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ROZPRAWA DOKTORSKA

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Spacery losowe w rzadkim losowym środowisku

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DOCTORAL THESIS

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On random walks in a sparse random environment

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Streszczenie

Przedmiotem niniejszej rozprawy jest model *spaceru losowego w rzadkim losowym środowisku* (RWSRE). Rozważamy cząsteczkę wykonującą prosty spacer losowy na osi liczb całkowitych. Porusza się ona symetrycznie za wyjątkiem pewnych punktów, wyznaczonych przez dwustronny proces odnowy, w których kładziemy losowy dryf. Innymi słowy, środowisko podzielone jest na bloki losowej długości; wewnątrz bloków cząsteczka wykonuje prosty symetryczny spacer losowy, zaś na ich krańcach występuje losowy dryf. RWSRE może być więc uważany za model pośredni między dwoma znanymi modelami: klasycznym, prostym symetrycznym spacerem losowym (SSRW) oraz spacerem losowym w losowym środowisku zadanym przez ciąg niezależnych, jednakowo rozłożonych zmiennych (RWRE). W zależności od rozkładu środowiska, RWSRE może posiadać cechy typowe albo dla SSRW, albo dla RWRE.

Jednym z celów pracy jest zbadanie, jak ta dychotomia przejawia się w granicznym zachowaniu spaceru. Pierwsza część rozprawy dotyczy twierdzeń granicznych typu *quenched* dla pozycji spaceru oraz czasów pierwszego przejścia. W pierwszej kolejności przedstawiamy mocne centralne twierdzenie graniczne typu *quenched* dla pozycji spaceru, uogólniając w ten sposób wyniki znane dla modelu RWRE. Następnie rozważamy przypadek, w którym rzadkość środowiska ma dominujący wpływ na graniczne zachowanie spaceru i przedstawiamy słabe twierdzenia graniczne typu *quenched* dla czasów pierwszego przejścia. W tym przypadku RWSRE przejawia cechy nieobserwowane dla RWRE.

Ostatnia część rozprawy dotyczy maksymalnych czasów lokalnych spaceru, tj. czasu, jaki cząsteczka spędza w swoich ulubionych punktach. Przedstawiamy twierdzenia graniczne typu *annealed* dla ciągu maksymalnych czasów lokalnych w dwóch przypadkach: dominującego dryfu i dominującej rzadkości. W pierwszym przypadku uzyskane wyniki mogą być uznane za uogólnienie twierdzeń znanych dla RWRE. W drugim przypadku, z powodu obecności w środowisku długich bloków, na których cząsteczka porusza się symetrycznie, natura jej ulubionych punktów jest znacząco inna.

Abstract

The subject of this thesis is a stochastic model called *a random walk in a sparse random environment* (RWSRE). We consider a single particle performing a nearest-neighbour random walk on the set of integers. The movement is symmetric apart from some sites marked by a two-sided renewal process, in which random drifts are imposed. That is, the environment is split into blocks of random lengths; within each block, the particle performs a symmetric random walk, while at the endpoints a random drift occurs. Therefore the RWSRE may be considered as being in-between two well-known models: a classic, simple symmetric random walk (SSRW) and a random walk in i.i.d. random environment (RWRE), and, depending on the distribution of the environment, it may manifest properties resembling one or the other.

One of the goals of the thesis is to examine how this dichotomy may be observed in the limiting behaviour of the walk. The first part of the thesis concerns quenched limit theorems for the position of the walk and the sequence of first passage times. We begin by presenting the case in which the strong quenched central limit theorem holds for the position of the walk, generalizing results known for the RWRE. Next we focus on the case in which the sparsity of the environment plays a dominant role in governing the limiting behaviour of the RWSRE and present weak quenched limit theorems for the sequence of first passage times. In this case the RWSRE exhibits properties not observed for the RWRE.

In the last part of the thesis we examine the sequence of maximal local times of the walk, i.e. the amount of time spent by the particle in its favourite sites. We present the annealed limit theorems for this sequence in two cases: the case of dominating drift and the case of dominating sparsity. In the former, we obtain results that may be seen as a generalization of those known for the RWRE. In the latter, the nature of the favourite sites is different because of the presence of long blocks on which the walk is symmetric.

