_q^+$ and $\boldsymbol{\Lambda}_q^-$. First, note that $\boldsymbol{\Lambda}_q^+ = \boldsymbol{\Lambda}_q^-$ thanks to the assumption of $\mu_1=\mu_2=0$ and equation (3.5), thus (3.6) becomes

$$(\boldsymbol{\Lambda}_q^+)^2 = \Delta_{\frac{2}{\boldsymbol{\sigma}^2}}\left[-\boldsymbol{Q}+q\mathbf{I}\right].$$

Since $-\alpha_1$ and $-\alpha_2$ are eigenvalues of $\boldsymbol{\Lambda}_q^+$, thus after some basic algebra, one reaches that

$$\boldsymbol{\Lambda}_q^+ = \boldsymbol{\Lambda}_q^- = \begin{pmatrix} \frac{-\sqrt{2\sigma_2^2(\alpha_1+\alpha_2)^2(q_{11}+q)-4q_{11}q_{22}}}{\sigma_1\sigma_2} & \frac{2q_{11}}{\sigma_1^2} \\ \frac{2q_{22}}{\sigma_2^2} & \frac{-\sqrt{2\sigma_1^2(\alpha_1+\alpha_2)^2(q_{22}+q)-4q_{11}q_{22}}}{\sigma_1\sigma_2} \end{pmatrix} \frac{1}{\alpha_1+\alpha_2}.$$

Finally, we will provide a graphical example of the scale matrix. Consider the following set of the parameters

$$\Delta_{\boldsymbol{\sigma}} = \begin{pmatrix} 1 & 0 \\ 0 & 1.2 \end{pmatrix}, \quad \Delta_{\boldsymbol{\mu}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} -0.05 & 0.05 \\ 0.1 & -0.1 \end{pmatrix}, \quad \text{and} \quad q = 0.05.$$

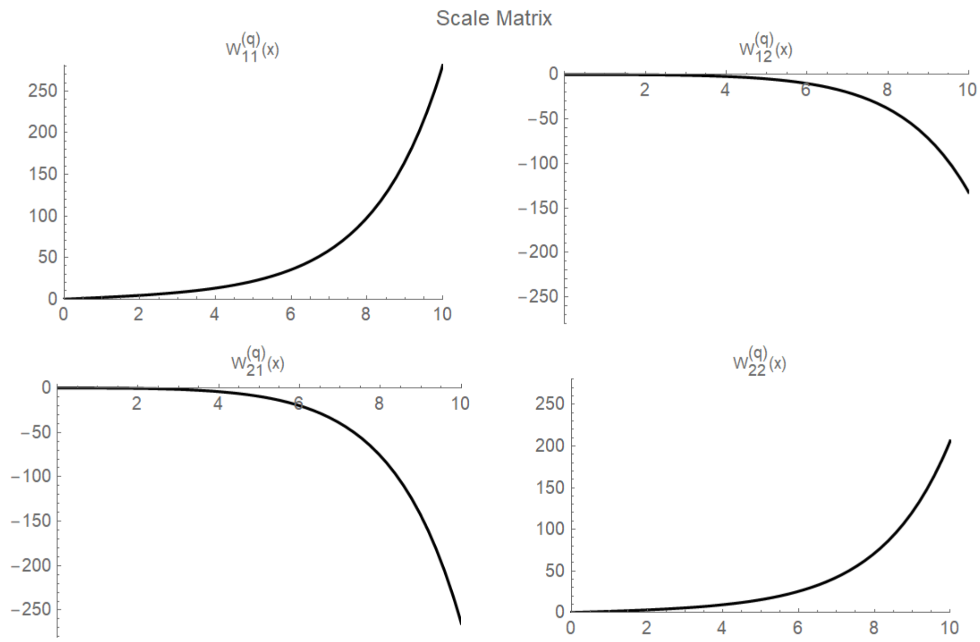

Figure 3.2: Entries of scale matrix function $\boldsymbol{W}^{(q)}$

Using the formula (3.9), the scale matrix $\boldsymbol{W}^{(q)}$ is plotted in Figure 3.2. We can see that this matrix's diagonal cells have the same shape as the one-dimensional scale functions, where off-diagonal ones are reflected in shape. In the following examples, the plots of ω-matrices are provided to be compared with these traditional ones.

3.2 ω-killing

3.2.1 Definition

Now, we will proceed to the so-called ω-killing. As we mentioned in the introduction to this chapter, the concept was introduced by Li and Palmowski [41] for the spectrally negative Lévy process. Let us recall our motivation from (1.2) when we introduced the concept of the two-sided exit problem. We said that the following expectation for $0 \le x \le c$

$$\mathbb{E}_x\left[e^{-q\tau_c^+}1_{\{\tau_c^+<\tau_0^-\}}\right], \tag{3.10}$$

can be seen as an unit payment when the process reaches level c, but only if we do not reach level 0 before. We discount this payment with the factor of $q > 0$. Now, the problem arises when the discounting structure is not flat. For example, one can assume that the discounting factor depends on the position of the process. Therefore, roughly speaking, one would like to consider

$$\mathbb{E}_x\left[e^{-\int_0^{\tau_c^+} q(X_s)ds}1_{\{\tau_c^+<\tau_0^-\}}\right],$$

where now q is a function of the position of the process. In the context of the Markov additive processes, one can also assume that the discounting structure depends on the state of J. Therefore, all the above leads us to the following definition of the ω function.

Definition 3.2.1. *Let $\omega : E \times \mathbb{R} \to \mathbb{R}^+$ be a function defined as $\omega(i,x) = \omega_i(x)$, where for a fixed $i \in E$, $\omega_i : \mathbb{R} \to \mathbb{R}^+$ is a bounded, nonnegative measurable function and its value formulates the matrix $\boldsymbol{\omega}(x) := diag(\omega_1(x), \ldots, \omega_N(x))$. Let $\lambda > 0$ be the upper bound of $|\omega_i(x)|$ on $[0,\infty)$ for all $i \in E$.*

Let us note that in our setting, we assume that ω is bounded, differently from Li and Palmowski [41] where the respective function was locally bounded. We have addressed one point of view on the two-sided exit problems. Let us recall the second one, related to the exponential killing. One can see (3.10) as a probability that the process will not be exponentially killed before it reaches level c but before going down to level 0. Now, we would like to extend this into the context of the ω function. Let us define the following stopping time

$$\tau_\omega := \inf\{t \geq 0 : \int_0^t \omega_{J_s}(X_s)ds > e_1\}, \tag{3.11}$$

where e_1 is an independent exponential random variable with parameter 1 and the function ω satisfy Definition 3.2.1. If one set $X_t = \vartheta$ for $t > \tau_\omega$ then we have a new way of killing the stochastic process. When one sets $\omega \equiv q$, then ω-killing is the same as exponential killing. Now, we can use this concept in terms of the Omega model. Traditionally, in this model, one assumes that the process gets penalties when it is in the so-called *red-zone*, and it is bankrupt if the penalties are too big. Moreover, we assume that if the process crosses a specific level, it is bankrupted immediately. Usually, in the literature, the *red-zone* is of the form $[-d, 0]$ where $-d$ is the level of immediate bankruptcy. Therefore, a small modification of (3.11) leads to

$$\tau_\omega^d := \inf\{t \geq 0 : \int_0^t \omega_{J_s}(X_s)ds > e_1 \vee X_t < -d\}, \tag{3.12}$$

where $d \in \mathbb{R}$. To be consistent with the literature, one needs to set $d > 0$, and we will do the same in further examples. However, from a theoretical point of view, there is no need to impose such restrictions in the general sense. Now, one can observe that the following exit problem

$$\mathbb{E}_x \left[e^{-\int_0^{\tau_c^+} \omega_{J_s}(X_s)ds} \mathbb{1}_{\{\tau_c^+ < \tau_{-d}^-\}} \right],$$

is the unit payment due to reaching level c before being ruined. Moreover, to incorporate discounting one need to modify ω function to $\omega^q(\cdot,\cdot) \equiv \omega(\cdot,\cdot) + q$, for some $q > 0$. Therefore, there is a natural need to solve such exit problems. However, the Omega model is only a motivation. Such exit problems can be used in many different settings.

3.2.2 Exit problems for ω-killing and spectrally negative Lévy processes

We will start the topic of exit problems for ω-killing in the case of the spectrally negative Lévy process. As we mentioned, it can be seen as a MAP with state space J that has only one element. In this part, we would like to present some results from the work from Li and Palmowski [41] where this model was considered. Mainly, the authors considered two types of exit problems: type A and type B. The first is when one is interested in crossing intervals by the upper boundary and later by the lower. We consider both before ruin/bankruptcy time. The authors showed that the following functions for $x \in [0, c]$

$$\mathcal{A}(x,c) := \mathbb{E}_x \left[e^{-\int_0^{\tau_c^+} \omega(X_s)ds}, \tau_c^+ < \tau_0^- \right],$$

$$\mathcal{B}(x,c) := \mathbb{E}_x \left[e^{-\int_0^{\tau_0^-} \omega(X_s)ds}, \tau_0^- < \tau_c^+ \right],$$

can be characterized by the two families of functions $\{\mathcal{W}^{(\omega)}(x), x \in \mathbb{R}\}$ and $\{\mathcal{Z}^{(\omega)}(x), x \in \mathbb{R}\}$ defined as uniquely solutions to the following equations

$$\mathcal{W}^{(\omega)}(x) = W(x) + \int_0^x W(x-y)\omega(y)\mathcal{W}^{(\omega)}(y)dy,$$
$$\mathcal{Z}^{(\omega)}(x) = 1 + \int_0^x W(x-y)\omega(y)\mathcal{Z}^{(\omega)}(y)dy.$$

The authors showed that

$$\mathcal{A}(x,c) = \frac{\mathcal{W}^{(\omega)}(x)}{\mathcal{W}^{(\omega)}(c)},$$
$$\mathcal{B}(x,c) = \mathcal{Z}^{(\omega)}(x) - \frac{\mathcal{W}^{(\omega)}(x)}{\mathcal{W}^{(\omega)}(c)}\mathcal{Z}^{(\omega)}(c).$$

Moreover, the authors also gave a more generalised scale function to allow for shifting. Namely, they defined functions $\mathcal{W}^{(\omega)}(x,y)$ and $\mathcal{Z}^{(\omega)}(x,y)$ on $\mathbb{R}\times\mathbb{R}$ being the solutions of following equations

$$\mathcal{W}^{(\omega)}(x,y) = W(x-y) + \int_y^x W(x-z)\omega(y)\mathcal{W}^{(\omega)}(z,y)dz \tag{3.13}$$

and

$$\mathcal{Z}^{(\omega)}(x,y) = 1 + \int_y^x W(x-z)\omega(z)\mathcal{Z}^{(\omega)}(z,y)dz.$$

Thanks to this the authors got that for $-\infty < z \leq x \leq y < \infty$

$$\mathbb{E}_x \left[e^{-\int_0^{\tau_y^+} \omega(X_s)ds}, \tau_y^+ < \tau_z^- \right] = \frac{\mathcal{W}^{(\omega)}(x,z)}{\mathcal{W}^{(\omega)}(y,z)},$$
$$\mathbb{E}_x \left[e^{-\int_0^{\tau_z^-} \omega(X_s)ds}, \tau_z^- < \tau_y^+ \right] = \mathcal{Z}^{(\omega)}(x,z) - \frac{\mathcal{W}^{(\omega)}(x,z)}{\mathcal{W}^{(\omega)}(y,z)}\mathcal{Z}^{(\omega)}(y,z).$$

In case of one-sided exit problems, they got, among other results, that for fixed level $d > 0$ and $x \geq -d$

$$\mathbb{E}_x \left[e^{\int_0^\infty \omega(s)ds}; \tau_{-d}^- = \infty \right] = c_{\mathcal{W}^{-1}(\infty,-d)}\mathcal{W}^{(\omega)}(x,-d), \tag{3.14}$$

where $c_{\mathcal{W}^{-1}(\infty,-d)} = \left[\lim_{c\to\infty} \mathcal{W}^{(\omega)}(c,-d) \right]^{-1}$.

3.2.3 Probability of Omega bankruptcy for Crámer-Lundberg process

One can use representations for ω-killed exit problems to obtain some results related to the Omega model or different applications. In this section, we will give a solution to the probability of Omega bankruptcy, which is the ruin time in the Omega model. The result is interesting on its own but also shows how one can work with the ω-killed exit problems. We will use the Crámer-Lundberg process with exponential claims as the underlying process. Recall from Section 1.4.2 that we define this process such that for every $t \geq 0$

$$X_t = x + p_t - \sum_{i=1}^{N_t} U_i,$$

where $x \in \mathbb{R}, p > 0$, $\{U_i\}_{i=1}^{\infty}$ is an *i.i.d.* sequence of exponential random variables with the parameter $\mu > 0$, and $\{N_t\}_{t \geq 0}$ is a homogeneous Poisson process with the intensity $\lambda > 0$. We also assume that the Poisson process and the exponential random variables are mutually independent. Before proceeding to the results, we would like to mention that the below calculations are done in the same manner as in Li and Palmowski [41] when the linear Brownian motion was an underlying process. The authors achieved the formula for the probability of bankruptcy in the Omega model and showed that this probability is a linear function of classical ruin probability. As shown below, we get a similar result. Therefore, let us proceed with the calculations.
We know that the Laplace exponent for the Cramér-Lundberg process with exponential claims is equal to

$$\psi(\alpha) = p\alpha - \frac{\lambda\alpha}{\mu + \alpha}.$$

Thus one can get the formula for $W^{(q)}$, namely

$$W^{(q)}(x) = \frac{1}{p}\Big(A^{+}e^{q^{+}x} - A^{-}e^{q^{-}x}\Big),$$

where

$$A^{\pm} = \frac{\mu + q^{\pm}}{q^{+} - q^{-}}, \quad q^{\pm} = \frac{q + \lambda - \mu p \pm \sqrt{(q + \lambda - \mu p)^2 + 4pq\mu}}{2p}.$$

From (1.3) we know that

$$\varphi(x) = \mathbb{P}_x(\tau_0^- < \infty) = 1 - \psi'(0+)W(x),$$

if $0 < \psi'(0+) = p - \frac{\lambda}{\mu}$. Note that this assumption is the well known net profit condition. In the case of this process, this is equivalent to the fact that drift is strictly positive. We will also consider this assumption here. From the above, one can see that we need to calculate the formula for the (0)-scale function. Therefore, note that

$$q^{+} = \frac{\lambda - \mu p}{p}, \quad q^{-} = 0,$$

thus

$$W(x) = \frac{1}{\lambda - \mu p}\left(\frac{\lambda}{p}e^{\left(\frac{\lambda - \mu p}{p}\right)x} - \mu\right).$$

Therefore, we can go back to the representation of the probability of classical ruin time

$$\varphi(x) = 1 - \left(p - \frac{\lambda}{\mu}\right) W(x) = \frac{\lambda}{\mu p} e^{\left(\frac{\lambda - \mu p}{p}\right)x}. \tag{3.15}$$

Let us consider the Omega ruin time, defined in (3.12), as a bankruptcy time in this model. At first, we would like to proceed to general calculations with some restrictions. Assume that the function ω satisfies the following conditions for $-d < 0$

- $\omega(x) \geq 0$ for $x \in [-d, 0]$ and zero otherwise,
- $\omega(x)$ is differentiable continuously function on $[-d, 0]$, where at the ending points, we use left and right derivative/limit, respectively.

Recall from (3.13) that (with $y = -d$)

$$\mathcal{W}^{(\omega)}(x, -d) = W(x+d) + \int_0^{x+d} W(x+d-y)\omega(y-d)\mathcal{W}^{(\omega)}(y-d, -d)dy. \tag{3.16}$$

Moreover, we are only interested in $x \geq -d$ as below $-d$ the process is killed. The following proposition will give us the possibility for numerical calculations for the above scale function.

Proposition 3.2.2. *Function $\mathcal{W}^{(\omega)}$ satisfy the following differential equation for $x \in [-d, 0]$*

$$p\mathcal{W}^{(\omega)''}(x, -d) - \Big[\omega(x) + (\lambda - \mu p)\Big]\mathcal{W}^{(\omega)'}(x, -d) - \Big[\mu\omega(x) + \omega^{'}(x)\Big]\mathcal{W}^{(\omega)}(x, -d) = 0,$$

with $\mathcal{W}^{(\omega)}(-d, -d) = \frac{1}{p}, \quad \mathcal{W}^{(\omega)'}(-d, -d) = \frac{\lambda + \omega(-d)}{p^2}.$

Proof. Take $z = x + d \geq 0$ and denote $g(z) := \mathcal{W}^{(\omega)}(z - d, -d) = \mathcal{W}^{(\omega)}(x, -d)$. Then from (3.16), we have that

$$g(z) = W(z) + \int_0^z W(z-y)\omega(y-d)g(y)dy. \tag{3.17}$$

Observe, since $q^- = 0$, that

$$\left(\frac{d}{dz} - q^+\right)\frac{d}{dz}W(x) = 0,$$

and

$$\left(\frac{d}{dz} - q^+\right)\frac{d}{dz}g(z) = \frac{1}{p}\left[\mu\omega(z-d)g(z) + \omega^{'}(z-d)g(z) + \omega(z-d)g^{'}(z)\right],$$

thus

$$pg^{''}(z) - \Big[\omega(z-d) + (\lambda - \mu p)\Big]g^{'}(z) - \Big[\mu\omega(z-d) + \omega^{'}(z-d)\Big]g(z) = 0,$$

with the initial values $g(0) = \frac{1}{p}$ and $g^{'}(0) = \frac{\lambda + \omega(-d)}{p^2}$. To end this proof, one must go back to the x-domain. □

Second proposition will be related to the probability of the Omega bankruptcy time. Note that the probability of Omega bankruptcy is equal to

$$\varphi^{(\omega)}(x) = \mathbb{P}_x\left(\tau_\omega^d < \infty\right) = 1 - \mathbb{E}_x\left[e^{-\int_0^\infty \omega(s)ds}; \tau_{-d}^- = \infty\right],$$

and from (3.14) we know that for $x \geq -d$

$$\mathbb{E}_x\left[e^{-\int_0^\infty \omega(s)ds}; \tau_{-d}^- = \infty\right] = c_{\mathcal{W}^{-1}(\infty,-d)}\mathcal{W}^{(\omega)}(x,-d),$$

with $c_{\mathcal{W}^{-1}(\infty,-d)} = \left[\lim_{c\to\infty} \mathcal{W}^{(\omega)}(c,-d)\right]^{-1}$.

Proposition 3.2.3. *Function $\varphi^{(\omega)}(x)$ is given by*

$$\varphi^{(\omega)}(x) := \varphi(x)\frac{\mu p^2}{\lambda(\mu p - \lambda)}c_{\mathcal{W}^{-1}(\infty,-d)}\mathcal{W}^{(\omega)'}(0,-d), \quad \textit{for } x \geq 0,$$

where as before $\varphi(x) = \mathbb{P}_x\left(\tau_0^- < \infty\right) = \frac{\lambda}{\mu p}e^{\frac{\lambda-\mu p}{p}x}$ is the classical ruin probability.