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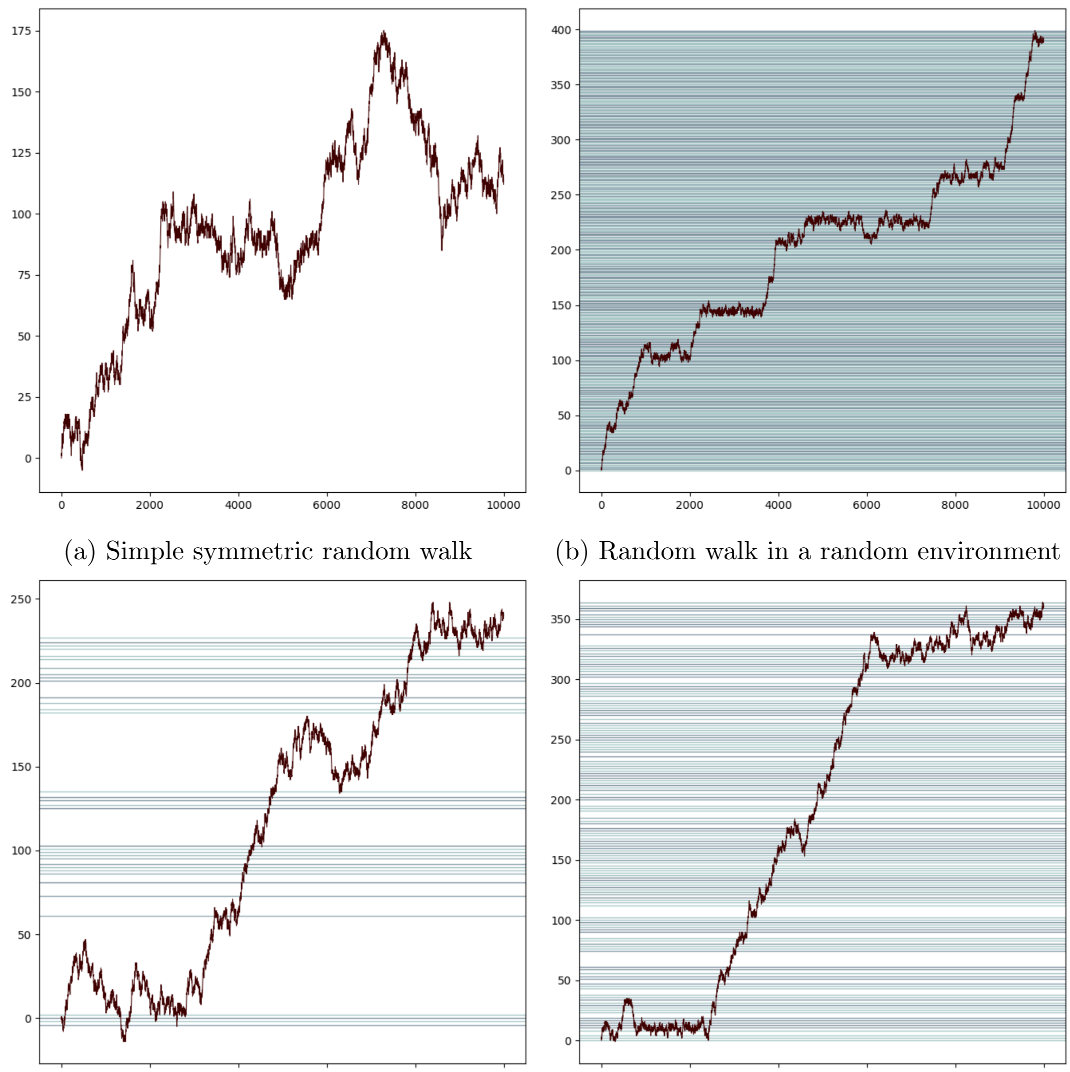
Contents

Chapter 1

Introduction

One of the most classic stochastic processes is a simple random walk on the set of integers. In this model, a particle moves along the axis, every unit time jumping with fixed probability p to its right neighbour and with probability $1-p$ to the left one. The process is a time and space homogeneous Markov chain, that is its increments are independent of the past and the transitions do not depend on time nor the current position of the process. It is well known that the walk is recurrent if and only if it is symmetric, i.e. $p = 1/2$; its asymptotic properties follow from classic results such as the strong law of large numbers, central limit theorem or Cramér's theorem. However, the homogeneity present in this model is not always desired. In many applications one would like to consider obstacles in the environment that would change the behaviour of the particle in some, possibly random, sites. As it turns out, even a small perturbation in the environment affects the asymptotic properties of the walk. In 1981, Harrison and Shepp [16] described the behaviour of the simple symmetric random walk in a slightly disturbed environment, replacing only the probability of passing from 0 to 1 by some fixed $p_0 \in (0,1)$. They observed that the scaling limit is not the Brownian motion, as is the case in homogeneous environment, but the skew Brownian motion.

Another way of perturbing the environment was proposed in the seventies by Solomon [28]. In his model, the drift occurs at every site and is chosen randomly with some distribution P. The *random walk in a random environment* (RWRE) defined this way has been well studied since. Various authors described how the choice of P determines such properties of the walk as its transience and asymptotic speed [28, 1], limit theorems [15, 18], or large deviations [9, 5]. As it transpires, the randomness of the environment leads to phenomena not observed in the classic model. For example, the walk may be sub-ballistic, i.e. transient, but with sub-linear speed. Moreover, under suitable assumptions on P, the position of the walk no longer satisfies a central limit theorem. Its scaling limit is closely related to the limit of a sequence of first passage times, which lies in the domain of attraction of a stable law, with the parameters and scaling determined by the distribution of the environment. In this case the rate of large deviations is of polynomial order, in opposition to the classic, exponential rate present under the assumptions of Cramér's theorem. This change in the behaviour of the particle is caused, roughly speaking, by the traps occurring in the environment, i.e. sites with unfavourable drift that have impact on the movement of the particle which is strong enough to be seen in the macroscopic scale of the limit theorems.



(a) Simple symmetric random walk

(b) Random walk in a random environment

(c) Random walk in a sparse random environment

(d) Random walk in a sparse random environment

Figure 1.0.1: Exemplary trajectories of random walks. Horizontal lines indicate sites with random drift; the darker the line, the stronger the drift to $-\infty$.

Solomon's model may be also seen as, in a way, homogeneous, since the drifts are usually assumed to be independent and identically distributed, or at least stationary and ergodic. It is therefore natural to consider slightly different construction of the environment, in which the symmetric movement of the particle would be perturbed only in some sparsely located sites. The model we intend to study was proposed first in [20] by Matzavinos, Rotershtein, and Seol, and is called *a random walk in a sparse random environment* (RWSRE). Instead of putting random drift at every site in the environment, we begin by marking a subset of integers by the positions of a renewal process. The random drift is imposed only in the marked sites, and the particle moves symmetrically everywhere else. This model may be seen as an interpolation between the classic symmetric random walk and Solomon's random walk in a random environment, or as a generalization of the latter. The asymptotic properties of the walk are driven not only by the drift, but also by the sparsity, i.e. the lengths of intervals on which the movement is symmetric. Therefore, we may expect that, depending on the interplay between the drift and the sparsity, RWSRE should manifest properties resembling either a simple symmetric random walk, or RWRE. Indeed, this dichotomy was already observed by Buraczewski et al. in [6, 7] in the context of annealed limit theorems for the transient RWSRE. Under suitable assumptions, the traps in the environment that have the largest impact on the movement of the particle are of the same nature as those appearing in RWRE, i.e. are caused by unfavourable drift. In such setting, the sequence of first passage times lies in the domain of attraction of a stable law, whose parameters, like in RWRE, are determined by the distribution of the drift. However, under different assumptions, ones that favour long distances between marked sites, the drift no longer plays the dominant role in governing the movement of the particle. In most sites it behaves like a simple symmetric random walk, and this change is reflected in the shape of limit theorems. The limiting variable may be once again stable, but its appearance, as well as its parameters, is caused by the presence of long blocks on which the walk is symmetric. When the environment is strongly sparse, the limiting variable is no longer stable; an additional term appears that comes from the random movement of the particle in the unmarked sites.

One of our objectives will be to describe how this phenomenon is reflected in the quenched limit theorems, i.e. to investigate asymptotic behaviour of the walk in randomly chosen, fixed environment. The first result, stated in Theorem 3.2.1, is a strong quenched central limit theorem for the position of the walk. By adapting the techniques used previously for the RWRE, we show that under relatively strong assumptions, for almost every environment, properly scaled position of the walk converges in distribution to the standard normal variable. Then, in Chapter 4, we pass to the setting of dominating sparsity, in particular strong sparsity, in which the limiting behaviour of RWSRE is significantly different from that observed for RWRE. In this case the scaling limit of the first passage times is determined by asymptotic properties of the renewal process that was used to mark the points with drifts, as well as the random movement of the particle in the blocks between them, while the influence of the drifts is negligible. Moreover, the limit theorem is no longer strong, i.e. the scaled sequence of first passage times does not converge in distribution for almost every environment. Instead, one should consider the weak limit of the sequence $(\mu_{n,\omega})_{n\in\mathbb{N}}$ given by

$$\mu_{n,\omega}(\cdot) = \mathrm{P}_\omega\left[(T_n - \mathrm{E}_\omega T_n)/\kappa_n \in \,\cdot\,\right]$$

for appropriate scaling κ_n, where T_n is the time by which RWSRE reaches site n, P_ω is the distribution of the walk in fixed environment ω, and E_ω is the expected value with respect to P_ω. Since the environment is random, $(\mu_{n,\omega})_{n\in\mathbb{N}}$ is a sequence of random measures and as such may have a weak limit. The main result of Chapter 4 states that, under suitable assumptions on the distribution of the environment,

$$\mu_{n,\omega} \Rightarrow G(N),$$

where $\Rightarrow$ denotes weak convergence. Here N is a Poisson point process with intensity determined by the renewal process used to mark the sites with drift (that is, the presence of N arises from the sparsity of the environment), while G is a measurable map defined with the help of the Brownian motion (that is, G comes from the limiting behaviour of a simple symmetric random walk). Precise results are stated in Theorems 4.2.1, 4.2.2, and 4.2.3.