Proof. First, we make again substitution $z = x + d$ and from (3.17) and the fact that $\omega(z) = 0$ for $z > d$ one can get the following

$$g^{'}(z) = \frac{\lambda}{p^2}e^{(\frac{\lambda-\mu p}{p})z}\left[1 + \int_0^d e^{-(\frac{\lambda-\mu p}{p})y}\omega(y-d)g(y)dy\right],$$

and

$$g(z) = g(d) + \frac{p}{\mu p - \lambda}\left[1 - e^{\frac{\lambda-\mu p}{p}(z-d)}\right]g^{'}(d).$$

Thus, when we get back to the x-domain using $g(z) = \mathcal{W}^{(\omega)}(x,-d)$, then above equation gives

$$\mathcal{W}(x,-d) = \mathcal{W}^{(\omega)}(0,-d) + \frac{p}{\mu p - \lambda}\left[1 - e^{\frac{\lambda-\mu p}{p}x}\right]\mathcal{W}^{(\omega)'}(0,-d).$$

Hence,

$$c_{\mathcal{W}^{-1}(\infty,-d)} = \frac{1}{\mathcal{W}^{(\omega)}(0,-d) + \frac{p}{\mu p-\lambda}\mathcal{W}^{(\omega)'}(0,-d)}.$$

Therefore

$$\begin{aligned}\varphi^{(\omega)}(x) =& 1 - c_{\mathcal{W}^{-1}(\infty,-d)}\mathcal{W}^{(\omega)}(x,-d) = \frac{p}{\mu p - \lambda}\left[e^{\frac{\lambda-\mu p}{p}x}\right]\mathcal{W}^{(\omega)'}(0,-d)c_{\mathcal{W}^{-1}(\infty,-d)} \\ =& \varphi(x)\frac{\mu p^2}{\lambda(\mu p-\lambda)}\mathcal{W}^{(\omega)'}(0,-d)c_{\mathcal{W}^{-1}(\infty,-d)}.\end{aligned} \tag{3.18}$$

□

Note, that we still need to calculate $\mathcal{W}^{(\omega)}(0,-d)$ and $\mathcal{W}^{(\omega)'}(0,-d)$. We will use numerical methods to approximate them. The main question here is how $\varphi^{(\omega)}(x)$ is related to the probability of the classical ruin time. In particular if $\omega(x) \equiv 0$ then

$$\varphi^{(\omega)}(x-d) = \varphi(x).$$

This is due to the translation of process X by the constant d (more precisely, due to the spatial homogeneity of the process). Observe that

$$\varphi(x+d) \leq \varphi^{(\omega)}(x) \leq \varphi(x). \tag{3.19}$$

We aim to show these inequalities in the picture, but we need to fix the ω function to give numerical examples. We will consider the shape of the ω function mentioned in the introductory section. Namely, let us take

$$\omega(x) = \Big[\gamma_0 + \gamma_1(x+d)\Big]\mathbf{1}_{\{x\in[-d,0]\}}.$$

As we mention before, we need to calculate $\mathcal{W}^{(\omega)}(0,-d)$ and $\mathcal{W}^{(\omega)'}(0,-d)$ with the use of numerical methods. Note that if we set

$$\gamma_0 = -\gamma_1 d,$$

then ω will be continuous at zero. It gives us a more reasonable interpretation because the penalty will decrease continuously to zero. Thus,

$$\omega(x) = \gamma_1 x \mathbf{1}_{\{x\in[-d,0]\}}.$$

It is straightforward that $\gamma_1 \leq 0$ because we have an assumption that $\omega(x) \geq 0$ for all x. For such the model, we have from Proposition 3.2.2 the following differential equation. For $x \in [-d,0]$ it holds

$$p\mathcal{W}^{(\omega)''}(x,-d) - \Big[\gamma_1 x + (\lambda - \mu p)\Big]\mathcal{W}^{(\omega)'}(x,-d) - \gamma_1\Big[\mu x + 1\Big]\mathcal{W}^{(\omega)}(x,-d) = 0, \tag{3.20}$$

with the initial values $\mathcal{W}^{(\omega)}(-d;-d) = \frac{1}{p}$ and $\mathcal{W}^{(\omega)'}(-d;-d) = \frac{\lambda-\gamma_1 d}{p^2}$.
Before we proceed to numerical examples, let us recall the basic procedure for dealing with such differential equations using numerical methods. For reference see e.g. Burden and Faires [10].
At the beginning we can set $f(x) := \mathcal{W}^{(\omega)}(x;-d)$ and $h(x) := \mathcal{W}^{(\omega)'}(x;-d)$ for $x \in [-d,0]$. Then (3.20) became

$$\begin{cases} \frac{df}{dx} = h(x), \\ \frac{dh}{dx} = \frac{(\gamma_1 x + (\lambda-\mu p))}{p}h(x) + \frac{\gamma_1(\mu x+1)}{p}f(x), \end{cases}$$

with $f(-d) = \frac{1}{p}$ and $h(-d) = \frac{\lambda-\gamma_1 d}{p^2}$. On the interval of interest, this system has a unique solution h and f due to the Lipschitz condition with respect to dependent variables and continuity. Therefore, we can easily obtain an approximation for $\mathcal{W}^{(\omega)}(0,-d)$ and $\mathcal{W}^{(\omega)'}(0,-d)$ using some of iterative methods (e.g. Runge-Kutta methods).
Now, we are ready to give a picture which shows relations between probabilities in the inequalities (3.19). Thus, let us fix the following

$$\lambda = 1, \quad \mu = 1, \quad \gamma_1 = -0.2, \quad d = 3, \quad p = 1.25,$$

and consider the picture below:

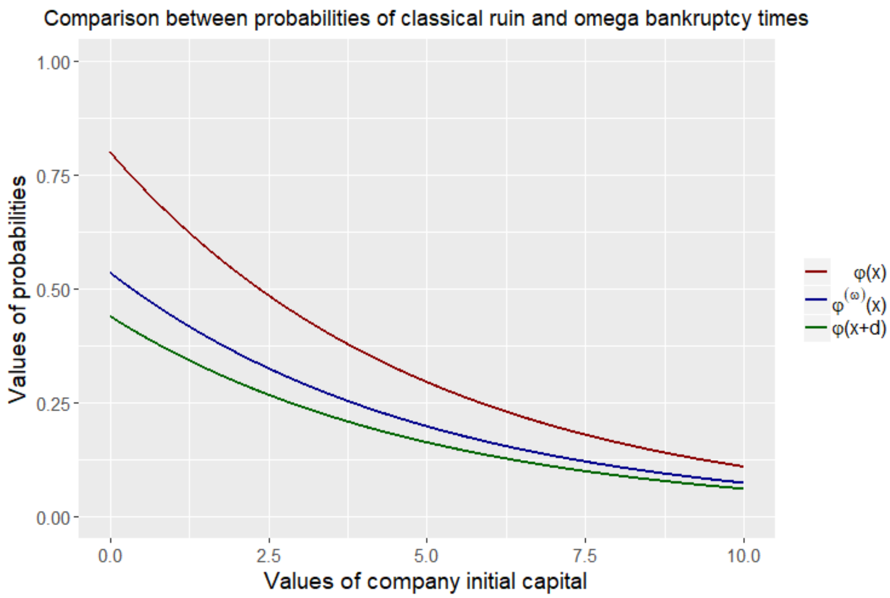

Figure 3.3: Comparison between $\varphi(x)$, $\varphi^{(\omega)}(x)$ and $\varphi(x+d)$

Above, we can see the relations between these three probabilities and the trivial observation that if we increase capital, probabilities become smaller exponentially fast. Note that if we increase the values of the penalty function ω, the probability of Omega bankruptcy time will become closer to the classical ruin time. If we behave conversely, we will be close to the $\varphi(x+d)$.

3.3 Exit problems for MAP and ω-killing

3.3.1 ω-scale matrices

Before presenting our main results, we shall devote a little time to establish some necessary notations. Our main aim is to represent the fluctuation identities for MAPs with ω-killing in terms of new ω-scale matrices defined as the unique solutions to the following equations

$$\mathcal{W}^{(\omega)}(x) = \mathbf{W}(x) + \mathbf{W} * \left(\boldsymbol{\omega}\mathcal{W}^{(\omega)}\right)(x), \tag{3.21}$$
$$\mathcal{Z}^{(\omega)}(x) = \mathbf{I} + \mathbf{W} * \left(\boldsymbol{\omega}\mathcal{Z}^{(\omega)}\right)(x),$$

where $f * g(x) = \int_0^x f(x-y)g(y)dy$ denotes the convolution of two matrix functions f and g. The following lemma shows that the above ω-scale matrices $\mathcal{W}^{(\omega)}$ and $\mathcal{Z}^{(\omega)}$ are well-defined and exist uniquely.

Lemma 3.3.1. *For every $i, j \in E$, let us assume that h_{ij} is a locally bounded function and ω_i is a bounded function on $\mathbb{R}$. There exists a unique solution to the following equation*

$$\mathbf{H}(x) = \mathbf{h}(x) + \mathbf{W} * (\boldsymbol{\omega}\mathbf{H})(x), \tag{3.22}$$

where $\mathbf{H}(x) = \mathbf{h}(x)$ *for* $x < 0$. *Furthermore, for any fixed* $\delta > 0$, $\mathbf{H}$ *satisfies (3.22) if and only if* $\mathbf{H}$ *satisfies*

$$\mathbf{H}(x) = \mathbf{h}_\delta(x) + \mathbf{W}^{(\delta)} * ((\boldsymbol{\omega} - \delta\mathbf{I})\mathbf{H})(x), \tag{3.23}$$

where $\mathbf{h}_\delta(x) = \mathbf{h}(x) + \delta\mathbf{W}^{(\delta)} * \mathbf{h}(x)$.

Proof. To prove the uniqueness of the solution, we will show that $\mathbf{H}(x) = \mathbf{0}$ is the only solution to

$$\mathbf{H}(x) = \int_0^x \mathbf{W}(x-y)\boldsymbol{\omega}(y)\mathbf{H}(y)dy. \tag{3.24}$$

Taking the Laplace transform on both sides of (3.24) (with an argument s_0), we get

$$\widetilde{\mathbf{H}}(s_0) = \widetilde{\mathbf{W}}(s_0)\,\widetilde{\boldsymbol{\omega}\mathbf{H}}(s_0).$$

Recall that λ is the upper bound of $|\omega_i(y)|$ on $[0,\infty)$ for all $i \in E$. Using (3.2), we obtain that the matrix norm of $\widetilde{\mathbf{H}}(s_0)$ fulfills the inequality

$$\|\widetilde{\mathbf{H}}(s_0)\| \le \lambda\|\widetilde{\mathbf{W}}(s_0)\|\|\widetilde{\mathbf{H}}(s_0)\| = \lambda\|\mathbf{F}^{-1}(s_0)\|\|\widetilde{\mathbf{H}}(s_0)\|. \tag{3.25}$$

Next, we will show that there exists s_0 such that

$$\|\mathbf{F}(s)^{-1}\| < \frac{1}{2\lambda}, \qquad \text{for all } s \ge s_0. \tag{3.26}$$

To do so, we recall the expression for $\mathbf{F}(\alpha)$:

$$\mathbf{F}(\alpha) = \mathrm{diag}(\psi_1(\alpha), \dots, \psi_N(\alpha)) + \mathbf{Q} \circ \mathbb{E}(e^{\alpha U_{ij}})_{i,j\in E}.$$

Observe that its diagonal goes to infinity, as α goes to infinity, and each element (entry-wise) other than the diagonal is bounded by the (fixed) q_{ij}.
We now prove that using the induction argument with respect to the dimension of $\mathbf{F}(\alpha)$,

$$\mathbf{F}^{-1}(\alpha) \to \mathbf{0}, \text{ as } \alpha \to \infty.$$

Define a series sub-matrices of $\mathbf{F}(\alpha)$, for $m = 1, 2, \dots, N$,

$$\mathbf{F}_m(\alpha)^{-1} := \mathbf{F}(\alpha)^{-1}_{m\times m} = \left(\{F_{ij}(\alpha)\}_{i,j=1}^m\right)^{-1},$$

and in what follows, we will show that

$$\mathbf{F}_m^{-1}(\alpha) \to \mathbf{0}_{m\times m}, \text{ as } \alpha \to \infty. \tag{3.27}$$

Clearly, $\mathbf{F}_N(\alpha)^{-1} = \mathbf{F}(\alpha)^{-1}$.
When $m = 1$, $\mathbf{F}_1(\alpha)^{-1} = \frac{1}{\psi_1(s_0)+q_{11}}$, which makes (3.27) hold obviously, and s_0 in (3.26) is chosen such that $\frac{1}{\psi_1(s_0)+q_{11}} < \frac{1}{2\lambda}$. Assume (3.27) holds for the dimension $m = k-1$. Then in the dimension $m = k$, we have

$$\mathbf{F}_k(\alpha)^{-1} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1},$$

where

$$\mathbf{A}_{(k-1)\times(k-1)} = \mathbf{F}_{k-1}(\alpha),$$
$$\mathbf{B}_{(k-1)\times 1} = (q_{1k}\mathbb{E}(e^{\alpha U_{1k}}), \dots, q_{(k-1)k}\mathbb{E}(e^{\alpha U_{(k-1)k}}))^T,$$
$$\mathbf{C}_{1\times(k-1)} = (q_{k1}\mathbb{E}(e^{\alpha U_{k1}}), \dots, q_{k(k-1)}\mathbb{E}(e^{\alpha U_{k(k-1)}})),$$

and

$$\mathbf{D}_{1\times 1} = \psi_k(\alpha) - q_{kk}.$$

Using the property for the inverse of the block matrix

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{pmatrix},$$

it is clear that each block for $\mathbf{F}_k(\alpha)^{-1}$ goes to $\mathbf{0}$ as $\alpha \to \infty$, since

$$\mathbf{A}^{-1} = \mathbf{F}_{k-1}(\alpha)^{-1} \to \mathbf{0}_{(k-1)\times(k-1)},$$

$$(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} = \frac{1}{\psi_k(\alpha) - q_{kk} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}} \to 0,$$

and $\mathbf{B}$, $\mathbf{C}$ have bounded (non-negative) elements. This completes the proof of (3.26). Plugging (3.26) into (3.25) gives

$$\|\widetilde{\mathbf{H}}(s_0)\| = 0, \text{ i.e., } \mathbf{H}(x) = \mathbf{0},$$

which completes the proof of uniqueness of the solution of Equation (3.22).
To prove the existence of the solution of Equation (3.22), we construct a series of matrices $\{\mathbf{H}_m\}$, which converge to the unique solution. Define the operator $\mathcal{G}$ on a matrix, for $z > 0$,

$$\mathcal{G}\widetilde{\mathbf{K}}(z) := \int_0^\infty e^{-zx} \int_0^x e^{-s_0(x-y)}\mathbf{W}(x-y)\boldsymbol{\omega}(y)\mathbf{K}(y)dydx = \widetilde{\mathbf{W}}(s_0+z)\ \widetilde{\boldsymbol{\omega}\mathbf{K}}(z).$$

Then,

$$\mathcal{G}^{(m+1)}\widetilde{\mathbf{K}}(z) := \mathcal{G}(\mathcal{G}^{(m)})\widetilde{\mathbf{K}}(z),$$
$$\widetilde{\mathbf{H}}_0(z) := \int_0^\infty e^{-zx}e^{-s_0x}\mathbf{h}(x)dx = \widetilde{\mathbf{h}}_0(s_0+z),$$
$$\widetilde{\mathbf{H}}_{m+1}(z) := \widetilde{\mathbf{H}}_0(z) + \mathcal{G}\widetilde{\mathbf{H}}_m(z).$$

$\mathcal{G}$ is a linear operator such that $\|\mathcal{G}\widetilde{\mathbf{K}}(z)\| < \frac{1}{2}\|\widetilde{\mathbf{K}}(z)\|$ for $z > 0$. Therefore, for $m > l$, we have

$$\|\widetilde{\mathbf{H}}_m(z) - \widetilde{\mathbf{H}}_l(z)\| = \|\sum_{k=l+1}^{m} \mathcal{G}^{(k)}\widetilde{\mathbf{H}}_0(z)\| < 2^{-l}\|\widetilde{\mathbf{H}}_0(z)\|,$$

which means $\{\widetilde{\mathbf{H}}_m(z), z > 0\}_{m\geq 0}$ forms a Cauchy sequence (entry-wise) that admits a limit $\widetilde{\mathfrak{H}}(z)$ for any $z > 0$ satisfying

$$\widetilde{\mathfrak{H}}(z) = \widetilde{\mathbf{H}}_0(z) + \mathcal{G}\widetilde{\mathfrak{H}}(z) = \widetilde{\mathbf{h}}_0(s_0+z) + \widetilde{\mathbf{W}}(s_0+z)\ \widetilde{\boldsymbol{\omega}\mathfrak{H}}(z).$$

Using the uniqueness of Laplace transform, we have

$$\mathfrak{H}(x) = e^{-s_0 x}\mathbf{h}_0(x) + \int_0^x e^{-s_0(x-y)}\mathbf{W}(x-y)\boldsymbol{\omega}(y)\mathfrak{H}(y)dy,$$

which shows that $\mathbf{H}(x) = e^{s_0 x}\mathfrak{H}(x)$ is the solution to (3.22).
As for the second statement in this lemma, we see that if $\mathbf{H}$ satisfies (3.23), by letting $\delta = 0$, we obtain (3.22) immediately. Now we only need to show that if $\mathbf{H}$ is the solution to (3.22), it is also the solution to (3.23). We convolute both sides of (3.22) with $\delta\mathbf{W}^{(\delta)}$ (on the left),

$$\begin{aligned}\delta\mathbf{W}^{(\delta)} * \mathbf{H}(x) &= \delta\mathbf{W}^{(\delta)} * \mathbf{h}(x) + \delta\mathbf{W}^{(\delta)} * \mathbf{W} * (\boldsymbol{\omega}\mathbf{H})(x) \\ &= \delta\mathbf{W}^{(\delta)} * \mathbf{h}(x) + (\mathbf{W}^{(\delta)} - \mathbf{W}) * (\boldsymbol{\omega}\mathbf{H})(x),\end{aligned}$$

where in the last step we used the identity $\mathbf{W}^{(\delta)} - \mathbf{W} = \delta\mathbf{W}^{(\delta)} * \mathbf{W}$ (which can be seen from the Laplace transform). Therefore,

$$\mathbf{H}(x) = \mathbf{h}(x) + \delta\mathbf{W}^{(\delta)} * \mathbf{h}(x) + \mathbf{W}^{(\delta)} * ((\boldsymbol{\omega} - \delta\mathbf{I})\mathbf{H})(x),$$

which completes the proof.

□

We further introduce more general scale matrices $\mathcal{W}^{(\omega)}(x,y)$ and $\mathcal{Z}^{(\omega)}(x,y)$ to allow shifting:

$$\mathcal{W}^{(\omega)}(x,y) = \mathbf{W}(x-y) + \int_y^x \mathbf{W}(x-z)\boldsymbol{\omega}(z)\mathcal{W}^{(\omega)}(z,y)dz, \tag{3.28}$$

$$\mathcal{Z}^{(\omega)}(x,y) = \mathbf{I} + \int_y^x \mathbf{W}(x-z)\boldsymbol{\omega}(z)\mathcal{Z}^{(\omega)}(z,y)dz. \tag{3.29}$$

Also note that $\mathcal{W}^{(\omega)}(x,0) = \mathcal{W}^{(\omega)}(x)$, $\mathcal{Z}^{(\omega)}(x,0) = \mathcal{Z}^{(\omega)}(x)$, as well as

$$\mathcal{W}^{(\omega^*)}(x-y) = \mathcal{W}^{(\omega)}(x,y), \quad \text{and} \quad \mathcal{Z}^{(\omega^*)}(x-y) = \mathcal{Z}^{(\omega)}(x,y), \tag{3.30}$$

with $\omega^*(\cdot,z) = \omega(\cdot, z+y)$.
Based on the fact that $\mathbf{W}^{(\delta)} - \mathbf{W} = \delta\mathbf{W}^{(\delta)} * \mathbf{W}$ and $\mathbf{Z}^{(\delta)} - \mathbf{Z} = \delta\mathbf{W}^{(\delta)} * \mathbf{Z}$, it is straightforward to check that

$$\mathcal{W}^{(\omega)}(x,y) = \mathbf{W}^{(\delta)}(x-y) + \int_y^x \mathbf{W}^{(\delta)}(x-z)(\boldsymbol{\omega}(z) - \delta\mathbf{I})\mathcal{W}^{(\omega)}(z,y)dz, \tag{3.31}$$

$$\mathcal{Z}^{(\omega)}(x,y) = \mathbf{Z}^{(\delta)}(x-y) + \int_y^x \mathbf{W}^{(\delta)}(x-z)(\boldsymbol{\omega}(z) - \delta\mathbf{I})\mathcal{Z}^{(\omega)}(z,y)dz.$$

To solve the one-sided upward problem (i.e., to get Corollary 3.3.7 (i)), we have to assume additionally that

$$\omega_i(x) \equiv \beta \geq 0, \quad \text{for all } x \leq 0 \text{ and } i \in E. \tag{3.32}$$

Hence we define a matrix function $\mathcal{H}^{(\omega)}$ which satisfies the following integral equation

$$\mathcal{H}^{(\omega)}(x) = e^{-\mathbf{R}^\beta x} + \int_0^x \mathbf{W}^{(\beta)}(x-z)(\boldsymbol{\omega}(z) - \beta\mathbf{I})\mathcal{H}^{(\omega)}(z)dz. \tag{3.33}$$

3.3.2 Exit problems and resolvents

This section establishes our main results on fluctuation identities and resolvents for spectrally negative ω-killed MAPs. Our main fluctuation identities will be related to the type A and type B two-sided exit problems. The following matrices can characterise them for $d \leq x \leq c$

$$\mathbf{A}_d^{(\omega)}(x,c) := \mathbb{E}_x\left[e^{-\int_0^{\tau_c^+}\omega_{J_s}(X_s)ds}, \tau_c^+ < \tau_d^-, J_{\tau_c^+}|J_0\right],$$

$$\mathbf{B}_d^{(\omega)}(x,c) := \mathbb{E}_x\left[e^{-\int_0^{\tau_d^-}\omega_{J_s}(X_s)ds}, \tau_d^- < \tau_c^+, J_{\tau_d^-}|J_0\right].$$

Theorem 3.3.2. ***(Two-sided exit problem for type A issue)***
For $d \leq x \leq c$

$$\mathbf{A}_d^{(\omega)}(x,c) = \mathcal{W}^{(\omega)}(x,d)\mathcal{W}^{(\omega)}(c,d)^{-1},$$

where matrix $\mathcal{W}^{(\omega)}$ is given in (3.28) and $\mathcal{W}^{(\omega)-1}$ is its inverse.

Proof. In what follows, we prove the case of $d = 0$, and then the general result holds using the shifting argument and the identity (3.30).
First, applying the strong Markov property of X at τ_y^+ and using the fact that X has no positive jumps, we get that:

$$\mathbf{A}^{(\omega)}(x,z) = \mathbf{A}^{(\omega)}(x,y)\mathbf{A}^{(\omega)}(y,z), \tag{3.34}$$

for all $0 \leq x \leq y \leq z$.
Following the similar argument as in Li and Palmowski [41], we recall that $\lambda > 0$ is the arbitrary upper bound of $\omega_i(x)$ (for all $x \in \mathbb{R}$ and $1 \leq i \leq N$). Let $\Upsilon = \{\Upsilon_t, t \geq 0\}$ be a Poisson point process on $\mathbb{R}_+ \times [0,\lambda]$ with a characteristic measure $\mu(dt,dy) = \lambda dt\ \frac{1}{\lambda}1_{\{[0,\lambda]\}}(y)dy$. Hence $\Upsilon = \{(T_k, M_k), k = 1,2,\dots\}$ is a doubly stochastic marked Poisson process with jump intensity λ, jumps epochs T_k and marks M_k being uniformly distributed on $[0,\lambda]$. Moreover, we construct Ψ to be independent of X. Therefore, for $T^\omega := \inf\{T_k > 0 : M_k < \omega_{J_{T_k}}(X_{T_k});\ \text{for } k \geq 1\}$, we have

$$\begin{aligned}
A_{ij}^{(\omega)}(x,c) =& \mathbb{P}_{x,i}\left(\tau_c^+ < \tau_0^- \wedge T^\omega, J_{\tau_c^+} = j\right)\\
=& \mathbb{P}_{x,i}\left(\#_k\{M_k < \omega_{J_{T_k}}(X_{T_k}) \text{ for } T_k < \tau_c^+, \tau_c^+ < \tau_0^-, J_{\tau_c^+} = j\} = 0\right).
\end{aligned}$$

In this case, there are two scenarios following. Either there is no T_k, which occurs before reaching level c or the first jump time T_1 occurs in state m, and the process renews from state m. Hence,

$$\begin{aligned}
A_{ij}^{(\omega)}(x,c) =& \mathbb{P}_{x,i}(T_1 > \tau_c^+, \tau_c^+ < \tau_0^-, J_{\tau_c^+} = j)\\
&+ \sum_{m=1}^N \mathbb{E}_{x,i}\left[A_{mj}^{(\omega)}(X_{T_1},c), T_1 < \tau_c^+ \wedge \tau_0^-, M_1 > \omega_m(X_{T_1}), J_{T_1} = m\right]\\
=& \mathbb{E}_{x,i}[e^{-\lambda\tau_c^+}; \tau_c^+ < \tau_0^-, J_{\tau_c^+} = j]\\
&+ \int_0^\infty \sum_{m=1}^N \mathbb{E}_{x,i}\left[X_{T_1} \in dy, T_1 < \tau_c^+ \wedge \tau_0^-, J_{T_1} = m\right]\frac{\lambda - \omega_m(y)}{\lambda}A_{mj}^{(\omega)}(y,c),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}\mathbf{A}^{(\omega)}(x,c) =&\mathbb{E}_x[e^{-\lambda\tau_c^+}, \tau_c^+ < \tau_0^-, J_{\tau_c^+}] \\ &+\int_0^c \mathbb{E}_x\left[X_{T_1}\in dy, T_1<\tau_c^+\wedge\tau_0^-, J_{T_1}\right]\frac{1}{\lambda}\left(\lambda\mathbf{I}-\boldsymbol{\omega}(y)\right)\mathbf{A}^{(\omega)}(y,c),\end{aligned}$$

where

$$\mathbb{E}_x[e^{-\lambda\tau_c^+}, \tau_c^+ < \tau_0^-, J_{\tau_c^+}] = \mathbf{W}^{(\lambda)}(x)\mathbf{W}^{(\lambda)}(c)^{-1},$$

and

$$\frac{1}{\lambda}\mathbb{E}_x\left[X_{T_1}\in dy, T_1<\tau_c^+\wedge\tau_0^-, J_{T_1}\right] = \left(\mathbf{W}^{(\lambda)}(x)\mathbf{W}^{(\lambda)}(c)^{-1}\mathbf{W}^{(\lambda)}(c-y)-\mathbf{W}^{(\lambda)}(x-y)\right)dy,$$

are given in Ivanovs and Palmowski [29] and Ivanovs [28], respectively.
Taking the last increment to the other side of the above equality and applying relation (3.34) gives

$$\begin{aligned}&\left(\mathbf{I}+\int_0^x \mathbf{W}^{(\lambda)}(x-y)\left(\lambda\mathbf{I}-\boldsymbol{\omega}(y)\right)\mathbf{A}^{(\omega)}(y,x)dy\right)\mathbf{A}^{(\omega)}(x,c) \\ =&\mathbf{W}^{(\lambda)}(x)\mathbf{W}^{(\lambda)}(c)^{-1}\left(\mathbf{I}+\int_0^c \mathbf{W}^{(\lambda)}(c-y)\left(\lambda\mathbf{I}-\boldsymbol{\omega}(y)\right)\mathbf{A}^{(\omega)}(y,c)dy\right).\end{aligned} \tag{3.35}$$

By defining

$$\mathcal{W}^{(\omega)}(x)^{-1} := \mathbf{W}^{(\lambda)}(x)^{-1}\left(\mathbf{I}+\int_0^x \mathbf{W}^{(\lambda)}(x-y)\left(\lambda\mathbf{I}-\boldsymbol{\omega}(y)\right)\mathbf{A}^{(\omega)}(y,x)dy\right), \tag{3.36}$$

we obtain the required identity

$$\mathbf{A}^{(\omega)}(x,c) = \mathcal{W}^{(\omega)}(x)\mathcal{W}^{(\omega)}(c)^{-1}.$$

Invertibility of the matrix $\mathcal{W}^{(\omega)}(x)^{-1}$ is given in the proposition at the end of this proof. After replacing $\mathbf{A}^{(\omega)}(y,x) = \mathcal{W}^{(\omega)}(y)\mathcal{W}^{(\omega)}(x)^{-1}$ in (3.36), we have

$$\begin{aligned}\mathbf{W}^{(\lambda)}(x) =& \left(\mathbf{I}+\int_0^x \mathbf{W}^{(\lambda)}(x-y)\left(\lambda\mathbf{I}-\boldsymbol{\omega}(y)\right)\mathbf{A}^{(\omega)}(y,x)dy\right)\mathcal{W}^{(\omega)}(x) \\ =&\mathcal{W}^{(\omega)}(x)+\int_0^x \mathbf{W}^{(\lambda)}(x-y)\left(\lambda\mathbf{I}-\boldsymbol{\omega}(y)\right)\mathcal{W}^{(\omega)}(y)dy.\end{aligned}$$

Now using the identity $\mathbf{W}^{(\delta)}-\mathbf{W} = \delta\mathbf{W}*\mathbf{W}^{(\delta)}$, it is straightforward to show

$$\mathcal{W}^{(\omega)}(x) = \mathbf{W}(x)+\int_0^x \mathbf{W}(x-y)\boldsymbol{\omega}(y)\mathcal{W}^{(\omega)}(y)\,dy.$$

Now, we are left with the proof of the inevitability of the matrix function $\mathcal{W}^{(\omega)}(x)^{-1}$.

Proposition 3.3.3. *The matrix $\mathcal{W}^{(\omega)}(x)^{-1}$ is invertible for any $x>0$.*

Proof. From (3.36), one can see that it is enough to prove that the matrix

$$\mathbf{P}(x) := \mathbf{I} + \int_0^x \mathbf{W}^{(\lambda)}(x-y)\left(\lambda \mathbf{I} - \boldsymbol{\omega}(y)\right)\mathbf{A}^{(\omega)}(y,x)dy,$$

is invertible for every $x \geq 0$. Using a similar argument as in Kyprianou and Palmowski [38], note that for all $y > 0$, there exists some $N \times N$ sub-stochastic invertible intensity matrix $\boldsymbol{\Lambda}^{\omega,*}(y)$ such that

$$\mathbb{P}_x\left(\tau_c^+ < \tau_0^- \wedge T^\omega, J_{\tau_c^+} | J_0\right) = \exp\left(\int_x^c \boldsymbol{\Lambda}^{\omega,*}(y)\, dy\right). \tag{3.37}$$

This observation implies that the matrix $\mathbf{A}^{(\omega)}(x,c)$ is invertible for any $x, c \in \mathbb{R}_+$ such that $0 < x \leq c$. The matrix $\mathbf{A}^{(\omega)}(x,c)$ is also continuous (entry wise) with respect of c. Now, assume that there exists $c > 0$ such that matrix $\mathbf{P}(x)$ is invertible for some $0 < x < c$ and is singular for $x = c$. Then from relation (3.35) we get contradiction, because the left-hand side of it is invertible (as a product of invertible matrices) and the right-hand side is singular from the assumption. Hence, only two scenarios are possible: the matrix $\mathbf{P}(x)$ is invertible for all $x > 0$ or it is singular for all $x > 0$. Finally, since $\mathbf{P}(0) = \mathbf{I}$ and $\mathbf{P}(x)$ is continuous in $x \geq 0$ we obtain that $\mathbf{P}(x)$ must be invertible for all $x \geq 0$. □

This proposition completes the proof of the theorem. □

One needs to define ω-type resolvents for the type B problem. Namely, let $\{(X_t, J_t)\}_{t\geq 0}$ be a MAP with the lifetime ξ (it means that the MAP is killed after this time), transition probabilities and q-resolvent measures, given, respectively by

$$Q_{t,ij} f_j(x) = \mathbb{E}_{x,i}\left[f_j(X_t), t < \xi, J_t = j\right]$$

and

$$K_{ij}^{(q)} f_j(x) = \int_0^\infty e^{-qt} Q_{t,ij} f_j(x) dt,$$

where $\{f_j\}_{j=1}^N$ is a set of nonnegative, bounded, continuous functions on $\mathbb{R}$ such that $\sup_{i,j} K_{ij}^{(0)} f_j(x) < \infty$. Then the ω-type resolvent $K_{ij}^{(\omega)}$ is defined by

$$K_{ij}^{(\omega)} f_j(x) := \int_0^\infty Q_{t,ij}^{(\omega)} f_j(x) dt,$$

where

$$Q_{t,ij}^{(\omega)} f_j(x) := \mathbb{E}_{x,i}\left[\exp\left(-\int_0^t \omega_{J_s}(X_s)ds\right) f_j(X_t); t < \xi, J_t = j\right].$$

The next lemma is a helpful tool used further to get the representation of the matrix $\boldsymbol{B}^{(\omega)}(x,c)$.

Lemma 3.3.4. *The matrix* $\mathbf{K}^{(\omega)}\boldsymbol{f}(x) = \{K_{ij}^{(\omega)} f_j(x)\}_{i,j=1}^N$ *satisfies the following equality*

$$\mathbf{K}^{(\omega)}\boldsymbol{f}(x) = \mathbf{K}^{(0)}\left(\boldsymbol{f} - \boldsymbol{\omega}\mathbf{K}^{(\omega)}\boldsymbol{f}\right)(x),$$

where $\boldsymbol{f} = diag(f_1, ..., f_N)$.

Proof. As before without loss of generality, we assume that $\omega_i(x)$ is bounded by some $\lambda > 0$ for all $x \in \mathbb{R}$ and $i \in E$. The finiteness of $K_{ij}^{(\omega)} f_j(x)$ comes from the fact that $K_{ij}^{(\omega)} f_j(x) < K_{ij}^{(0)} f_j(x)$ for all $1 \le i \le N$. Using similar arguments as in the proof of Theorem 3.3.2, we have

$$\begin{aligned}
&Q_{t,ij}^{(\omega)} f_j(x)\\
=&\mathbb{E}_{x,i}\left[f_{J_t}(X_t); t < \xi \text{ and } M_k > \omega_{J_{T_k}}(X_{T_k}) \text{ for all } T_k < t, J_t = j\right]\\
=&\mathbb{E}_{x,i}\left[f_{J_t}(X_t); t < \xi, T_1 > t, J_t = j\right] + \sum_{l=1}^N \int_0^t \mathbb{E}_{x,i}\left[Q_{t-s,lj}^{(\omega)} f_j(X_s), M_1 > \omega_l(X_s), J_s = l\right] \mathbb{P}(T_1 \in ds)\\
=&\mathbb{E}_{x,i}\left[e^{-\lambda t} f_j(X_t); t < \xi, J_t = j\right] + \sum_{l=1}^N \int_0^t \mathbb{E}_{x,i}\left[(\lambda - \omega_l(X_s)) Q_{t-s,lj}^{(\omega)} f_j(X_s), J_s = l\right] e^{-\lambda s} ds\\
=&Q_{t,ij}^{(\lambda)} f_j(x) + \sum_{l=1}^N \int_0^t Q_{s,il}^{(\lambda)}\left((\lambda - \omega_l) Q_{t-s,lj}^{(\omega)} f_j\right)(x) ds.
\end{aligned}$$

Note that the superscript λ denotes a counterpart for fixed $\omega_i(x) \equiv \lambda$. Equivalently, in a matrix form, we have

$$\mathbf{Q}_t^{(\omega)} \boldsymbol{f}(x) = \mathbf{Q}_t^{(\lambda)} \boldsymbol{f}(x) + \int_0^t \mathbf{Q}_s^{(\lambda)}\left((\lambda \mathbf{I} - \boldsymbol{\omega}) \mathbf{Q}_{t-s}^{(\omega)} \boldsymbol{f}\right)(x) ds,$$

where by matrix compounding, we mean $(\mathbf{A}(\mathbf{B})(x))_{ij} = \sum_{m=1}^N A_{im} B_{mj}(x)$. Thus,

$$\mathbf{K}^{(\omega)} \boldsymbol{f}(x) = \int_0^\infty \mathbf{Q}_t^{(\omega)} \boldsymbol{f}(x) dt = \mathbf{K}^{(\lambda)} \boldsymbol{f}(x) + \mathbf{K}^{(\lambda)}\left((\lambda \mathbf{I} - \boldsymbol{\omega}) \mathbf{K}^{(\omega)} \boldsymbol{f}\right)(x). \tag{3.38}$$

Using the resolvent identity $\lambda \mathbf{K}^{(0)}(\mathbf{K}^{(\lambda)}) = \mathbf{K}^{(0)} - \mathbf{K}^{(\lambda)}$, we have

$$\lambda \mathbf{K}^{(0)}\left(\mathbf{K}^{(\omega)} \boldsymbol{f}\right)(x) = (\mathbf{K}^{(0)} - \mathbf{K}^{(\lambda)}) \boldsymbol{f}(x) + (\mathbf{K}^{(0)} - \mathbf{K}^{(\lambda)})\left((\lambda \mathbf{I} - \boldsymbol{\omega}) \mathbf{K}^{(\omega)} \boldsymbol{f}\right)(x). \tag{3.39}$$

Comparing (3.38) with (3.39) completes the proof.

□

Having the above lemma, we are ready to prove the two-sided exit problem for the type B issue.