Another object of our interest will be the favourite sites of a transient RWSRE. More precisely, we will study the annealed limit theorems for the maximal local times, i.e. the amount of time spent by the particle in its favourite site. A natural presumption is that this maximal time should be obtained at a site with strong, unfavourable drift that forces the particle to make many attempts to cross the site. As we will see, this is indeed true for RWRE and RWSRE under suitable assumptions on the distribution of the environment. In the complementary case, however, when it is the sparsity that governs the limiting behaviour of the walk, the maximal local time is obtained when the particle crosses a particularly long block between the marked sites. As can be seen in Theorems 5.2.1 and 5.2.2, in both cases the limiting distribution is Fréchet, that is, for every $x > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\max_{k\leqslant n} L_k(n)/\kappa_n \leqslant x\right] = e^{-cx^{-\gamma}},$$

where $\mathbb{P}$ denotes the annealed measure and $L_k(n)$ is the number of times the walk visits site k before reaching n. However, the scaling sequence κ_n and the parameter γ depend entirely either on the distribution of drifts, or on the renewal process used to mark the sites. Interestingly, this change of regime – between the domination of drift and sparsity – occurs under different assumptions than for the first passage times. The reason, as may be seen from the arguments used in the two cases, is the fact that first passage times and local times of a simple symmetric random walk are asymptotically of different orders, while for a RWRE they are comparable.

The dissertation is organised as follows: in the remaining part of this chapter we define our model formally and provide a summary of notation used throughout the thesis. Elementary properties of the model are described in Chapter 2. Chapters 3 and 4 concern quenched limit theorems for the position of the walk and first passage times, and in Chapter 5 we present annealed limit theorems for maximal local times.

1.1 Random walk in a (sparse) random environment

Let $\Omega = (0,1)^{\mathbb{Z}}$ and let $\mathcal{F}$ be the corresponding cylindrical σ-algebra. A random element $\omega = (\omega_n)_{n\in\mathbb{Z}}$ of $(\Omega, \mathcal{F})$ distributed according to a probability measure P is called a *random environment*. Let $\mathcal{X} = \mathbb{Z}^{\mathbb{N}}$ be the set of possible paths of a random walk on $\mathbb{Z}$, with corresponding cylindrical σ-algebra $\mathcal{G}$. Then any $\omega \in \Omega$ and $i \in \mathbb{Z}$ gives rise to a measure P_ω^i on $\mathcal{X}$ such that $\mathrm{P}_\omega^i[X_0 = i] = 1$ and

$$\mathrm{P}_\omega^i\left[X_{n+1} = j | X_n = k\right] = \begin{cases} \omega_k & \text{if } j = k+1, \\ 1-\omega_k & \text{if } j = k-1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1.1}$$

where $X = (X_n)_{n\in\mathbb{N}} \in \mathcal{X}$. That is, under P_ω^i, X is a nearest-neighbour random walk starting from i with transition probabilities given by the sequence ω. In particular, it is a time-homogeneous Markov chain.

Since the environment itself is random, it is natural to consider a measure $\mathbb{P}^i$ on $(\Omega \times \mathcal{X}, \mathcal{F} \otimes \mathcal{G})$ such that

$$\mathbb{P}^i\left[F \times G\right] = \int_F \mathrm{P}_\omega^i[G]\, \mathrm{P}(d\omega) \tag{1.1.2}$$

for any $F \in \mathcal{F}, G \in \mathcal{G}$. We shall write $\mathrm{P}_\omega = \mathrm{P}_\omega^0$ and $\mathbb{P} = \mathbb{P}^0$. Observe that under $\mathbb{P}$ the walk X may exhibit a long-time dependencies and thus no longer be a Markov chain.