Theorem 3.3.5. ***(Two-sided exit problem for type B issue)***
For $d \le x \le c$,

$$\mathbf{B}_d^{(\omega)}(x, c) = \mathcal{Z}^{(\omega)}(x, d) - \mathcal{W}^{(\omega)}(x, d) \mathcal{W}^{(\omega)}(c, d)^{-1} \mathcal{Z}^{(\omega)}(c, d),$$

where $\mathcal{Z}^{(\omega)}$ is given in (3.29) *and invertibility of $\mathcal{W}^{(\omega)}$ is given in Theorem 3.3.2.*

Proof. Again we prove the case of $d = 0$, and then the general result holds true using the shifting argument as well as the identity (3.30). For $i, j \in E$, define

$$B_{ij}^{(\omega)}(x) := \lim_{c \to \infty} B_{ij}^{(\omega)}(x, c) = \mathbb{E}_{x,i}\left[e^{-\int_0^{\tau_0^-} \omega_{J_s}(X_s) ds}, \tau_0^- < \infty, J_{\tau_0^-} = j\right]. \tag{3.40}$$

Note that for any $i, j \in E$ and $x, c \in \mathbb{R}$ such that $x < c$ matrix function $B_{ij}^{(\omega)}(x,c)$ is monotone in c, and it is bounded by $0 \le B_{ij}^{(\omega)}(x,c) \le \mathbb{P}_{x,i}\left(\tau_0^- < \tau_c^+, J_{\tau_0^-} = j\right) \le 1$, so the limit in (3.40) exists and is finite. The strong Markov property and spectrally negativity of X give that

$$\mathbf{B}^{(\omega)}(x,c) = \mathbf{B}^{(\omega)}(x) - \mathbf{A}^{(\omega)}(x,c)\mathbf{B}^{(\omega)}(c). \tag{3.41}$$

To identify $\mathbf{B}^{(\omega)}(x)$, we use Lemma 3.3.4 with $\xi = \tau_0^-$ and $\boldsymbol{f}(\cdot) = \boldsymbol{\omega}(\cdot)$. Hence

$$\begin{aligned}\mathbf{I}(x) - \mathbf{B}^{(\omega)}(x) =& \mathbb{E}_x\left[\int_0^{\tau_0^-} \omega_{J_t}(X_t)\exp\left(-\int_0^t \omega_{J_s}(X_s)ds\right)dt, t < \tau_0^-, J_t\right] \\ =& \int_0^\infty \mathbb{E}_x\left[\omega_{J_t}(X_t)\exp\left(-\int_0^t \omega_{J_s}(X_s)ds\right), t < \tau_0^-, J_t\right]dt \\ =& \int_0^\infty \left(\mathbf{W}(x)e^{\mathbf{R}y} - \mathbf{W}(x-y)\right)\left[\boldsymbol{\omega}(y) - \boldsymbol{\omega}(\mathbf{I} - \mathbf{B}^{(\omega)})(y)\right]dy, \end{aligned} \tag{3.42}$$

where the potential measure

$$\mathbf{K}^{(0)}\left(1_{(0,\infty)}(X_t \in dy)\right)(x) = \boldsymbol{U}_{(0,\infty)}(x,dy) = \left(\mathbf{W}(x)e^{\mathbf{R}y} - \mathbf{W}(x-y)\right)dy,$$

was obtained in Ivanovs [28] with $\mathbf{R} = \mathbf{R}^0$. We may rewrite it as

$$\mathbf{B}^{(\omega)}(x) = \mathbf{I}(x) - \mathbf{W}(x)\mathbf{C}_{B^{(\omega)}} + \int_0^x \mathbf{W}(x-y)\boldsymbol{\omega}(y)\mathbf{B}^{(\omega)}(y)dy, \tag{3.43}$$

where

$$\mathbf{C}_{B^{(\omega)}} = \int_0^\infty e^{\mathbf{R}y}\boldsymbol{\omega}(y)\mathbf{B}^{(\omega)}(y)dy. \tag{3.44}$$

Note that $0 \le B_{ij}^{(\omega)}(y) \le 1$ and recall that $0 \le \omega_i(x) \le \lambda$. Hence the last increment on the right-hand side of equation (3.43) is finite, and then matrix $\mathbf{C}_{B^{(\omega)}}$ is well defined and finite. From the definitions of ω-scale matrices we have

$$\mathbf{B}^{(\omega)}(x) = \mathcal{Z}^{(\omega)}(x) - \mathcal{W}^{(\omega)}(x)\mathbf{C}_{B^{(\omega)}}. \tag{3.45}$$

Equation (3.41) completes the proof. □

Remark 3.3.6. When $d = 0$, we use simplified notations

$$\mathbf{A}^{(\omega)}(x,c) := \mathbf{A}_0^{(\omega)}(x,c),$$

and

$$\mathbf{B}^{(\omega)}(x,c) := \mathbf{B}_0^{(\omega)}(x,c).$$

Now, taking the limits $d \to -\infty$ and $c \to \infty$ (as well as $d = 0$) in Theorems 3.3.2 and 3.3.5 respectively, we obtain the following corollary regarding to the one-sided exit problem.

Corollary 3.3.7. ***(One-sided exit problem)***

(i) Under the assumption (3.32), for $x \leq c$,

$$\mathbb{E}_x\left[e^{-\int_0^{\tau_c^+}\omega_{J_s}(X_s)ds}, \tau_c^+ < \infty, J_{\tau_c^+}|J_0\right] = \mathcal{H}^{(\omega)}(x)\mathcal{H}^{(\omega)}(c)^{-1},$$

for invertible matrix function $\mathcal{H}^{(\omega)}$ given in (3.33).

(ii) For $x \geq 0$ and $\lambda > 0$,

$$\mathbb{E}_x\left[e^{-\int_0^{\tau_0^-}\omega_{J_s}(X_s)ds}, \tau_0^- < \infty, J_{\tau_0^-}|J_0\right] = \mathcal{Z}^{(\omega)}(x) - \mathcal{W}^{(\omega)}(x)\mathbf{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)},$$

where matrix

$$\mathbf{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)} := \lim_{c\to\infty}\mathcal{W}^{(\omega)}(c)^{-1}\mathcal{Z}^{(\omega)}(c),$$

exists and has finite entries.

Proof. Proof of the case (i) First we will prove that

$$\lim_{d\to-\infty}\mathcal{W}^{(\omega)}(x,d)\mathcal{W}^{(\omega)}(c,d)^{-1} = \mathcal{H}^{(\omega)}(x)\mathcal{H}^{(\omega)}(c)^{-1}. \tag{3.46}$$

Then the result will follow from Theorem 3.3.2. Recall that for $x \geq d$ and any fixed $\beta \geq 0$, we have:

$$\mathcal{W}^{(\omega)}(x,d) = \mathbf{W}^{(\beta)}(x-d) + \int_0^x \mathbf{W}^{(\beta)}(x-z)(\boldsymbol{\omega}(z) - \beta\mathbf{I})\mathcal{W}^{(\omega)}(z,d)dz.$$

Moreover, for $x = 0$,

$$\mathcal{W}^{(\omega)}(0,d)e^{-\mathbf{R}^\beta d} = \mathbf{W}^{(\beta)}(-d)e^{-\mathbf{R}^\beta d}.$$

Hence from (3.3) we have

$$\lim_{d\to-\infty}\mathcal{W}^{(\omega)}(0,d)e^{-\mathbf{R}^\beta d} = \lim_{d\to-\infty}\mathbf{W}^{(\beta)}(-d)e^{-\mathbf{R}^\beta d} = \mathbf{L}^\beta.$$

From Theorem 3.3.2, for $x > 0$,

$$\mathbb{E}\left[e^{-\int_0^{\tau_x^+}\omega_{J_s}(X_s)ds}, \tau_x^+ < \tau_d^-, J_{\tau_x^+}|J_0\right]\mathcal{W}^{(\omega)}(x,d) = \mathcal{W}^{(\omega)}(0,d).$$

Since the above expectation is increasing with respect to d, the following limit is well-defined and finite for every $x > d$

$$\begin{aligned}&\lim_{d\to-\infty}\mathbb{E}\left[e^{-\int_0^{\tau_x^+}\omega_{J_s}(X_s)ds}, \tau_x^+ < \tau_d^-, J_{\tau_x^+}|J_0\right]\mathcal{W}^{(\omega)}(x,d)e^{-\mathbf{R}^\beta d}\\ =&\mathbb{E}\left[e^{-\int_0^{\tau_x^+}\omega_{J_s}(X_s)ds}, \tau_x^+ < \infty, J_{\tau_x^+}|J_0\right]\lim_{d\to-\infty}\mathcal{W}^{(\omega)}(x,d)e^{-\mathbf{R}^\beta d} = \mathbf{L}^\beta.\end{aligned}$$

Note also that, since matrix $\mathbf{L}^\beta$ is invertible as it was note above Equation (3.3), from the above equation, it follows that the matrix $\lim_{d\to-\infty}\mathcal{W}^{(\omega)}(x,d)e^{-\mathbf{R}^\beta d}$ is also invertible. Taking

$$\mathcal{H}^{(\omega)}(x) := \lim_{d\to-\infty}\mathcal{W}^{(\omega)}(x,d)e^{-\mathbf{R}^\beta d}(\mathbf{L}^\beta)^{-1}.$$

completes the proof of the first part of the corollary. To show that the above form of $\mathcal{H}^{(\omega)}(x)$ satisfies (3.33), note that

$$\mathcal{W}^{(\omega)}(x,d)e^{-\mathbf{R}^{\beta}d} = \left(\mathbf{W}^{(\beta)}(x-d) + \int_0^x \mathbf{W}^{(\beta)}(x-z)(\boldsymbol{\omega}(z)-\beta\mathbf{I})\mathcal{W}^{(\omega)}(z,d)dz\right)e^{-\mathbf{R}^{\beta}d}.$$

Then, the result follows by taking the limit $d \to -\infty$ and applying the dominated convergence theorem.

Proof of the case (*ii*) The proof follows by taking the limit (3.40), which exists and is finite. Moreover, the limit

$$\lim_{c\to\infty} \mathcal{W}^{(\omega)}(c)^{-1}\mathcal{Z}^{(\omega)}(c) = \mathbf{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)} = \mathbf{C}_{B^{(\omega)}}$$

is finite by (3.44). This completes the proof. □

Next, we present the representations of four ω-type resolvents. These types of identities are usually used to describe the position of the Lévy process right before exit from some interval or half-line based on the so-called compensation formula; see Kyprianou [35, Chap. 5] for detalis.

Theorem 3.3.8. ***(Resolvents)***

(i) For $d \le x \le c$,

$$\begin{aligned}\boldsymbol{U}^{(\omega)}_{(d,c)}(x,dy) &:= \int_0^\infty \mathbb{E}_x\left[\exp\left(-\int_0^t \omega_{J_s}(X_s)ds\right), X_t \in dy, t < \tau_d^- \wedge \tau_c^+, J_t|J_0\right]dt \\ &= \left(\mathcal{W}^{(\omega)}(x,d)\mathcal{W}^{(\omega)}(c,d)^{-1}\mathcal{W}^{(\omega)}(c,y) - \mathcal{W}^{(\omega)}(x,y)\right)dy.\end{aligned}$$

(ii) For $x \ge 0$ *and* $\lambda > 0$,

$$\begin{aligned}\boldsymbol{U}^{(\omega)}_{(0,\infty)}(x,dy) &:= \int_0^\infty \mathbb{E}_x\left[\exp\left(-\int_0^t \omega_{J_s}(X_s)ds\right), X_t \in dy, t < \tau_0^-, J_t|J_0\right]dt \\ &= \left(\mathcal{W}^{(\omega)}(x)\mathbf{C}_{\mathcal{W}(\infty)^{-1}\mathcal{W}(\infty)}(y) - \mathcal{W}^{(\omega)}(x,y)\right)dy,\end{aligned}$$

where

$$\mathbf{C}_{\mathcal{W}(\infty)^{-1}\mathcal{W}(\infty)}(y) := \lim_{c\to\infty} \mathcal{W}^{(\omega)}(c)^{-1}\mathcal{W}^{(\omega)}(c,y)$$

is a well-defined and finite matrix.

(iii) For $x, y \le c$,

$$\begin{aligned}\boldsymbol{U}^{(\omega)}_{(-\infty,c)}(x,dy) :&= \int_0^\infty \mathbb{E}_x\left[\exp\left(-\int_0^t \omega_{J_s}(X_s)ds\right), X_t \in dy, t < \tau_c^+, J_t|J_0\right]dt \\ &= \left(\mathcal{H}^{(\omega)}(x)\mathcal{H}^{(\omega)}(c)^{-1}\mathcal{W}^{(\omega)}(c,y) - \mathcal{W}^{(\omega)}(x,y)\right)dy.\end{aligned}$$

(iv) For $x \in \mathbb{R}$,

$$\begin{aligned}\boldsymbol{U}^{(\omega)}_{(-\infty,\infty)}(x,dy) :&= \int_0^\infty \mathbb{E}_x\left[\exp\left(-\int_0^t \omega_{J_s}(X_s)ds\right), X_t \in dy, J_t|J_0\right]dt \\ &= \left(\mathcal{H}^{(\omega)}(x)\mathbf{C}_{\mathcal{H}(\infty)^{-1}\mathcal{W}(\infty)}(y) - \mathcal{W}^{(\omega)}(x,y)\right)dy,\end{aligned}$$

where matrix $\mathbf{C}_{\mathcal{H}(\infty)^{-1}\mathcal{W}(\infty)} = \lim_{c\to\infty} \mathcal{H}^{(\omega)}(c)^{-1}\mathcal{W}^{(\omega)}(c,y)$ *exists and has finite entries.*

Proof. Proof of the case (i)
Using Lemma 3.3.4, we have

$$\boldsymbol{U}^{(\omega)}_{(d,c)}\boldsymbol{f}(x) := \int_0^\infty \mathbb{E}_x\left[f_{J_t}(X_t)\exp\left(-\int_0^t \omega_{J_s}(X_s)ds\right), t<\tau_d^-\wedge\tau_c^+, J_t|J_0\right]dt$$
$$= \int_d^c \boldsymbol{U}_{(d,c)}(x,dy)\left(\boldsymbol{f}(y)-\boldsymbol{\omega}(y)\boldsymbol{U}^{(\omega)}_{(d,c)}\boldsymbol{f}(y)\right), \tag{3.47}$$

where $\boldsymbol{U}_{(d,c)}(x,dy)$ is the potential measure of the MAP without ω-killing, given in Theorem 1 of Ivanovs [28]:

$$\boldsymbol{U}_{(d,c)}(x,dy) = \big(\mathbf{W}(x-d)\mathbf{W}(c-d)^{-1}\mathbf{W}(c-y)-\mathbf{W}(x-y)\big)\,dy.$$

Hence, we can rewrite Equation (3.47) as

$$\boldsymbol{U}^{(\omega)}_{(d,c)}\boldsymbol{f}(x) = \mathbf{W}(x-d)\mathbf{C}_U - \int_d^x \mathbf{W}(x-y)\boldsymbol{f}(y)dy + \int_d^x \mathbf{W}(x-y)\boldsymbol{\omega}(y)\boldsymbol{U}^{(\omega)}_{(d,c)}\boldsymbol{f}(y)dy,$$

where $\mathbf{C}_U = \int_d^c \mathbf{W}(c-d)^{-1}\mathbf{W}(c-y)\left(\boldsymbol{f}(y)-\boldsymbol{\omega}(y)\boldsymbol{U}^{(\omega)}_{(d,c)}\boldsymbol{f}(y)\right)dy$. Multiplying Equation (3.28) by $\mathbf{C}_U$ gives that

$$\mathcal{W}^{(\omega)}(x,d)\mathbf{C}_U = \mathbf{W}(x-d)\mathbf{C}_U + \int_d^x \mathbf{W}(x-y)\boldsymbol{\omega}(y)\mathcal{W}^{(\omega)}(y,d)\mathbf{C}_U dy,$$

and define the operator $\mathcal{R}^{(\omega)}\boldsymbol{f}(x) := \int_d^x \mathcal{W}^{(\omega)}(x,y)\boldsymbol{f}(y)dy$, which leads to

$$\mathcal{R}^{(\omega)}\boldsymbol{f}(x) = \int_d^x \mathbf{W}(x-y)\boldsymbol{f}(y)dy + \int_d^x \mathbf{W}(x-y)\boldsymbol{\omega}(y)\mathcal{R}^{(\omega)}\boldsymbol{f}(y)dy.$$

Therefore, by the uniqueness property in Lemma 3.3.1, we have

$$\boldsymbol{U}^{(\omega)}_{(d,c)}\boldsymbol{f}(x) = \mathcal{W}^{(\omega)}(x,d)\mathbf{C}_U - \mathcal{R}^{(\omega)}\boldsymbol{f}(x).$$

To find the constant matrix $\mathbf{C}_U$, we use the boundary condition $\boldsymbol{U}^{(\omega)}_{(d,c)}\boldsymbol{f}(c)=0$. One completes the proof by denoting the density of $\boldsymbol{U}^{(\omega)}_{(d,c)}\boldsymbol{f}(x)$ as $\boldsymbol{U}^{(\omega)}_{(d,c)}(x,dy)$.
Proof of the case (ii)
This identity follows directly from Theorem 3.3.8 (i) by taking the limit and using (3.3) together with the dominated convergence theorem.
Proof of the case (iii)
The formula follows by taking the limit $\lim_{d\to-\infty}$ in Theorem 3.3.8 (i) and then using (3.46).
Proof of the case (iv)
This identity follows from Theorem 3.3.8 (iii) by taking the limit $c\to\infty$. Since $\mathcal{H}^{(\omega)}(c)^{-1}\mathcal{W}^{(\omega)}(c,y)$ is monotonic in c then the result holds. □

3.4 Dividends in the Omega ruin model

In this section, we demonstrate one application of the previously obtained results on the dividend problem. We assume that the company's reserve process is governed by a MAP (X, J). We consider a dividend barrier strategy (at c) and define the cumulative dividends paid up to time t as follows

$$L_t^c = \sup_{s \le t}[X_s - c] \vee 0.$$

With the barrier dividend strategy, we work with the controlled risk process $U = \{U_t : t \ge 0\}$ such that

$$U_t^c := X_t - L_t^c.$$

Moreover, we assume that this company pays dividends according to the barrier strategy until Omega ruin time. Let us recall that we defined it as

$$\tau_\omega^d = \inf\{t \ge 0 : \int_0^t \omega_{J_s}(U_s^c)ds > \mathrm{e}_1 \text{ or } U_t^c < -d\},$$

where e_1 is an independent exponential random variable (with mean 1) and a fixed level $d \in \mathbb{R}$ is a threshold. Usually, one set $d > 0$, and for all $i \in E$ and for $-d \le x \le 0$, the (typically decreasing) function $\omega_i(x) \ge 0$ can be interpreted as a bankruptcy rate. However, from the theoretical point of view, there is no need for such restriction. For example, if one set $d = 0$ and $\omega_i(x) \ge 0$, for all $i \in E$ for some region above 0, then we will have some form of early warning zone before classical ruin. Also, in the proof of the value function representation, we will use $d = 0$ and then the general result will be followed by simple shifting. In particular, we say that the process U^c is in the time t in the so-called "red zone" if $\omega_{J_s}(U_t^c) > 0$.
Thus, ruin can occur in two situations. The first is when the process crosses a fixed level $-d$ (for $d = 0$, we have a case of classical ruin time). The second possibility is when bankruptcy happens in the "red zone", and the intensity of this bankruptcy is a function of the current level of the additive regulated component U^c and the Markov chain J. In other words the probability of bankruptcy within an infinitesimal time dt in the "red zone" is $\omega_{J_t}(x)dt$. For more details related to the Omega ruin time, we refer to Gerber *et al.* [23] and Li and Palmowski [41]. In the following theorem, we examine the case of $d = 0$ and then consider a general d in the corollary.