The process X defined above is called *a random walk in a random environment* and was introduced by Solomon [28]. A well-studied case is ω being an i.i.d. sequence, which gives rise to *a random walk in i.i.d. random environment.*

We will consider a specific choice of the environment that was introduced first by Matzavinos, Roitershtein, and Seol in [20]. Consider an i.i.d. sequence $((\xi_k, \lambda_k))_{k\in\mathbb{Z}} \in (\mathbb{N}_+ \times (0,1))^{\mathbb{Z}}$ and define, for any $n, k \in \mathbb{Z}$,

$$S_n = \begin{cases} \sum_{j=1}^n \xi_j, & n > 0, \\ 0, & n = 0, \\ -\sum_{j=n+1}^0 \xi_j, & n < 0; \end{cases} \qquad \omega_k = \begin{cases} \lambda_n & \text{if } k = S_n \text{ for some } n \in \mathbb{Z}, \\ 1/2 & \text{otherwise.} \end{cases} \tag{1.1.3}$$

The random walk evolving in an environment ω defined by (1.1.3) is called *a random walk in a sparse random environment*. We shall refer to the random sites S_n as *marked points* and write (ξ, λ) for a generic element of the sequence $((\xi_k, \lambda_k))_{k\in\mathbb{Z}}$.

The term *sparse* refers to the fact that, unless $\xi = 1$ almost surely, the random drift is put only in the marked sites, while in the blocks between them, whose lengths are given by the sequence $(\xi_k)_{k\in\mathbb{Z}}$, the particle performs a simple symmetric random walk. In other words, the impurities are put *sparsely* on $\mathbb{Z}$. However, if $\xi = 1$ almost surely, then we obtain once again a random walk in i.i.d. environment. Therefore the RWSRE model may be seen as an interpolation between a simple symmetric random walk and a walk in i.i.d. environment, or as a generalization of the latter.

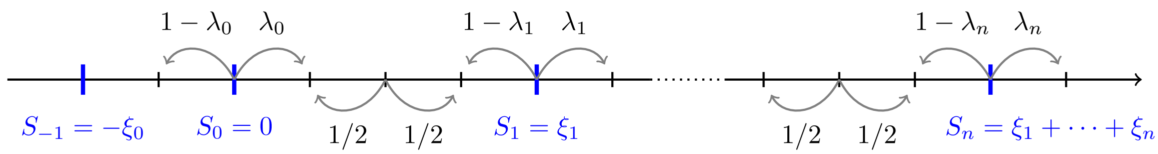


Figure 1.1.1: Exemplary realization of the sparse random environment. Sites with random drift are marked in blue. Under P_ω, the walk obeys transition rules indicated by arrows.

1.2 Notation and basic definitions

We provide a list of definitions and notation we shall use throughout the thesis.

- For $p \in (0,1)$, by $Geo(p)$ we denote the geometric distribution with parameter p supported on $\{0,1,\dots\}$, i.e. if $G \sim Geo(p)$, then

$$\mathbb{P}[G = k] = p(1-p)^k \quad \text{for } k = 0,1,2,\dots$$

- For a topological space Z by $\mathcal{M}_1(Z)$ we denote the space of probability measures on Z with the Borel σ-algebra. For our purposes we will take Z to be an Euclidean space or its subspace; $\mathcal{M}_1(Z)$ equipped with the Prokhorov distance is then a separable metric space which inherits completeness from Z. Similarly, by $\mathcal{M}_p(Z)$ we will denote the space of point measures on Z, equipped with the topology of vague convergence.
- For two functions $f, g : \mathbb{R} \to \mathbb{R}$, we write $f(t) \sim g(t)$ whenever $f(t)/g(t) \to 1$ as $t \to \infty$. A function ℓ is slowly varying at infinity if $\ell(ct) \sim \ell(t)$ as $t \to \infty$ for any constant $c > 0$.
- For two numbers a, b we denote $a \wedge b := \min(a,b)$ and $a \vee b := \max(a,b)$.

We will frequently make use of the following notation given in terms of the variables $((\xi_k, \lambda_k))_{k\in\mathbb{Z}}$: for $i, j, k \in \mathbb{Z}$,

$$\rho_k = \frac{1-\lambda_k}{\lambda_k};$$

$$\Pi_{i,j} = \prod_{k=i}^{j} \rho_k \quad \text{with the convention that an empty product equals 1;}$$

$$R_{i,j} = \sum_{k=i}^{j} \xi_k \Pi_{i,k-1}, \qquad R_i = \sum_{k=i}^{\infty} \xi_k \Pi_{i,k-1}, \qquad \bar{R}_i = \sum_{k=i}^{\infty} \Pi_{i,k-1};$$

$$W_{i,j} = \sum_{k=i}^{j} \xi_k \Pi_{k,j}, \qquad W_j = \sum_{k=-\infty}^{j} \xi_k \Pi_{k,j}.$$

Chapter 2

Random walk in a sparse random environment

In this chapter we present basic properties of RWSRE, in particular invoke known results on the transience criteria and the asymptotic speed of the walk. We also study quenched mean and variance of the hitting times.