Theorem 3.4.1. *Assume that dividends are discounted at a constant force of interest $\delta > 0$ and $d = 0$. The expected discounted present value of the dividends paid before Omega ruin ($\tau_\omega := \tau_\omega^0$) under a constant dividend barrier, c is given by*

$$\mathbf{v}_c(x) := \mathbb{E}_x\left[\int_0^{\tau_\omega} e^{-\delta t} dL_t, J_{\tau_\omega} | J_0\right] = \begin{cases} \mathcal{W}^{(\delta+\omega)}(x)\mathcal{W}^{(\delta+\omega)\prime}(c)^{-1}, & \text{for} \quad 0 < x \le c, \\ (x-c) + \mathcal{W}^{(\delta+\omega)}(c)\mathcal{W}^{(\delta+\omega)\prime}(c)^{-1}, & \text{for} \quad x > c, \end{cases}$$

where the invertible matrix function fulfils

$$\mathcal{W}^{(\delta+\omega)\prime}(c) = \mathbf{W}'(c) + \int_0^c \mathbf{W}'(c-y)(\omega(y) + \delta\mathbf{I})\mathcal{W}^{(\delta+\omega)}(y)dy + \mathbf{W}(0)(\omega(c) + \delta\mathbf{I})\mathcal{W}^{(\delta+\omega)}(c).$$

Proof. We start with the case of $0 < x \le c$. Conditioning on reaching the level c first, we have

$$\mathbf{v}_c(x) = \mathbf{A}^{(\omega)}(x,c)\mathbf{v}_c(c) = \mathcal{W}^{(\delta+\omega)}(x)\mathcal{W}^{(\delta+\omega)}(c)^{-1}\mathbf{v}_c(c).$$

As a first step, we will find a lower bound for $\mathbf{v}_c(c)$. For $m \in \mathbb{N}$, consider that the dividend is not paid until reaching the level $c + \frac{1}{m}$

$$\begin{aligned}\mathbf{v}_c(c) \ge& \mathbb{E}_c\left[e^{-\int_0^{\tau^+_{c+1/m}}(\delta+\omega_{J_s}(X_s))ds}, \tau^+_{c+\frac{1}{m}} < \tau_0^-, J_{\tau^+_{c+\frac{1}{m}}}|J_0\right]\mathbf{v}_c\left(c+\frac{1}{m}\right)\\ =&\mathbb{E}_c\left[e^{-\int_0^{\tau^+_{c+1/m}}(\delta+\omega_{J_s}(X_s))ds}, \tau^+_{c+\frac{1}{m}} < \tau_0^-, J_{\tau^+_{c+\frac{1}{m}}}|J_0\right]\left(\mathbf{v}_c(c)+\frac{1}{m}\mathbf{I}\right),\end{aligned}$$

where the last equality is due to the dividend of $\frac{1}{m}$ paid immediately and the fact that the drop in surplus will not cause the state transition.
On the other hand, an upper bound can be found as

$$\begin{aligned}\mathbf{v}_c(c) \le& \mathbb{E}_c\left[e^{-\int_0^{\tau^+_{c+1/m}}(\delta+\omega_{J_s}(X_s))ds}, \tau^+_{c+\frac{1}{m}} < \tau_0^-, J_{\tau^+_{c+\frac{1}{m}}}|J_0\right]\left(\mathbf{v}_c(c)+\frac{1}{m}\mathbf{I}\right)\\ &+\frac{1}{m}\mathbb{E}_c\left[\int_0^{\tau^+_{c+1/m}} e^{-\delta t}dt\ e^{-\int_0^{\tau^+_{c+1/m}}\omega_{J_s}(X_s)ds}, \tau^+_{c+\frac{1}{m}} < \tau_0^-, J_{\tau_0^-}|J_0\right]\\ &+\mathbb{E}_c\left[\int_0^{\tau_\omega} e^{-\delta t}dL_t^c, \tau_\omega < \tau^+_{c+\frac{1}{m}}, J_{\tau_\omega}|J_0\right],\end{aligned}$$

where L_t^c will be bounded by $\frac{1}{m}$ for the process starting from level c to level $c+\frac{1}{m}$, i.e.,

$$\mathbb{E}_c\left[\int_0^{\tau_\omega} e^{-\delta t}dL_t^c, \tau_\omega < \tau^+_{c+\frac{1}{m}}, J_{\tau_\omega}|J_0\right] \le \frac{1}{m}\mathbb{P}_c\left(\tau_\omega < \tau^+_{c+\frac{1}{m}}, J_{\tau_\omega}|J_0\right).$$

Note that as $m \to \infty$, the following two limits approach to $\mathbf{0}$:

$$\lim_{m\to\infty}\mathbb{E}_c\left[\int_0^{\tau^+_{c+1/m}} e^{-\delta t}dt\ e^{-\int_0^{\tau^+_{c+1/m}}\omega_{J_s}(X_s)ds}, \tau^+_{c+\frac{1}{m}} < \tau_0^-, J_{\tau_0^-}|J_0\right] = \mathbf{0},$$

and

$$\lim_{m\to\infty}\mathbb{P}_c\left(\tau_\omega < \tau^+_{c+\frac{1}{m}}, J_{\tau_\omega}|J_0\right) = \mathbf{0}.$$

See Renaud and Zhou [53] and Czarna *et al.* [16] for more details.
Therefore, by matching the upper and lower bounds, we have

$$\begin{aligned}\mathbf{v}_c(c) =& \mathbb{E}_c\left[e^{-\int_0^{\tau^+_{c+1/m}}(\delta+\omega_{J_s}(X_s))ds}, \tau^+_{c+\frac{1}{m}} < \tau_0^-, J_{\tau^+_{c+\frac{1}{m}}}|J_0\right]\left(\mathbf{v}_c(c)+\frac{1}{m}\mathbf{I}\right)+o\left(\frac{1}{m}\right)\\ =& \mathcal{W}^{(\delta+\omega)}(c)\mathcal{W}^{(\delta+\omega)}(c+\frac{1}{m})^{-1}\left(\mathbf{v}_c(c)+\frac{1}{m}\mathbf{I}\right)+o\left(\frac{1}{m}\right),\end{aligned}$$

and hence with some rearangements,

$$\left(\frac{\mathcal{W}^{(\delta+\omega)}(c+\frac{1}{m})-\mathcal{W}^{(\delta+\omega)}(c)}{1/m}\right)\mathcal{W}^{(\delta+\omega)}(c)^{-1}\mathbf{v}_c(c)=\mathbf{I}+o\left(\frac{1}{m}\right).$$

Letting $m\to\infty$, it turns out

$$\mathcal{W}^{(\delta+\omega)\prime}(c)\mathcal{W}^{(\delta+\omega)}(c)^{-1}\mathbf{v}_c(c)=\mathbf{I},$$

where matrix

$$\mathcal{W}^{(\delta+\omega)\prime}(c)=\mathbf{W}'(c)+\int_0^c\mathbf{W}'(c-y)(\omega(y)+\delta\mathbf{I})\mathcal{W}^{(\delta+\omega)}(y)dy+\mathbf{W}(0)(\omega(c)+\delta\mathbf{I})\mathcal{W}^{(\delta+\omega)}(c),$$

is well-defined since the scale matrix $\mathbf{W}$ is almost everywhere differentiable, see Kyprianou and Palmowski [38]. Furthermore, one can observe that, from representation (3.37), the above matrix is invertible for any $c>0$ and then $\mathbf{v}_c(c)=\mathcal{W}^{(\delta+\omega)}(c)\mathcal{W}^{(\delta+\omega)\prime}(c)^{-1}$.
To end this proof, note that for $x>c$, one is immediately paying a dividend of size $x-c$ (and this will not cause the state transition), therefore

$$\mathbf{v}_c(x)=(x-c)+\mathbf{v}_c(c)=(x-c)+\mathcal{W}^{(\delta+\omega)}(c)\mathcal{W}^{(\delta+\omega)\prime}(c)^{-1}.$$

□

Applying the shifting argument to Theorem 3.4.1, we have the representation for the value function for a general d.

Corollary 3.4.2. *For $\delta>0$, the expected present value of the dividend paid before Omega ruin (τ_ω^d) under a constant dividend barrier c is*

$$\mathbf{v}_c^d(x):=\mathbb{E}_x\left[\int_0^{\tau_\omega^d}e^{-\delta t}dL_t,J_{\tau_\omega^d}|J_0\right]=\begin{cases}\mathcal{W}^{(\delta+\omega)}(x,-d)\mathcal{W}^{(\delta+\omega)\prime}(c,-d)^{-1}, & \textit{for}\quad -d<x\le c,\\ (x-c)+\mathcal{W}^{(\delta+\omega)}(c,-d)\mathcal{W}^{(\delta+\omega)\prime}(c,-d)^{-1}, & \textit{for}\quad x>c,\end{cases}$$

where the invertible matrix function satisfy

$$\begin{aligned}\mathcal{W}^{(\delta+\omega)\prime}(c,-d):=&\mathbf{W}'(c+d)+\int_{-d}^c\mathbf{W}'(c-y)(\omega(y)+\delta\mathbf{I})\mathcal{W}^{(\delta+\omega)}(y,-d)dy\\&+\mathbf{W}(0)(\omega(c)+\delta\mathbf{I})\mathcal{W}^{(\delta+\omega)}(c,-d).\end{aligned}$$

3.5 Examples

This section aims to demonstrate some explicit examples of ω-scale matrices when the ω function is specified. We would like to present relations between $\mathcal{W}^{(\omega)}$ and $\boldsymbol{W}^{(q)}$, for some $q\ge 0$, as well as numerical examples which help to understand better the nature of the explored matrix-valued functions. All the examples will assume that the underlying process is the Markov modulated Brownian motion (MMBM). Thus, we will often refer to the Section 3.1.3

3.5.1 Constant state-dependent discount rates

Consider the special case where $\omega_i(x) \equiv \omega_i$ is a constant for all $x \in \mathbb{R}$ and $i \in E$. Therefore, the discounting structure depends on the state of the chain J only. Let us state the following proposition.

Proposition 3.5.1. *Let $\omega_i(x) \equiv \omega_i$ for all $x \in \mathbb{R}$ and $i \in E$. The ω-scale matrix has the Laplace transform*

$$\widetilde{\mathcal{W}}^{(\omega)}(s) = (\mathbf{F}(s) - \boldsymbol{\omega})^{-1}.$$

Proof. Taking the Laplace transform on both sides of (3.21), we have

$$\widetilde{\mathcal{W}}^{(\omega)}(s) = \widetilde{\mathbf{W}}(s) + \widetilde{\mathbf{W}}(s)\boldsymbol{\omega}\widetilde{\mathcal{W}}^{(\omega)}(s),$$

which gives

$$\widetilde{\mathcal{W}}^{(\omega)}(s) = \left(\mathbf{I} - \widetilde{\mathbf{W}}(s)\boldsymbol{\omega}\right)^{-1} \widetilde{\mathbf{W}}(s) = (\mathbf{F}(s) - \boldsymbol{\omega})^{-1}.$$

□

As an example of such ω-scale matrix, we take again the model of MMBM with the following parameters: $\omega_1(x) = \omega_1$, $\omega_2(x) = \omega_2$, $\Delta_{\boldsymbol{\sigma}}$, $\Delta_{\boldsymbol{\mu}}$ and $\boldsymbol{Q}$ are given in (3.8). one can use the inverse of the Laplace transform to get that

$$\mathcal{W}^{(\omega)}(x) = \begin{pmatrix} 2(q_{22}+\omega_2) - \alpha_2^2\sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11}+\omega_1) - \alpha_2^2\sigma_1^2 \end{pmatrix} \frac{e^{\alpha_2 x} - e^{-\alpha_2 x}}{(\alpha_1^2 - \alpha_2^2)\alpha_2\sigma_1^2\sigma_2^2}$$
$$- \begin{pmatrix} 2(q_{22}+q_2) - \alpha_1^2\sigma_2^2 & 2q_{11} \\ 2q_{22} & 2(q_{11}+\delta_1) - \alpha_1^2\sigma_1^2 \end{pmatrix} \frac{e^{\alpha_1 x} - e^{-\alpha_1 x}}{(\alpha_1^2 - \alpha_2^2)\alpha_1\sigma_1^2\sigma_2^2},$$

where

$$\alpha_1 = \frac{\sqrt{M_\omega + \sqrt{(M_\omega)^2 - 4\sigma_1^2\sigma_2^2 K_\omega}}}{\sigma_1\sigma_2}, \quad \alpha_2 = \frac{\sqrt{M_\omega - \sqrt{(M_\omega)^2 - 4\sigma_1^2\sigma_2^2 K_\omega}}}{\sigma_1\sigma_2},$$
$$M_\omega = \sigma_1^2(q_{22}+\omega_2) + \sigma_2^2(q_{11}+\omega_1), \quad K_\omega = q_{11}\omega_2 + \omega_1 q_{22} + \omega_1\omega_2.$$

Note that, for $\omega_1 = \omega_2 = q$, the result is consistent with the previous result for the (q)-scale matrix $\boldsymbol{W}^{(q)}$ in (3.9). Now, let us consider the following setting of the parameters

$$\Delta_{\boldsymbol{\sigma}} = \begin{pmatrix} 1 & 0 \\ 0 & 1.2 \end{pmatrix}, \quad \Delta_{\boldsymbol{\mu}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} -0.05 & 0.05 \\ 0.1 & -0.1 \end{pmatrix},$$
$$\omega_1(x) = 0.05, \quad \omega_2(x) = 0.25,$$

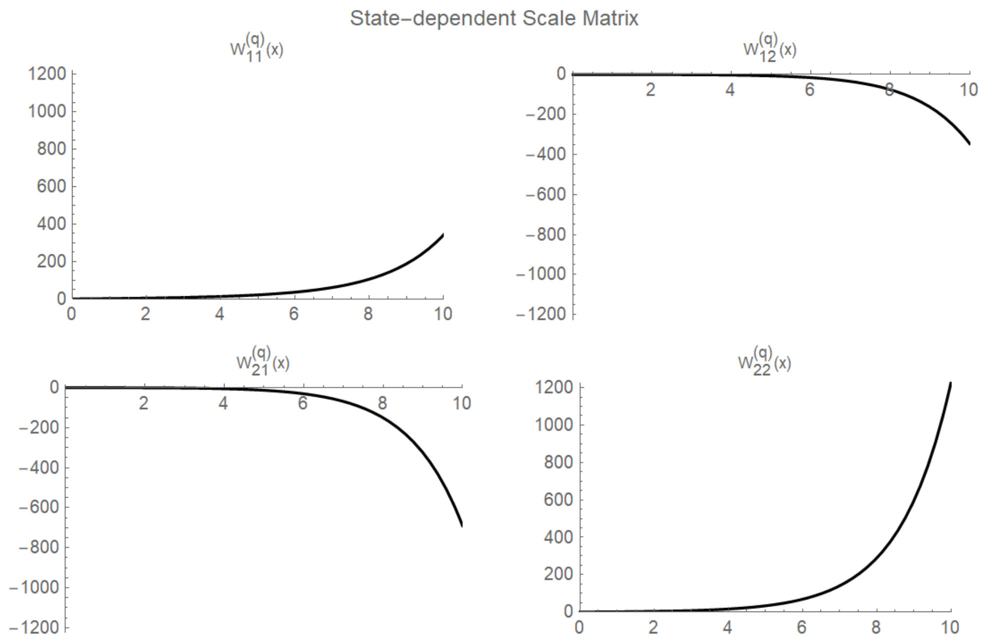

Figure 3.4: Entries of ω-scale matrix function $\mathcal{W}^{(\omega)}$ for state-dependent ω function

In Figure 3.4 one can observe that ω-scale matrix has similar shape as $\boldsymbol{W}^{(q)}$.

3.5.2 Step ω-scale matrix

In this example, we consider the ω function as a positive step function which depends only on the position of the process X. Such an assumption is motivated by the situation where the company has a discount structure depending on its current financial status (or used as an indication of the economic environment). Li and Palmowski [41] showed that, in the case of spectrally negative Lévy processes, such ω-scale functions have recurrent nature. The same observation holds for MAPs.

Proposition 3.5.2. *Assume that ω function is of the form*

$$\omega(i,x) := \omega(x) = p_0 + \sum_{j=1}^{n}(p_j - p_{j-1})1_{\{x>x_j\}}, \qquad \textit{for all } i \in E,$$

where $n \in \mathbb{N}$, $\{p_j\}_{j=0}^n$ is a fixed sequence and $\{x_j\}_{j=1}^n$ is an increasing sequence dividing $\mathbb{R}$ into $(n+1)$ parts. Then the ω-matrix $\mathcal{W}^{(\omega)}(x,y)$ satisfies

$$\mathcal{W}^{(\omega)}(x,y) = \mathcal{W}_n^{(\omega)}(x,y),$$

for $x > y$, where $\mathcal{W}_n^{(\omega)}(x,y)$ is defined recursively as follows:

$$\mathcal{W}_0^{(\omega)}(x,y) = \mathbf{W}^{(p_0)}(x-y),$$

and

$$\mathcal{W}_{k+1}^{(\omega)}(x,y) = \mathcal{W}_k^{(\omega)}(x,y) + (p_{k+1} - p_k)\int_{x_{k+1}}^{x} \mathbf{W}^{(p_{k+1})}(x-z)\mathcal{W}_k^{(\omega)}(z,y)dz,$$

for $x > x_{k+1}$ and $k = 0, 1, \ldots, n-1$.

Proof. Denote $\omega^{(k)}(x) := p_0 + \sum_{j=1}^{k}(p_j - p_{j-1})1_{\{x>x_j\}}$ with $\omega^{(0)}(x) = p_0$. From Equation (3.31), we get that

$$\mathcal{W}_k^{(\omega)}(x,y) = \mathbf{W}^{(p_{k+1})}(x-y) + \int_y^x (\omega^{(k)}(z) - p_{k+1})\mathbf{W}^{(p_{k+1})}(x-z)\mathcal{W}_k^{(\omega)}(z,y)dz, \tag{3.48}$$

and

$$\mathcal{W}_{k+1}^{(\omega)}(x,y) = \mathbf{W}^{(p_{k+1})}(x-y) + \int_y^x (\omega^{(k+1)}(z) - p_{k+1})\mathbf{W}^{(p_{k+1})}(x-z)\mathcal{W}_{k+1}^{(\omega)}(z,y)dz. \tag{3.49}$$

Note that $\omega^{(k+1)}(z) - p_{k+1} = 0$ for $z > x_{k+1}$ and $\omega^{(k+1)}(z) = \omega^{(k)}(z)$ for $z \le x_{k+1}$. Thus from Lemma 3.3.1, we have

$$\mathcal{W}_{k+1}^{(\omega)}(x,y) = \mathcal{W}_k^{(\omega)}(x,y),$$

for $x \le x_{k+1}$. Equation (3.49) could be rewritten as

$$\begin{aligned}\mathcal{W}_{k+1}^{(\omega)}(x,y) &= \mathbf{W}^{(p_{k+1})}(x-y) + \int_y^{x_k} (\omega^{(k+1)}(z) - p_{k+1})\mathbf{W}^{(p_{k+1})}(x-z)\mathcal{W}_{k+1}^{(\omega)}(z,y)dz \\ &= \mathbf{W}^{(p_{k+1})}(x-y) + \int_y^{x_k} (\omega^{(k)}(z) - p_{k+1})\mathbf{W}^{(p_{k+1})}(x-z)\mathcal{W}_{k+1}^{(\omega)}(z,y)dz \\ &= \mathcal{W}_k^{(\omega)}(x,y) - \int_{x_k}^{x} (\omega^{(k)}(z) - p_{k+1})\mathbf{W}^{(p_{k+1})}(x-z)\mathcal{W}_k^{(\omega)}(z,y)dz,\end{aligned}$$

where the last step uses (3.48). The proof is completed by noticing that $\omega^{(k)}(z) - p_{k+1} = p_k - p_{k+1}$ for $z > x_{k+1}$. $\square$

Note also that similar considerations will lead to the same result for the second ω-scale matrix $\mathcal{Z}^{(\omega)}$.

In the following proposition, we will compute the matrix $\mathcal{W}^{(\omega)}$ for one particular case.