2.1 Some properties of the simple symmetric random walk

In this section we present some facts concerning the simple symmetric random walk which will be used repeatedly in the next chapters.

Let, for the use of the next lemma, $\widetilde{X} = (\widetilde{X}_n)_{n\in\mathbb{N}}$ denote a simple symmetric random walk on $\mathbb{Z}$. Let $\mathbb{P}^i$ be the probability on the underlying probability space conditioned on the event $\{\widetilde{X}_0 = i\}$, and let $\mathbb{E}^i$ denote the corresponding expected value. Let $\widetilde{T}_n = \inf\{k \in \mathbb{N} : \widetilde{X}_k = n\}$ be the first passage time of $\widetilde{X}$. It is well known that every $\widetilde{T}_n$ is finite a.s. The next lemma gathers facts on $\widetilde{X}$ that will be of use to us.

Lemma 2.1.1. *The following hold for any* $0 < i < N$*:*

$$\mathbb{P}^i[\widetilde{T}_N < \widetilde{T}_0] = \frac{i}{N}, \tag{2.1.1}$$

$$\mathbb{E}^i[\widetilde{T}_N \wedge \widetilde{T}_0] = i(N-i). \tag{2.1.2}$$

$$\mathbb{E}^i\left[\widetilde{T}_N \, \mathbb{1}_{\widetilde{T}_N<\widetilde{T}_0}\right] = \frac{i(N-i)(N+i)}{3N}, \tag{2.1.3}$$

$$\mathbb{E}^i\left[\widetilde{T}_0 \, \mathbb{1}_{\widetilde{T}_0<\widetilde{T}_N}\right] = \frac{i(2N-i)(N-i)}{3N}, \tag{2.1.4}$$

$$\mathbb{E}^i\left[(\widetilde{T}_N \wedge \widetilde{T}_0)^2\right] = \frac{1}{3}\left(i^4 - 2i^3N + 2i^2 + iN^3 - 2iN\right). \tag{2.1.5}$$

Proof. The first two relations are well-known results on the gambler's ruin problem and can be obtained by solving recursive equations or by applying Doob's stopping theorem to martingales $\widetilde{X}_n$ and $\widetilde{X}_n^2 - n$.

To show (2.1.3), consider a Doob transform of $\widetilde{X}$ with the harmonic function $h(i) = i/N$. That is, consider a Markov chain on $\{0, 1, \dots, N\}$ with transition probabilities given by

$$P(i,j) = \begin{cases} \frac{i+1}{2i}, & j = i+1, \\ \frac{i-1}{2i}, & j = i-1, \\ 0 & \text{otherwise} \end{cases}$$

for $i \in \{1, \dots, N-1\}$ and $P(N,N) = 1$. Since $h(i) = \mathbb{P}^i[\widetilde{T}_N < \widetilde{T}_0]$, this Markov chain has the same distribution as the process $\widetilde{X}$ conditioned on $\{\widetilde{T}_N < \widetilde{T}_0\}$, stopped upon reaching site N. Therefore the sequence $a_i := \mathbb{E}^i[\widetilde{T}_N | \widetilde{T}_N < \widetilde{T}_0]$ satisfies

$$a_i = 1 + \frac{i-1}{2i} a_{i-1} + \frac{i+1}{2i} a_{i+1}, \quad a_N = 0.$$

Solving the equation gives $a_i = (N+i)(N-i)/3$, which implies (2.1.3). Now (2.1.4) follows by symmetry.

To obtain (2.1.5), we may apply Doob's stopping theorem to the martingale $M_n = \widetilde{X}_n^4 - 6n\widetilde{X}_n^2 + 3n^2 + 2n$. For $\widetilde{T} = \widetilde{T}_0 \wedge \widetilde{T}_N$ we get

$$i^4 = \mathbb{E}^i M_0 = \mathbb{E}^i M_{\widetilde{T}} = N^4 \mathbb{P}^i[\widetilde{T}_N < \widetilde{T}_0] - 6N^2 \mathbb{E}^i[\widetilde{T}\, \mathbb{1}_{\widetilde{T}_N < \widetilde{T}_0}] + 3\mathbb{E}^i \widetilde{T}^2 + 2\mathbb{E}^i \widetilde{T}.$$

Applying the above formulae and solving for $\mathbb{E}^i \widetilde{T}^2$ gives (2.1.5).