Proposition 3.5.3. *Let (X,J) be a Markov modulated Brownian motion with $\mu_i \in \mathbb{R}$ and $\sigma_i^2 > 0$ for all $i \in E$. Assume that $\{p_j\}_{j=0}^n = \{p_0, p_1\}$ and $\{x_j\}_{j=1}^n = \{x_1\}$ with p_0, p_1, x_1 being positive numbers. Then, for $x \le x_1$,*

$$\mathcal{W}^{(\omega)}(x,y) = \boldsymbol{W}^{(p_0)}(x-y),$$

and for $x > x_1$,

$$\begin{aligned}\mathcal{W}^{(\omega)}(x,y) =& \Big(e^{-\mathbf{\Lambda}_{p_1}^+(x-x_1)}\Big(\mathbf{\Lambda}_{p_1}^+ + \mathbf{\Lambda}_{p_1}^-\Big)^{-1}\mathbf{\Lambda}_{p_1}^- + e^{\mathbf{\Lambda}_{p_1}^-(x-x_1)}\Big(\mathbf{\Lambda}_{p_1}^+ + \mathbf{\Lambda}_{p_1}^-\Big)^{-1}\mathbf{\Lambda}_{p_1}^+\Big) \\ &\cdot \boldsymbol{W}^{(p_0)}(x_1-y) - \boldsymbol{W}^{(p_1)}(x-x_1)\Delta_{\frac{\sigma^2}{2}}\boldsymbol{W}^{(p_0)\prime}(x_1-y).\end{aligned}$$

Proof. Note that the case for $x \le x_1$ is a straightforward conclusion from Proposition 3.5.2. For $x > x_1$, from previous Proposition and (3.4), we have

$$\begin{aligned}\mathcal{W}_1(x,y) =& \mathcal{W}_0(x,y) + (p_1 - p_0)\int_{x_1}^x \boldsymbol{W}^{(p_1)}(x-z)\mathcal{W}_0(z,y)dz \\ =& \boldsymbol{W}^{(p_0)}(x-y) + (p_1-p_0)\int_{x_1}^x \Big(e^{-\mathbf{\Lambda}_{p_1}^+(x-z)}\Xi_{p_1}e^{-\mathbf{\Lambda}_{p_0}^+(z-y)} \\ &- e^{-\mathbf{\Lambda}_{p_1}^+(x-z)}\Xi_{p_1}e^{\mathbf{\Lambda}_{p_0}^-(z-y)} - e^{\mathbf{\Lambda}_{p_1}^-(x-z)}\Xi_{p_1}e^{-\mathbf{\Lambda}_{p_0}^+(z-y)} + e^{\mathbf{\Lambda}_{p_1}^-(x-z)}\Xi_{p_1}e^{\mathbf{\Lambda}_{p_0}^-(z-y)}\Big)dz\ \Xi_{p_0}.\end{aligned} \tag{3.50}$$

We start by identifying the following integral appearing in Equation (3.50):

$$\int_{x_1}^{x}\left(e^{-\mathbf{\Lambda}^+_{p_1}(x-z)}\Xi_{p_1}e^{-\mathbf{\Lambda}^+_{p_0}(z-y)}\right)dz. \tag{3.51}$$

Consider (3.51) as a function $M_1 : A \to \mathbb{R}^{N\times N}$, where

$$A = \{(x, y) : x \geq x_1, x > y\},$$

and $N \times N$ is the dimension of the matrix $\boldsymbol{W}^{(p_0)}$. Then

$$M_1(x,y) = \int_{x_1}^{x}\left(e^{-\mathbf{\Lambda}^+_{p_1}(x-z)}\Xi_{p_1}e^{-\mathbf{\Lambda}^+_{p_0}(z-y)}\right)dz = e^{-\mathbf{\Lambda}^+_{p_1}x}\int_{x_1}^{x}\left(e^{\mathbf{\Lambda}^+_{p_1}z}\Xi_{p_1}e^{-\mathbf{\Lambda}^+_{p_0}z}\right)dz\ e^{\mathbf{\Lambda}^+_{p_0}y}.$$

By taking partial derivatives of M_1 with respect to x and y, we get

$$\begin{cases} \frac{\partial M_1(x,y)}{\partial x} = -\mathbf{\Lambda}^+_{p_1}M_1(x,y) + \Xi_{p_1}e^{-\mathbf{\Lambda}^+_{p_0}(x-y)}, \\ \frac{\partial M_1(x,y)}{\partial y} = M_1(x,y)\mathbf{\Lambda}^+_{p_0}, \end{cases}$$

with the boundary conditions

$$M_1(x_1,y) = \mathbf{0} \quad \text{and} \quad K_1(x) := M_1(x,x_1) = \int_{x_1}^{x}\left(e^{-\mathbf{\Lambda}^+_{p_1}(x-z)}\Xi_{p_1}e^{-\mathbf{\Lambda}^+_{p_0}(z-x_1)}\right)dz.$$

The derivative of $K_1(x)$ is equal to

$$K_1'(x) = -\mathbf{\Lambda}^+_{p_1}K_1(x) + \Xi_{p_1}e^{-\mathbf{\Lambda}^+_{p_0}(x-x_1)}, \tag{3.52}$$

with the boundary condition $K_1(x_1) = \mathbf{0}$. We will prove that the solution of the above differential equation is of the form

$$K_1(x) = \boldsymbol{C}e^{-\mathbf{\Lambda}^+_{p_0}(x-x_1)} - e^{-\mathbf{\Lambda}^+_{p_1}(x-x_1)}\boldsymbol{C}, \tag{3.53}$$

where $\boldsymbol{C}$ is some constant matrix. To do this, we need to verify our guess for $K_1(x)$ by plugging it into (3.52). After some calculation, one can prove that (3.53) is indeed the solution if the following equation holds

$$\mathbf{\Lambda}^+_{p_1}\boldsymbol{C} - \boldsymbol{C}\mathbf{\Lambda}^+_{p_0} = \Xi_{p_1}. \tag{3.54}$$

The above equality is an example of a well-known Sylvester equation. One usually needs to rely on numerical methods to solve equations of this type. However, in this case, one can make a guess and check for the formula for $\boldsymbol{C}$:

$$\boldsymbol{C} = -\left(\mathbf{\Lambda}^+_{p_1} + \mathbf{\Lambda}^-_{p_1}\right)^{-1}\left(\mathbf{\Lambda}^+_{p_0} + \mathbf{\Lambda}^-_{p_1}\right)\cdot\frac{1}{p_1 - p_0}.$$

Indeed, putting it to equation (3.54), one can verify that the following equations are equivalent

$$\begin{aligned}
&\Big[-\mathbf{\Lambda}^+_{p_1}\Big(\mathbf{\Lambda}^+_{p_1}+\mathbf{\Lambda}^-_{p_1}\Big)^{-1}\Big(\mathbf{\Lambda}^+_{p_0}+\mathbf{\Lambda}^-_{p_1}\Big)+\Big(\mathbf{\Lambda}^+_{p_1}+\mathbf{\Lambda}^-_{p_1}\Big)^{-1}\Big(\mathbf{\Lambda}^+_{p_0}+\mathbf{\Lambda}^-_{p_1}\Big)\mathbf{\Lambda}^+_{p_0}\Big]\frac{1}{p_1-p_0}=\Xi_{p_1},\\
&\Big[\Big(\mathbf{\Lambda}^+_{p_1}+\mathbf{\Lambda}^-_{p_1}\Big)\mathbf{\Lambda}^+_{p_1}\Big(\mathbf{\Lambda}^+_{p_1}+\mathbf{\Lambda}^-_{p_1}\Big)^{-1}\Big(\mathbf{\Lambda}^+_{p_0}+\mathbf{\Lambda}^-_{p_1}\Big)-\Big(\mathbf{\Lambda}^+_{p_0}+\mathbf{\Lambda}^-_{p_1}\Big)\mathbf{\Lambda}^+_{p_0}\Big]\cdot\frac{1}{p_1-p_0}=\Delta_{\frac{2}{\sigma^2}},\\
&\Big[\Big(\Delta_{\frac{2\mu}{\sigma^2}}+\mathbf{\Lambda}^-_{p_1}\Big)\Big(\mathbf{\Lambda}^+_{p_0}+\mathbf{\Lambda}^-_{p_1}\Big)-(\mathbf{\Lambda}^+_{p_0})^2-\mathbf{\Lambda}^-_{p_1}\mathbf{\Lambda}^+_{p_0}\Big]\cdot\frac{1}{p_1-p_0}=\Delta_{\frac{2}{\sigma^2}},\\
&\Big[\Delta_{\frac{2\mu}{\sigma^2}}\mathbf{\Lambda}^+_{p_0}+\Delta_{\frac{2\mu}{\sigma^2}}\mathbf{\Lambda}^-_{p_1}+(\mathbf{\Lambda}^-_{p_1})^2-(\mathbf{\Lambda}^+_{p_0})^2\Big]\cdot\frac{1}{p_1-p_0}=\Delta_{\frac{2}{\sigma^2}},\\
&\Big[\Big(\boldsymbol{Q}-p_0\mathbf{I}\Big)-\Big(\boldsymbol{Q}-p_1\mathbf{I}\Big)\Big]\cdot\frac{1}{p_1-p_0}=\mathbf{I},\\
&\mathbf{I}=\mathbf{I}.
\end{aligned}$$

In the second line of the above calculations, we apply the definition of Ξ_{p_1}. Then, the third equation follows from the second by the relation (3.7). Finally, to get the fifth equation, we make use of (3.6) as well as (3.7). Therefore, $K_1(x)$ is a solution to the differential equation (3.52). It is now straightforward to check the expression for $M_1(x,y)$, i.e.,

$$M_1(x,y)=\boldsymbol{C}e^{-\mathbf{\Lambda}^+_{p_0}(x-y)}-e^{-\mathbf{\Lambda}^+_{p_1}(x-x_1)}\boldsymbol{C}e^{-\mathbf{\Lambda}^+_{p_0}(x_1-y)}.$$

Following similar reasoning as for the derivation of M_1, one can determine other integrals appearing in Equation (3.50), namely

$$\begin{aligned}
M_2(x,y)&=\int_{x_1}^{x}e^{-\mathbf{\Lambda}^+_{p_1}(x-z)}\Xi_{p_1}e^{\mathbf{\Lambda}^-_{p_0}(z-y)}dz=\boldsymbol{D}e^{\mathbf{\Lambda}^-_{p_0}(x-y)}-e^{-\mathbf{\Lambda}^+_{p_1}(x-x_1)}\boldsymbol{D}e^{\mathbf{\Lambda}^-_{p_0}(x_1-y)},\\
M_3(x,y)&=\int_{x_1}^{x}e^{\mathbf{\Lambda}^-_{p_1}(x-z)}\Xi_{p_1}e^{-\mathbf{\Lambda}^+_{p_0}(z-y)}dz=\boldsymbol{E}e^{-\mathbf{\Lambda}^+_{p_0}(x-y)}-e^{\mathbf{\Lambda}^-_{p_1}(x-x_1)}\boldsymbol{E}e^{-\mathbf{\Lambda}^+_{p_0}(x_1-y)},\\
M_4(x,y)&=\int_{x_1}^{x}e^{\mathbf{\Lambda}^-_{p_1}(x-z)}\Xi_{p_1}e^{\mathbf{\Lambda}^-_{p_0}(z-y)}dz=\boldsymbol{F}e^{\mathbf{\Lambda}^-_{p_0}(x-y)}-e^{\mathbf{\Lambda}^-_{p_1}(x-x_1)}\boldsymbol{F}e^{\mathbf{\Lambda}^-_{p_0}(x_1-y)},
\end{aligned}$$

where matrices $\boldsymbol{C},\boldsymbol{D},\boldsymbol{E},\boldsymbol{F}$ are given by

$$\begin{aligned}
\boldsymbol{C}&=-\Big(\mathbf{\Lambda}^+_{p_1}+\mathbf{\Lambda}^-_{p_1}\Big)^{-1}\Big(\mathbf{\Lambda}^+_{p_0}+\mathbf{\Lambda}^-_{p_1}\Big)\cdot\frac{1}{p_1-p_0},\\
\boldsymbol{D}&=\Big(\mathbf{\Lambda}^+_{p_1}+\mathbf{\Lambda}^-_{p_1}\Big)^{-1}\Big(\mathbf{\Lambda}^-_{p_0}-\mathbf{\Lambda}^-_{p_1}\Big)\cdot\frac{1}{p_1-p_0},\\
\boldsymbol{E}&=-\Big(\mathbf{\Lambda}^+_{p_1}+\mathbf{\Lambda}^-_{p_1}\Big)^{-1}\Big(\mathbf{\Lambda}^+_{p_0}-\mathbf{\Lambda}^+_{p_1}\Big)\cdot\frac{1}{p_1-p_0},\\
\boldsymbol{F}&=\Big(\mathbf{\Lambda}^+_{p_1}+\mathbf{\Lambda}^-_{p_1}\Big)^{-1}\Big(\mathbf{\Lambda}^-_{p_0}+\mathbf{\Lambda}^+_{p_1}\Big)\cdot\frac{1}{p_1-p_0}.
\end{aligned}$$

Plugging them back to (3.50), we have for $x > x_1$,

$$\begin{aligned}
\mathcal{W}_1^{(\omega)}(x,y) =& \Big(e^{-\boldsymbol{\Lambda}_{p_0}^{+}(x-y)} - e^{\boldsymbol{\Lambda}_{p_0}^{-}(x-y)}\Big)\Xi_{p_0} \\
&+ (p_1 - p_0)\Big(M_1(x,y) - M_2(x,y) - M_3(x,y) + M_4(x,y)\Big)\Xi_{p_0} \\
=& \Big[\Big(\mathbf{I} - (p_1 - p_0)\Big(\boldsymbol{E} - \boldsymbol{C}\Big)\Big)e^{-\boldsymbol{\Lambda}_{p_0}^{+}(x-y)} - \Big(\mathbf{I} - (p_1 - p_0)\Big(\boldsymbol{D} + \boldsymbol{F}\Big)\Big)e^{\boldsymbol{\Lambda}_{p_0}^{-}(x-y)} \\
&+ (p_1 - p_0)\Big(e^{\boldsymbol{\Lambda}_{p_1}^{-}(x-x_1)}\Big(\boldsymbol{E}e^{-\boldsymbol{\Lambda}_{p_0}^{+}(x_1-y)} - \boldsymbol{F}e^{\boldsymbol{\Lambda}_{p_0}^{-}(x_1-y)}\Big) \\
&- e^{-\boldsymbol{\Lambda}_{p_1}^{+}(x-x_1)}\Big(\boldsymbol{C}e^{-\boldsymbol{\Lambda}_{p_0}^{+}(x_1-y)} - \boldsymbol{D}e^{\boldsymbol{\Lambda}_{p_0}^{-}(x_1-y)}\Big)\Big)\Big]\Xi_{p_0} \\
=& \Big[e^{-\boldsymbol{\Lambda}_{p_1}^{+}(x-x_1)}\Big(\boldsymbol{\Lambda}_{p_1}^{+} + \boldsymbol{\Lambda}_{p_1}^{-}\Big)^{-1}\boldsymbol{\Lambda}_{p_1}^{-} + e^{\boldsymbol{\Lambda}_{p_1}^{-}(x-x_1)}\Big(\boldsymbol{\Lambda}_{p_1}^{+} + \boldsymbol{\Lambda}_{p_1}^{-}\Big)^{-1}\boldsymbol{\Lambda}_{p_1}^{+}\Big] \\
&\cdot \boldsymbol{W}^{(p_0)}(x_1 - y) - \boldsymbol{W}^{(p_1)}(x - x_1)\Delta_{\frac{\sigma^2}{2}}\boldsymbol{W}^{(p_0)\prime}(x_1 - y),
\end{aligned}$$

where we notice that

$$(p_1 - p_0)\Big(\boldsymbol{E} - \boldsymbol{C}\Big) = \mathbf{I}, \quad (p_1 - p_0)\Big(\boldsymbol{D} + \boldsymbol{F}\Big) = \mathbf{I}.$$

This completes the proof. Note that the uniqueness of this result is a straightforward conclusion from Lemma 3.3.1. □

Remark 3.5.4. *In general, if we choose to divide $\mathbb{R}$ into more intervals, a similar idea could be adopted for computing the ω-scale matrix.*

One can be interested in the shape of such a ω-scale matrix. Therefore, let us present a numerical approximation to this function, with the following choice of the parameters

$$\begin{aligned}
&\Delta_{\boldsymbol{\sigma}} = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.85 \end{pmatrix}, \quad \Delta_{\boldsymbol{\mu}} = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.1 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} -0.1 & 0.1 \\ 0.3 & -0.3 \end{pmatrix}, \\
&p_0 = 0.25, \quad p_1 = 0.03, \quad x_1 = 4.
\end{aligned}$$

Note that under the assumption of $\Delta_{\boldsymbol{\mu}} \neq \mathbf{0}$, we cannot use the formula (3.9). Instead, we use numerical package from Ivanovs [27] for the computations.

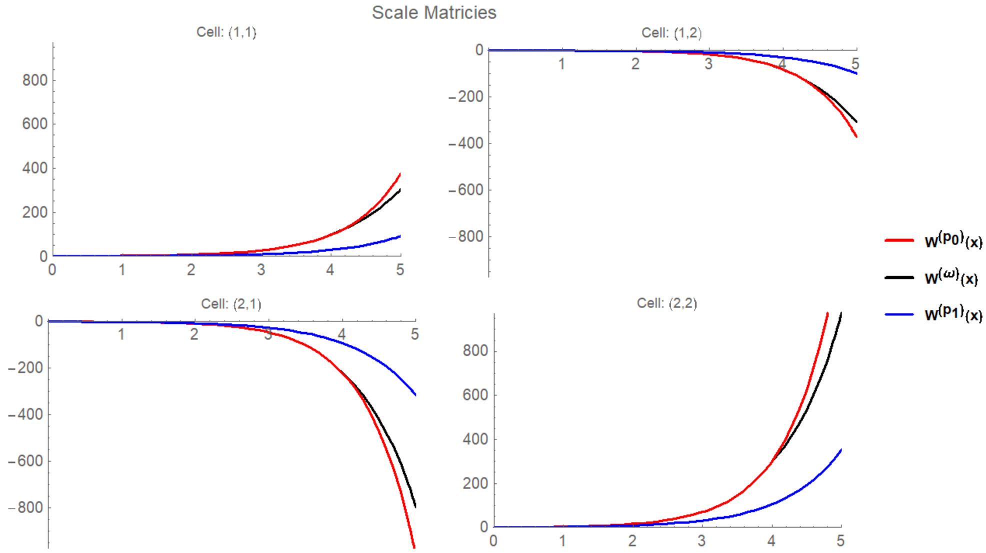

Figure 3.5: Comparison between entries of scale matrix $\boldsymbol{W}^{(p_0)}$,$\boldsymbol{W}^{(p_1)}$ and entries of ω-scale matrix function $\mathcal{W}^{(\omega)}$

From Figure 3.5 one can see that in every cell we have interesting relation that $\mathcal{W}^{(\omega)}$ lies between $\boldsymbol{W}^{(p_0)}$ and $\boldsymbol{W}^{(p_1)}$ and these functions are similar in shape.