□

2.2 Basic properties of the random walk in a sparse random environment

2.2.1 Estimates of certain processes related to the environment

Let, for $k \in \mathbb{Z}$,

$$\rho_k = \frac{1 - \lambda_k}{\lambda_k}.$$

Observe that $(\rho_k)_{k \in \mathbb{Z}}$ is, under P, a sequence of i.i.d. random variables. We shall write ρ for its generic element. Let, for integers $i \leqslant j$,

$$\Pi_{i,j} = \prod_{k=i}^{j} \rho_k, \qquad R_{i,j} = \sum_{k=i}^{j} \xi_k \Pi_{i,k-1}, \qquad W_{i,j} = \sum_{k=i}^{j} \xi_k \Pi_{k,j}, \tag{2.2.1}$$

with the convention that $\Pi_{i,j} = 1$ for $i > j$. We will also make use of the limits

$$R_i = \lim_{j \to \infty} R_{i,j} = \sum_{k=i}^{\infty} \xi_k \Pi_{i,k-1}, \qquad W_j = \lim_{i \to -\infty} W_{i,j} = \sum_{k=-\infty}^{j} \xi_k \Pi_{k,j}. \tag{2.2.2}$$

Note that if $\mathrm{E}\log\rho < 0$ and $\mathrm{E}\log\xi < \infty$, both series are convergent almost surely as one can see by a straightforward application of the law of large numbers and the Borel-Cantelli lemma (see [4, Theorem 2.1.3]). The sequences $(R_i)_{i\in\mathbb{Z}}$ and $(W_j)_{j\in\mathbb{Z}}$ obey the recursive formulae

$$R_i = \xi_i + \rho_i R_{i+1} \qquad \text{and} \qquad W_j = \rho_j\xi_j + \rho_j W_{j-1}. \tag{2.2.3}$$

We can therefore invoke the proof of [4, Lemma 2.3.1] to infer the following result on the existence of moments of R_i's and W_j's. In what follows we write R (respectively W) for a generic element of $(R_i)_{i\in\mathbb{Z}}$ (respectively $(W_j)_{j\in\mathbb{Z}}$).

Lemma 2.2.1. *Let $\eta > 0$. If $\mathrm{E}\rho^\eta < 1$, $\mathrm{E}\rho^\eta\xi^\eta < \infty$, and $\mathrm{E}\xi^\eta < \infty$, then $\mathrm{E}R^\eta$ and $\mathrm{E}W^\eta$ are both finite.*

2.2.2 Recurrence, transience, and the speed of the walk

Let $X = (X_n)_{n\in\mathbb{N}}$ be a random walk in a sparse random environment. For any $k \in \mathbb{Z}$, let

$$T_k = \inf\{n : X_n = k\}. \tag{2.2.4}$$

We shall refer to T_k's as the *hitting* or *first passage times.* The analysis of the sequence $T = (T_k)_{k\in\mathbb{Z}}$ gives insight into relevant properties of the RWSRE. We will consider first the hitting times along the marked sites, i.e. the sequence $(T_{S_k})_{k\in\mathbb{Z}}$.

As it turns out, the variables $R_{i,j}$ defined in (2.2.1) may be used to express exit probabilities of the walk.

Lemma 2.2.2. *For any $i < k < j$,*

$$\mathrm{P}_\omega^{S_k}[T_{S_i} > T_{S_j}] = \frac{R_{i+1,k}}{R_{i+1,j}}, \qquad \mathrm{P}_\omega^{S_k}[T_{S_i} < T_{S_j}] = \Pi_{i+1,k}\frac{R_{k+1,j}}{R_{i+1,j}}. \tag{2.2.5}$$

Proof. Obtaining the formulae (2.2.5) for any fixed environment ω is a matter of solving a simple recursive equation. For fixed $i < j$ consider

$$p_k = \mathrm{P}_\omega^{S_k}\left[T_{S_i} > T_{S_j}\right].$$

Then $p_i = 0, p_j = 1$. Conditioning on the first step of the walk and using Lemma 2.1.1, we obtain

$$\begin{aligned} p_k &= \lambda_k \mathrm{P}_\omega^{S_k+1}\left[T_{S_i} > T_{S_j}\right] + (1-\lambda_k)\mathrm{P}_\omega^{S_k-1}\left[T_{S_i} > T_{S_j}\right] \\ &= \lambda_k\left(\frac{p_{k+1}}{\xi_{k+1}} + \frac{\xi_{k+1}-1}{\xi_{k+1}}p_k\right) + (1-\lambda_k)\left(\frac{p_{k-1}}{\xi_k} + \frac{\xi_k - 1}{\xi_k}p_k\right) \end{aligned}$$

Solving this equation gives (2.2.5). □

In view of the asymptotic properties of the sequence $(R_{1,n})_{n\in\mathbb{N}}$ described in Section 2.2.1, Lemma 2.2.2 may be used to determine the conditions under which RWSRE is recurrent or transient. The following is Theorem 3.1 from [20].