3.5.3 Omega model

In Section 3.4, we examined the (Omega) dividend problem in general Markov additive model, where the formula for the value function was derived in terms of the ω-scale matrix. In this subsection, we will revisit this problem under MMBM and for a specific choice of the ω function, namely

$$\omega_i(x) := \omega(x) = \Big(\gamma_0 + \gamma_1(x+d)\Big)1_{\{-d\le x\le 0\}}, \quad \text{for all } i \in E,$$

where $\gamma_0 > 0$ and $\gamma_1 < 0$ are some constants such that the ω function is decreasing in x. A similar model for the Lévy-risk process was analysed in Li and Palmowski [41].
Let us fix a constant force of interest $\delta \ge 0$. Using (3.28) one can obtain that $\mathcal{W}^{(\omega+\delta)}$ satisfy the following equation for $x \in [-d, 0]$,

$$\begin{aligned}\mathcal{W}^{(\omega+\delta)}(x,-d) &= \mathbf{W}(x+d) + \int_{-d}^{x}(\omega(z)+\delta)\mathbf{W}(x-z)\mathcal{W}^{(\omega)}(z,-d)dz \\ &= \mathbf{W}(x+d) + \int_{0}^{x+d}(\omega(y-d)+\delta)\mathbf{W}(x+d-y)\mathcal{W}^{(\omega)}(y-d,-d)dy \\ &= \mathbf{W}^{(\gamma_0+\delta)}(x+d) + \gamma_1\int_{0}^{x+d} y\mathbf{W}^{(\gamma_0+\delta)}(x+d-y)\mathcal{W}^{(\omega)}(y-d,-d)dy.\end{aligned}$$

Now, let $z = x + d \ge 0$ and

$$\mathbf{G}(z) := \mathcal{W}^{(\omega+\delta)}(z-d,-d) = \mathcal{W}^{(\omega+\delta)}(x,-d). \tag{3.55}$$

Then we can rewrite equation for $\mathcal{W}^{(\omega+\delta)}$ as

$$\boldsymbol{G}(z) = \boldsymbol{W}^{(\gamma_0+\delta)}(z) + \gamma_1 \int_0^z y\boldsymbol{W}^{(\gamma_0+\delta)}(z-y)\boldsymbol{G}(y)dy.$$

From equation (3.4), we obtain the following equation for $\boldsymbol{W}^{(\gamma_0+\delta)}$

$$\Big(\frac{d}{dz} - \boldsymbol{C}_{\gamma_0+\delta}\Big)\Big(\frac{d}{dz} + \Lambda^+_{\gamma_0+\delta}\Big)\boldsymbol{W}^{(\gamma_0+\delta)}(z) = \boldsymbol{0}, \tag{3.56}$$

where $\boldsymbol{C}_{\gamma_0+\delta} = (\boldsymbol{\Lambda}^+_{\gamma_0+\delta} + \boldsymbol{\Lambda}^-_{\gamma_0+\delta})\boldsymbol{\Lambda}^-_{\gamma_0+\delta}(\boldsymbol{\Lambda}^+_{\gamma_0+\delta} + \boldsymbol{\Lambda}^-_{\gamma_0+\delta})^{-1}$.
Based on (3.55), for $z \in [0, d]$ (or equivalently for $x \in [-d, 0]$) we have

$$\Big(\frac{d}{dz} - \boldsymbol{C}_{\gamma_0+\delta}\Big)\Big(\frac{d}{dz} + \Lambda^+_{\gamma_0+\delta}\Big)\boldsymbol{G}(z) = \gamma_1 z \Delta_{\frac{2}{\boldsymbol{\sigma}^2}}\boldsymbol{G}(z), \tag{3.57}$$

with the boundary conditions $\boldsymbol{G}(0) = \boldsymbol{0}$ and $\boldsymbol{G}'(0) = \Delta_{\frac{2}{\boldsymbol{\sigma}^2}}$.
Let us rewrite the above differential matrix equation into the following form

$$\boldsymbol{G}''(z) + \Big(\boldsymbol{\Lambda}^+_{\gamma_0+\delta} - C_{\gamma_0+\delta}\Big)\boldsymbol{G}'(z) - \Big(\boldsymbol{C}_{\gamma_0+\delta}\boldsymbol{\Lambda}^+_{\gamma_0+\delta} + 2\gamma_1 z\Delta_{1/\boldsymbol{\sigma}^2}\Big)\boldsymbol{G}(z) = \boldsymbol{0},$$

which by (3.6) can be simplified to,

$$\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{G}''(z) + \Delta_{\boldsymbol{\mu}}\boldsymbol{G}'(z) + \boldsymbol{Q}\boldsymbol{G}(z) - (\omega_1(z)+\delta)\boldsymbol{G}(z) = \boldsymbol{0}, \quad \text{for } z \in [0, d].$$

Now, we will treat the case of $z \geq d$ (or equivalently for $x \geq 0$). We first rewrite the formula

$$\mathcal{W}^{(\omega+\delta)}(x; -d) = \boldsymbol{W}(x+d) + \int_0^{x+d} \omega(y-d)\boldsymbol{W}(x+d-y)\mathcal{W}^{(\omega+\delta)}(y-d; -d)dy, \quad \text{for } x \geq 0,$$

in terms of matrix $\boldsymbol{G}(z)$ with respect to $z \geq d$:

$$\boldsymbol{G}(z) = \boldsymbol{W}(z) + \int_0^d (\delta + (\gamma_0 + \gamma_1 y))\boldsymbol{W}(z-y)\boldsymbol{G}(y)dy + \delta \int_d^z \boldsymbol{W}(z-y)\boldsymbol{G}(y)dy.$$

Similar to (3.56) and (3.57), we have, respectively

$$\Big(\frac{d}{dz} - \boldsymbol{C}\Big)\Big(\frac{d}{dz} + \boldsymbol{\Lambda}^+\Big)\boldsymbol{W}(z) = \boldsymbol{0},$$

and

$$\Big(\frac{d}{dz} - \boldsymbol{C}\Big)\Big(\frac{d}{dz} + \boldsymbol{\Lambda}^+\Big)\boldsymbol{G}(z) = \delta\Delta_{\frac{2}{\boldsymbol{\sigma}^2}}\boldsymbol{G}(z), \quad \text{for } z \geq d,$$

where $\boldsymbol{C} = (\boldsymbol{\Lambda}^+ + \boldsymbol{\Lambda}^-)\boldsymbol{\Lambda}^-(\boldsymbol{\Lambda}^+ + \boldsymbol{\Lambda}^-)^{-1}$. Using (3.6) for $q = 0$, one can get that

$$\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{G}''(z) + \Delta_{\boldsymbol{\mu}}\boldsymbol{G}'(z) + \boldsymbol{Q}\boldsymbol{G}(z) - \delta\boldsymbol{G}(z) = \boldsymbol{0}, \quad \text{for } z > d.$$

Summarizing, $\boldsymbol{G}(z)$ satisfies the following differential equations:

$$\begin{aligned}\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{G}''(z) + \Delta_{\boldsymbol{\mu}}\boldsymbol{G}'(z) + \boldsymbol{Q}\boldsymbol{G}(z) - (\omega_1(z)+\delta)\boldsymbol{G}(z) &= \boldsymbol{0}, \quad \text{for } z \in [0, d],\\ \Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{G}''(z) + \Delta_{\boldsymbol{\mu}}\boldsymbol{G}'(z) + \boldsymbol{Q}\boldsymbol{G}(z) - \delta\boldsymbol{G}(z) &= \boldsymbol{0}, \quad \text{for } z > d,\end{aligned}$$

with the boundary conditions $\boldsymbol{G}(0) = \boldsymbol{0}$, and $\boldsymbol{G}'(0) = \Delta_{\frac{2}{\boldsymbol{\sigma}^2}}$.
Therefore from (3.55) for $x \in [-d, 0]$ we obtain

$$\Delta_{\frac{\boldsymbol{\sigma}^2}{2}} \mathcal{W}^{(\omega+\delta)\prime\prime}(x, -d) + \Delta_{\boldsymbol{\mu}} \mathcal{W}^{(\omega+\delta)\prime}(x, -d) - ((\omega(x+d) + \delta)\boldsymbol{I} - \boldsymbol{Q})\mathcal{W}^{(\omega+\delta)}(x, -d) = \boldsymbol{0},$$

and for $x > 0$,

$$\Delta_{\frac{\boldsymbol{\sigma}^2}{2}} \mathcal{W}^{(\omega+\delta)\prime\prime}(x, -d) + \Delta_{\boldsymbol{\mu}} \mathcal{W}^{(\omega+\delta)\prime}(x, -d) - (\delta\boldsymbol{I} - \boldsymbol{Q})\mathcal{W}^{(\omega+\delta)}(x, -d) = \boldsymbol{0},$$

with the boundary conditions $\mathcal{W}^{(\omega+\delta)}(-d, -d) = \boldsymbol{0}$ and $\mathcal{W}^{(\omega+\delta)\prime}(-d, -d) = \Delta_{\frac{2}{\boldsymbol{\sigma}^2}}$.
Before we proceed to numerical example, we recall that N is the cardinality of the state space E and $\mathcal{W}^{(\omega+\delta)}$ maps $\mathbb{R}$ into $\mathbb{R}^{N\times N}$. Thus, one can see that the differential equations for $\mathcal{W}^{(\omega+\delta)}$ can be treated as a (2×N)th-order system of the second-order initial-value problems. As usual, in such a setting, one can introduce new unknown functions which are derivative of the remaining functions. Then we obtain (4×N)th-order system of the first-order initial-value problems for which rich collections of iterative algorithms exist (e.g. Runge-Kutta methods). Let us focus on the uniqueness and the existence in the general case. For reference see e.g. Burden and Faires [10]. Namely, recall that every mth-order system of the first-order initial-value problems can be written in the form of

$$\frac{dy_i}{dt} = g_i(t, y_1, y_2, ..., y_m),$$

where for all $i \in \{1, 2, ..., m\}$, g_i is assumed to be defined on some set

$$D_i = \{(t, y_1, ..., y_m) : a \le t \le b, -\infty < y_k < \infty, \forall k = 1, 2, ..., m\}.$$

Then the system has a unique solution $y_1(t), y_2(t), ..., y_m(t)$, for $a \le t \le b$ if all g_i's are continuous on D_i and satisfy the Lipschitz condition with respect to $(y_1, y_2, ..., y_m)$.
In the framework of this section, we choose $a = -d$ and $b = t_{max}$ as a upper limit of our approximation. It is also clear that if we choose ω to be continuous, the above sufficient condition holds. For illustration, with the following parameters

$$\Delta_{\boldsymbol{\sigma}} = \begin{pmatrix} 1.2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \Delta_{\boldsymbol{\mu}} = \begin{pmatrix} 1.75 & 0 \\ 0 & 1.25 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} -0.4 & 0.4 \\ 0.2 & -0.2 \end{pmatrix},$$

$$\gamma_0 = 0.5, \quad \gamma_1 = -0.1, \quad d = 5, \quad t_{max} = 10 \quad \text{and} \quad \delta = 0.04.$$

we present Figure 3.6 showing entries of the numerical approximations of the matrix function $\mathcal{W}^{(\omega+\delta)}$. The main difference between the classical scale matrix and the (ω)-scale matrix is that here we have non-zero values in the interval $(-d, 0]$.

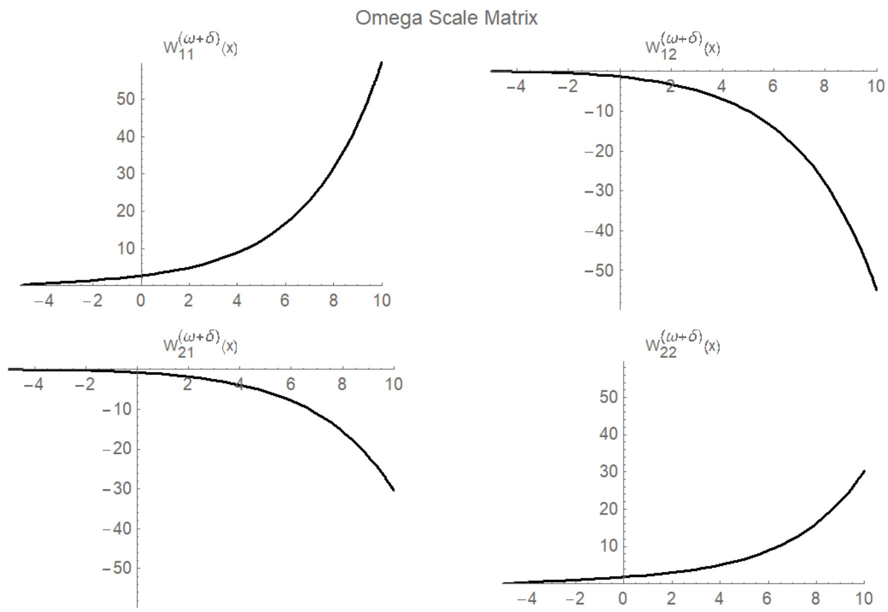

Figure 3.6: Entries of ω-scale matrix function $\mathcal{W}^{(\omega+\delta)}$

Practical applications of our models and results will rely heavily on numerical evaluation. For instance, one can use the numerical approach presented here to approximate the value function of the dividend strategy in the Omega model. Moreover, one can produce similar experiments for different choices of ω to capture the other discount structures or bankruptcy rates in the context, bringing ω-scale matrices closer to intuitions. To close this section, let us present a heat map of the value function of the dividend strategy.

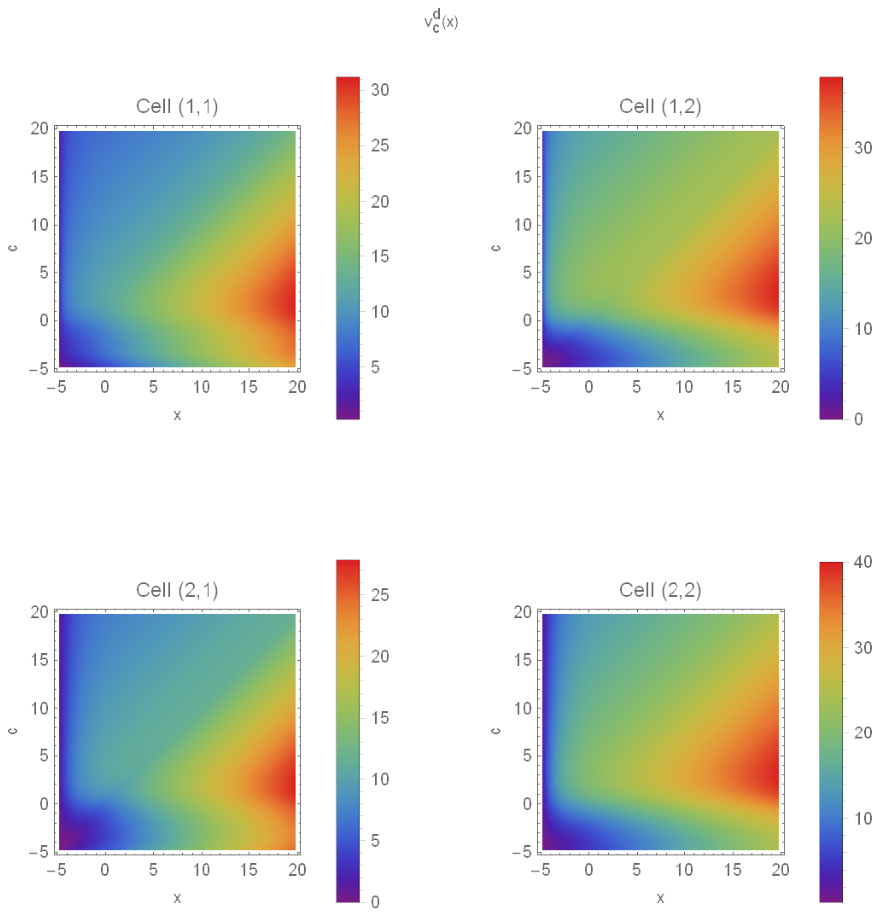

Figure 3.7: Approximation to the value function of the dividend strategy in the Omega model

One can observe that for every x, the same point c is optimal. However, these are only visual observations that make intuitions useful for solving optimal dividend problems.

3.5.4 Numerical approach to Omega model for Markov modulated Brownian motion

In this example, we will continue our analysis of the Omega model for Markov modulated Brownian motion. For clarity of numerical examples we will fix state space of J to $E = \{1, 2\}$. We aim to numerically compute the probability of bankruptcy in this model. From the definition of $\mathbf{\Lambda}_q^+$ we know that for $i, j \in E$

$$\mathbb{P}\Big(\tau_x^+ < e_q, J_{\tau_x^+} = j \Big| J_0 = i\Big) = \Big(e^{\mathbf{\Lambda}_q^+ x}\Big)_{ij}.$$

Matrix $\mathbf{\Lambda}_q^-$ play the same role for the process $(-X, J)$ (which is MMBM but with the drift vector $-\boldsymbol{\mu}$). One can use the next proposition to identify classical ruin time for MMBM. Note that the same can be proven just using spatial homogeneity. However, we want to show that Corollary 3.3.7 involves quantities that one can compute semi-explicitly.

Proposition 3.5.5. *For $x \geq 0$ and $q \geq 0$ we have that*

$$\boldsymbol{P}_x\Big(\tau_0^- < e_q, J_{\tau_0^-}\Big) = \boldsymbol{Z}^{(q)}(x) - \boldsymbol{W}^{(q)}(x)\boldsymbol{C}_{W(\infty)^{-1}Z(\infty)} = e^{\boldsymbol{\Lambda}_q^- x},$$

where e_q is independent exponential random variable with the parameter q (if $q = 0$ then we set $e_q = \infty$) and $\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)} := \lim_{c\to\infty} \boldsymbol{W}^{(q)}(c)^{-1}\boldsymbol{Z}^{(q)}(c)$.

We left the proof of this proposition at the end of this section due to long calculations.
After setting $q = 0$ in the above proposition, we get formula for classical ruin probability, namely for $x \geq 0$

$$\boldsymbol{P}_x\Big(\tau_0^- < \infty | J_0\Big) = e^{\boldsymbol{\Lambda}^- x}\vec{1},$$

where $\vec{1}$ is a column vector of ones of the size $N \times 1$. One can observe that for every $i \in E$

$$\mathbb{P}_x\Big(\tau_{-d}^- < \infty | J_0 = i\Big) \leq \mathbb{P}_x\Big(\tau_\omega^d < \infty | J_0 = i\Big) \leq \mathbb{P}_x\Big(\tau_0^- < \infty | J_0 = i\Big).$$

Before we state our numerical method to compute the probability of Omega bankruptcy, we need to consider the numerical method to obtain an approximation of the $\boldsymbol{\Lambda}_q^{\pm}$ matrices. Formally our approximation will be valid for $q \geq 0$, but in examples, we will be interested in the case of $q = 0$. Here we will quote the result from Breuer [9] where an iterative method was derived. One can use other methods, for example, involving spectral analysis of matrix $\boldsymbol{\Lambda}_q^{\pm}$, see D'Auria *et al.* [20].
Let us recall that $\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$ is an inverse of the function $\psi(\theta)$. In the case of linear Brownian motion, we have an explicit formula for this function, namely

$$\Phi(q) = \frac{-\mu + \sqrt{\mu^2 + 2q\sigma^2}}{\sigma^2}. \tag{3.58}$$

Let us denote $\Phi_i(q)$ as a function related to the linear Brownian motion with the parameters μ_i and σ_i for $i \in E$. Then we denote $\Delta_\Phi := diag(\Phi(q_i + q))_{i\in E}$, where $q_i = -q_{ii}$ and q_{ii} is (i,i)-entry of the matrix $\boldsymbol{Q}$.
Let $\boldsymbol{U}_0 := -\Delta_\Phi$ and $\boldsymbol{U}_{n+1} := g(\boldsymbol{U}_n)$ for $n > 0$ where row i of the matrix $g(\boldsymbol{U}_n)$ is defined as follows

$$\begin{aligned}(e_i)^T g(\boldsymbol{U}_n) :=& -\Phi_i(q_i+q)(e_i)^T + q_i\Big(\sum_{k\in E} p_{ik}(e_k)^T\Big)\Big[\Phi_i(q_i+q)\mathbf{I} + \boldsymbol{U}(n)\Big]\cdot \\ & \Big[-\frac{\sigma_i^2}{2}\boldsymbol{U}(n)^2 + \mu_i\boldsymbol{U}(n) + (q_i+q)\mathbf{I}\Big]^{-1},\end{aligned} \tag{3.59}$$

where e_i is vector of zeros despite i'th position (canonical vector) and p_{ik} is the probability that if process J exit from the state i then it will go to the state k. In Breuer [9], author proved that $\boldsymbol{U}_n$ converge to the matrix $\boldsymbol{\Lambda}_q^+$. As we mentioned before, to obtain a numerical method for matrix $\boldsymbol{\Lambda}_q^-$ we need to consider process $(-X, J)$ as a background for the above algorithm. As a first example, we will consider the following parameters

$$\Delta_{\boldsymbol{\mu}} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad \Delta_{\boldsymbol{\sigma}} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}, \quad q = 0, \quad x_0 = 1.$$

Then we can apply the above numerical method to derive an approximation of matrix $\mathbf{\Lambda}^-$ and, therefore, the probability of classical ruin time. Namely, we get the following approximation

$$\mathbf{\Lambda}^- \approx \begin{pmatrix} -4.819 & 0.618 \\ 3.915 & -23.08 \end{pmatrix}, \quad e^{\mathbf{\Lambda}^-} \approx \begin{pmatrix} 0.00915 & 0.000307 \\ 0.00195 & 0.0000654 \end{pmatrix}.$$

Note that the (i, j) cell of the matrix $e^{\mathbf{\Lambda}^-}$ is the probability that

$$\mathbb{P}_1(\tau_0^- < \infty, J(\tau_0^-) = j | J_0 = i).$$

Thus to obtain the desired probability, we need to sum up cells in the row i. One can be interested in how the probability of Omega ruin time differs for MMBM and Brownian Motion X^i with parameters μ_i and σ_i for $i \in E$. Recall that for Brownian Motion probability of classical ruin time is of the following form

$$\mathbb{P}_x(\tau_0^- < \infty) = e^{-\frac{2\mu}{\sigma^2}x}. \tag{3.60}$$

Let us proceed to a numerical example of the Omega bankruptcy for MMBM. Unfortunately, we do not yet have an analytical representation of scale matrices for Markov modulated Brownian motion. Therefore, we want to use numerical methods to obtain the approximation of Omega bankruptcy time. Therefore, we aim to approximate

$$(\varphi^{(\omega)}(x))_i = 1 - \mathbb{E}_x\left[e^{-\int_0^\infty \omega_{J_s}(X_s)ds}; \tau_{-d}^- = \infty | J_0 = i\right],$$

for all $i \in E$. Denote $\varphi_T^{(\omega)}(x)$ as the following vector of expected values (or just probabilities) for all $i \in E$

$$(\varphi_T^{(\omega)}(x))_i := 1 - \mathbb{E}_x\left[e^{-\int_0^T \omega_{J_s}(X_s)ds}; \tau_{-d}^- = \infty \middle| J_0 = i\right].$$

Therefore, this is a modification of our bankruptcy time so that we allow it to be killed by the penalty function only before time T. If we let $T \to \infty$ then $\varphi_T^{(\omega)}$ converge to $\varphi^{(\omega)}(x)$ entry-wise, by dominated convergence theorem. From now we will hold the assumption that $\kappa = \boldsymbol{\pi\mu} > 0$, thus roughly speaking after some long time process should be saved from the penalty. Therefore, we can use that to set big enough T to approximate our ruin time. Therefore, because we will approximate $\varphi^{(\omega)}$ using approximation of $\varphi_T^{(\omega)}$ we will face so-called cut-off error.
Thus, we turn our problem into an approximation of $\varphi_T^{(\omega)}(x)$, and for that, we will use Monte Carlo methods. First, however, we must consider a few problems related to our approximation method.

- How to simulate a sample path of the Markov modulated Brownian motion?
- How to deal with the different starting points of the process X?
- How big should parameter T be?
- How many simulations are sufficient to get a trustworthy approximation?

Simulation of the sample path of the MMBM

First, let us recall the method of simulation of the process J. Let us assume that $J_0 = i$. Then we know that time until J change the state from i to j is distributed like exponential distributed random variable with parameter $q_i = -q_{i,i}$. Then, when J is leaving the state i it can go to state j with the probability $p_{i,j}$. Note that these probabilities can be determinant from the matrix $\boldsymbol{Q}$. For more details, we refer to Norris [50]

Therefore, one can see that if we would like to simulate J until some time T, then we need to simulate random numbers from exponential distributions until their sum cross-level T.

Let us assume that we simulate sample path of the process J and $(X_0, J_0) = (0, i)$ for some $i \in E$. Then let $T_0 = 0, T_1, ...$ be a sequence of the successive jumps epoch of J (namely, the times when J change the state). In the interval $[T_n, T_{n+1})$ we now that J is constant and equal to some $i \in E$. Thus, in this interval, we can simulate increments of X the same as for linear Brownian Motion with the parameters μ_i and σ_i. In shorthand, we divide the time interval using occupation times of J and then use well-known methods for the simulation of linear Brownian Motion.

Different starting points of the process X

As we mentioned, we would like to simulate (X, J) efficiently with the different choices of X_0. Note that if we sample the random path of the process (X, J) with $X_0 = 0$, then we can translate this sample path of X by the constant x to obtain the sample path of the process (X, J) with $X_0 = x$. Therefore, we will have one simulation per every starting point after one simulation of the process (X, J) with the single starting point.

Choice of the parameter T

We need to choose such T that X_T will be "safe" with high probability. Note that if we choose T for $X_0 = 0$, then for a greater starting point, this T will also be sufficient (because the probability of ruin decreases when X_0 increase). Thus, we will only consider $X_0 = 0$ and will take the following criteria. Let us take (if such max arg is unique)

$$i = \max \arg_{k \in E} -\frac{2\mu_k}{\sigma_k^2}.$$

We will take such X^i for which the probability of classical ruin time is the highest from all possible $i \in E$. If the maximum is not unique, then take this i's, which satisfies this maximum and take this one with the smallest drift. Note that such X^i will have a higher probability of classical ruin than process X himself. Therefore, this will be our worst-case scenario. Note that X_T^i is distributed as $N(\mu_i T, \sigma_i^2 T)$. Therefore, with a high probability, we know that X_T^i will be greater than $\mu_i T - 3\sigma\sqrt{T}$. Thus we aim to set T big enough that the probability that linear Brownian Motion, which starts with value $\mu_i T - 3\sigma\sqrt{T}$, ever cross-level zero is less than some fixed ϵ. Then we must take the lowest value of T, which satisfy

$$e^{-\frac{2\mu_i\sqrt{T}}{\sigma_i^2}(\mu_i T - 3\sigma_i\sqrt{T})} \leq \epsilon,$$

due to (3.60). Let us assume that $\epsilon = 10^{-4}$.

Note that this method is somehow trivial and restricted. One can find another bound for T, which is better for numerical approximation.

Number of simulations

From the theory of Monte Carlo simulations, we know that the rate of convergence is $n^{-\frac{1}{2}}$, or to be more precise, on the significance level $\alpha = 0.05$ the relation between error (call it b), sample variance and the number of simulations is

$$b = \frac{1.96\widehat{S}_n}{\sqrt{n}}.$$

Thus to obtain n, we need to make some pilot simulations to get $\widehat{S}_n$ and then we also need to choose an acceptable error on the selected significance level.

Example of simulations

Finally, we are ready to approximate the probability of the Omega bankruptcy time. Let us take the same parameters chosen for the classical probability of ruin. Namely,

$$\Delta_{\boldsymbol{\mu}} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad \Delta_{\boldsymbol{\sigma}} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

In addition, for all $i \in E$ we take

$$\omega_i(x) = -0.02x\mathbf{1}_{\{x\in[-5,0]\}},$$

thus $d = 5$. For such parameters we get that $T = 68$ and $\widehat{S}_n \approx 0.07$. Therefore, we have that

$$b \approx \frac{0.137}{\sqrt{n}},$$

on the significance level $\alpha = 0.05$. We will show the result for the error of the size 10^{-3}. Then it is sufficient to take $N \approx 19000$. Let us consider the following picture

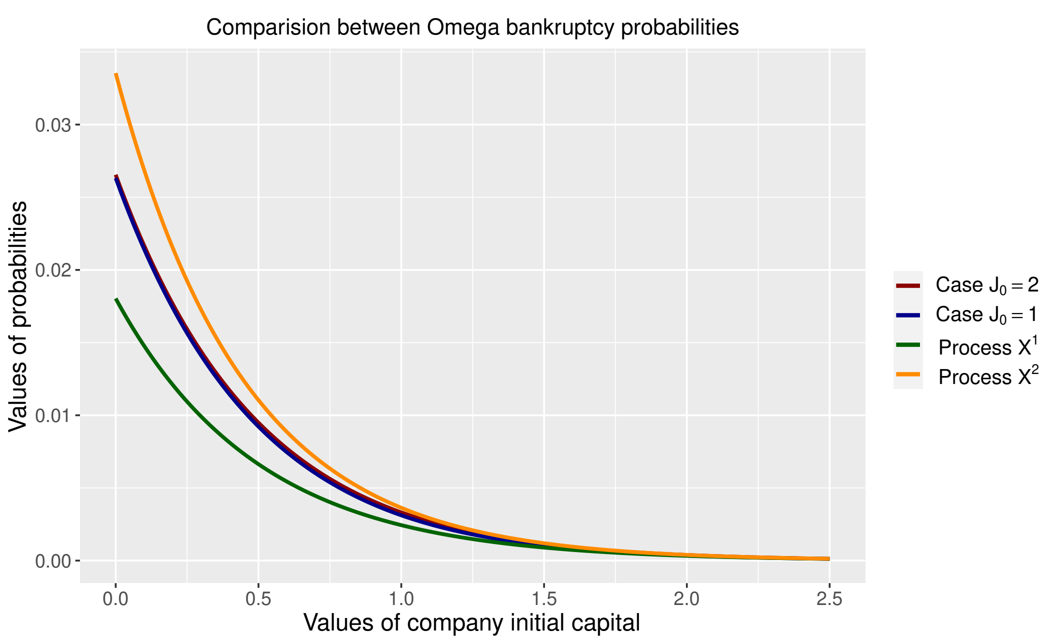

Figure 3.8: Comparison between Omega bankruptcy probabilities for MMBM with different values of J_0, X^1 and X^2

Note that we used the formula for the probability of Omega bankruptcy for Brownian motion from Li and Palmowski [41]. One can see that, like before, probabilities for MMBM lie between these for X^1 and X^2. However, here we can see a little difference between the cases for $J_0 = 1$ and $J_0 = 2$.

Proof of Proposition (3.5.5)

Recall from Corollary (3.3.7) that

$$\boldsymbol{E}_x\left[e^{-\int_0^{\tau_0^-}\omega_{J_s}(X_s)ds}, \tau_0^- < \infty, J_{\tau_0^-}\middle| J_0\right] = \boldsymbol{\mathcal{Z}}^{(\omega)}(x) - \boldsymbol{\mathcal{W}}(x)\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)},$$

If we take $\omega(i,x) = 0$ for all $i \in E$ and $x \geq 0$, then above turns to

$$\boldsymbol{P}_x\left[\tau_0^- < \infty, J_{\tau_0^-}\middle| J_0\right] = \boldsymbol{Z}^{(q)}(x) - \boldsymbol{W}^{(q)}(x)\boldsymbol{C}_{W(\infty)^{-1}Z(\infty)},$$

where $\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)} = \lim_{c\to\infty} \boldsymbol{W}^{(q)}(c)^{-1}\boldsymbol{Z}^{(q)}(c)$. Recall that

$$\boldsymbol{Z}^{(q)}(x) = \mathbf{I} - \int_0^x \boldsymbol{W}^{(q)}(z)dz\Big(\boldsymbol{Q} - q\mathbf{I}\Big).$$

Before we state the proof, let us recall some relation between $\boldsymbol{\Lambda}_q^+$, $\boldsymbol{\Lambda}_q^-$ and the model parameters. From Lemma 3.1.4, we know that

$$\boldsymbol{C}_q\boldsymbol{\Lambda}_q^+ = \Delta_{\frac{2}{\boldsymbol{\sigma}^2}}\Big[-\boldsymbol{Q} + q\mathbf{I}\Big], \tag{3.61}$$

where $\boldsymbol{C}_q = \Big(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-\Big)\boldsymbol{\Lambda}_q^-\Big(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-\Big)^{-1}$.
Recall also (see Ivanovs [27]) that in the case of Markov modulated Brownian motion, we have that

$$\lim_{x\to\infty} \boldsymbol{L}^{(q)}(x) = \boldsymbol{\Xi_q}. \tag{3.62}$$

One can see that our proof can be divided into a few parts. Thus, we will need the two lemmas.

Lemma 3.5.6. *For $x \geq 0$*

$$\int_0^x \boldsymbol{W}^{(q)}(z)dz\Big(\boldsymbol{Q} - q\boldsymbol{I}\Big) = \boldsymbol{I} - e^{-\boldsymbol{\Lambda}_q^+ x} - \boldsymbol{W}^{(q)}(x)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+.$$

Proof. After simple calculations, one can obtain the following

$$\begin{aligned}\int_0^x \boldsymbol{W}^{(q)}(z)dz\Big(\boldsymbol{Q} - q\mathbf{I}\Big) = &-\Big[(\boldsymbol{\Lambda}_q^+)^{-1}e^{-\boldsymbol{\Lambda}_q^+ x} + (\boldsymbol{\Lambda}_q^-)^{-1}e^{\boldsymbol{\Lambda}_q^- x}\Big]\boldsymbol{\Xi}_q\Big(\boldsymbol{Q} - q\mathbf{I}\Big) + \\ &\Big[(\boldsymbol{\Lambda}_q^+)^{-1} + (\boldsymbol{\Lambda}_q^-)^{-1}\Big]\boldsymbol{\Xi}_q\Big(\boldsymbol{Q} - q\mathbf{I}\Big).\end{aligned} \tag{3.63}$$

We will divide our calculations into two parts, namely

$$\begin{aligned}
&-\Big[(\boldsymbol{\Lambda}_q^+)^{-1}e^{-\boldsymbol{\Lambda}_q^+x}+(\boldsymbol{\Lambda}_q^-)^{-1}e^{\boldsymbol{\Lambda}_q^-x}\Big]\boldsymbol{\Xi}_q\Big(\boldsymbol{Q}-q\mathbf{I}\Big)\\
&=\Big[(\boldsymbol{\Lambda}_q^+)^{-1}e^{-\boldsymbol{\Lambda}_q^+x}+(\boldsymbol{\Lambda}_q^-)^{-1}e^{\boldsymbol{\Lambda}_q^-x}\Big]\left(\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-\right)^{-1}\Delta_{\frac{2}{\boldsymbol{\sigma}^2}}\Big(\boldsymbol{Q}-q\mathbf{I}\Big)\\
&\overset{(3.61)}{=}-\Big[(\boldsymbol{\Lambda}_q^+)^{-1}e^{-\boldsymbol{\Lambda}_q^+x}+(\boldsymbol{\Lambda}_q^-)^{-1}e^{\boldsymbol{\Lambda}_q^-x}\Big]\boldsymbol{\Lambda}_q^-\Big(\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-\Big)^{-1}\boldsymbol{\Lambda}_q^+\\
&=-\Big[e^{-\boldsymbol{\Lambda}_q^+\boldsymbol{x}}(\boldsymbol{\Lambda}_q^+)^{-1}\boldsymbol{\Lambda}_q^-+e^{\boldsymbol{\Lambda}_q^-x}\Big]\Big(\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-\Big)^{-1}\boldsymbol{\Lambda}_q^+\\
&=-\Big[e^{-\boldsymbol{\Lambda}_q^+\boldsymbol{x}}\Big((\boldsymbol{\Lambda}_q^+)^{-1}(\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-)-\mathbf{I}\Big)+e^{\boldsymbol{\Lambda}_q^-x}\Big]\Big(\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-\Big)^{-1}\boldsymbol{\Lambda}_q^+\\
&=-e^{-\boldsymbol{\Lambda}_q^+x}+\Big[e^{-\boldsymbol{\Lambda}_q^+x}-e^{\boldsymbol{\Lambda}_q^-x}\Big]\Big(\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-\Big)^{-1}\boldsymbol{\Lambda}_q^+=-e^{-\boldsymbol{\Lambda}_q^+x}-\boldsymbol{W}^{(q)}(x)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+,
\end{aligned}$$

and

$$\begin{aligned}
\Big[(\boldsymbol{\Lambda}_q^+)^{-1}+(\boldsymbol{\Lambda}_q^-)^{-1}\Big]\boldsymbol{\Xi}_q\Big(\boldsymbol{Q}-q\mathbf{I}\Big)&=-\Big[(\boldsymbol{\Lambda}_q^+)^{-1}+(\boldsymbol{\Lambda}_q^-)^{-1}\Big]\Big[\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-\Big]^{-1}\Delta_{\frac{2}{\boldsymbol{\sigma}^2}}\Big(\boldsymbol{Q}-q\mathbf{I}\Big)\overset{(3.61)}{=}\\
&=\Big[(\boldsymbol{\Lambda}_q^+)^{-1}+(\boldsymbol{\Lambda}_q^-)^{-1}\Big]\boldsymbol{\Lambda}_q^-\Big(\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-\Big)^{-1}\boldsymbol{\Lambda}_q^+=\\
&=(\boldsymbol{\Lambda}_q^+)^{-1}\Big[\boldsymbol{\Lambda}_q^-+\boldsymbol{\Lambda}_q^+\Big]\Big[\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-\Big]^{-1}\boldsymbol{\Lambda}_q^+=\mathbf{I}.
\end{aligned}$$

Above calculations ends the proof. □

Lemma 3.5.7. *We have that*

$$\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)}=-\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^-.$$

Proof. We have from the previous proposition, the fact that $\boldsymbol{\Xi}$ is invertible, the definition of scale matrix and (3.62) that

$$\begin{aligned}
\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)}&=\lim_{a\to\infty}\boldsymbol{W}^{-1}(a)\boldsymbol{Z}^{(q)}(a)=\lim_{a\to\infty}\boldsymbol{W}^{-1}(a)\Big[e^{-\boldsymbol{\Lambda}_q^+a}+\boldsymbol{W}^{(q)}(a)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+\Big]\\
&=\lim_{a\to\infty}\Big[\boldsymbol{L}^{(q)}(a)\Big]^{-1}+\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+=\Big(\boldsymbol{\Xi}_{\boldsymbol{q}}\Big)^{-1}+\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+\\
&=-\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\Big(\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-\Big)+\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+=-\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^-.
\end{aligned}$$

□

Now we are ready to prove the Proposition (3.5.5).

Proof of the Proposition (3.5.5).

$$\begin{aligned}
&\boldsymbol{Z}^{(q)}(x)-\boldsymbol{W}^{(q)}(x)\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)}=e^{-\boldsymbol{\Lambda}_q^+x}+\boldsymbol{W}^{(q)}(x)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^++\boldsymbol{W}^{(q)}(x)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^-=e^{-\boldsymbol{\Lambda}_q^+x}+\\
&\boldsymbol{W}^{(q)}(x)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\Big(\boldsymbol{\Lambda}_q^++\boldsymbol{\Lambda}_q^-\Big)=e^{-\boldsymbol{\Lambda}_q^+x}-\Big(e^{-\boldsymbol{\Lambda}_q^+x}-e^{\boldsymbol{\Lambda}_q^-x}\Big)=e^{\boldsymbol{\Lambda}_q^-x}.
\end{aligned}$$

□

3.6 Comments

This chapter solved ω-killed exit problems for spectrally negative Markov additive processes. Main results are Theorems 3.3.2,3.3.5, Corollary 3.3.7 and Theorem 3.3.8. We have given their representations of two-sided, one-sided exit problems and representations for resolvents, respectively. In many problems, similar representations turned out to be key tools. For example, we showed a semi-explicit representation of the value function in the issue of optimal dividend payments. The next step could be defining optimisation criteria and finding whether the barrier strategy is optimal. We know that, in the case of spectrally negative Lévy process, the optimal barrier stays on the level

$$a^* = \sup\{a \geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(x) \text{ for all } x \geq 0\}.$$

Let us note that in the case of spectrally negative MAP, the candidate for optimal level needs to depend on the initial distribution of J. It can be suspected that it will be some combination of optimal levels for $X^i, i \in E$.

Moreover, we noticed that the obtained results could be applied in different directions, mainly due to the variety of interpretations of the function ω. On the one hand, it can be used for some kind of killing of the processes (as in the case of the Omega model), and it can also be used for more advanced interest rate structures. The downside to the examples is that we only used the MMBM process. On the other hand, we wanted to focus primarily on the various examples of the ω function to show the variety in this matter.

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