Proposition 2.2.3. *Assume that* $\mathrm{E}\log\xi < \infty$. *Then the following holds* $\mathbb{P}$*-almost surely:*

- *if* $\mathrm{E}\log\rho < 0$, *then* $\lim_{n\to\infty} X_n = \infty$;
- *if* $\mathrm{E}\log\rho > 0$, *then* $\lim_{n\to\infty} X_n = -\infty$;
- *if* $\mathrm{E}\log\rho = 0$, *then* $\limsup_{n\to\infty} X_n = \infty$, $\liminf_{n\to\infty} X_n = -\infty$.

From now on we will consider only RWSRE that is transient to the right, therefore we will always assume

$$\mathrm{E}\log\rho \in [-\infty, 0) \quad \text{and} \quad \mathrm{E}\log\xi < \infty. \tag{2.2.6}$$

Note that the first condition in (2.2.6) excludes the degenerate case $\rho = 1$ a.s. in which X is a simple symmetric random walk. Under (2.2.6), the RWSRE satisfies a strong law of large numbers. The following result was first stated in [20] under more strict conditions, and then generalised in [7].

Proposition 2.2.4. *Assume that conditions* (2.2.6) *hold. Then*

$$X_n/n \to v, \quad T_n/n \to 1/v \quad \mathbb{P}-a.s., \tag{2.2.7}$$

where

$$v = \begin{cases} \frac{(1-\mathrm{E}\rho)\mathrm{E}\xi}{(1-\mathrm{E}\rho)\mathrm{E}\xi^2+2\mathrm{E}\rho\xi\mathrm{E}\xi} & \text{if } \mathrm{E}\rho<1,\ \mathrm{E}\rho\xi<\infty,\ \mathrm{E}\xi^2<\infty; \\ 0 & \text{otherwise,} \end{cases}$$

with the convention $1/0 = \infty$.

As we will see in Lemma 2.3.1, the conditions under which v is non-zero guarantee that $\mathbb{E}T_{S_1} < \infty$. The main point of the proof of Proposition 2.2.4 is an application of the ergodic theorem to obtain the strong law of large numbers for T_{S_n}/n, which further implies the convergence (2.2.7). For the full proof, we refer the reader to [20, Theorem 3.3] or [7, Proposition 2.1].

2.3 Hitting times of a random walk in a sparse random environment

In this section we examine the structure of the sequence of hitting times of a transient RWSRE under the quenched measure P_ω.

Let $\mathbb{T}_k = T_{S_k} - T_{S_{k-1}}$ be the time that the particle needs to hit k'th marked point S_k after reaching S_{k-1}. Note that $\mathbb{T}_k$'s are independent under P_ω for any fixed ω, but may be dependent under $\mathbb{P}$. The next lemma gives expressions on moments of $\mathbb{T}_k$ in the case of a transient walk.

Lemma 2.3.1. *Assume that* $\mathrm{E}\log\rho < 0, \mathrm{E}\log\xi < \infty$. *Then for any* $k \in \mathbb{Z}$, P*-almost surely,*

$$\mathrm{E}_\omega \mathbb{T}_k = \xi_k^2 + 2\xi_k W_{k-1}, \tag{2.3.1}$$

$$\begin{aligned}\mathrm{Var}_\omega \mathbb{T}_k = 8\xi_k &\sum_{i<k} \left(\xi_i^2 W_{i-1} + \xi_i W_{i-1}^2 + \frac{1}{3}\xi_i^3\right) \Pi_{i,k-1} \\ &+ \frac{2}{3}\xi_k^4 - \frac{2}{3}\xi_k^2 - \frac{4}{3}\xi_k W_{k-1} + \frac{8}{3}\xi_k^3 W_{k-1} + 4\xi_k^2 W_{k-1}^2.\end{aligned} \tag{2.3.2}$$

Proof. Observe that assumptions of the lemma guarantee that all the above series are convergent P-almost surely.

Denote by $T(i,j)$ the time needed to reach j when starting from i. Then $\mathbb{T}_{k+1}$ has the same distribution as $T(S_k, S_{k+1})$, which may be decomposed as follows: first, we consider the first step of the walk, i.e.

$$T(S_k, S_{k+1}) \overset{\mathrm{d}}{=} 1 + \mathbb{1}_{X_1 = S_k+1} T(S_k+1, S_{k+1}) + \mathbb{1}_{X_1 = S_k-1} T(S_k-1, S_{k+1}),$$

for $T(S_k+1, S_{k+1})$ and $T(S_k-1, S_{k+1})$ independent of X_1.

Next, we decompose $T(S_k+1, S_{k+1})$ with respect to the point by which the walk exits interval $[S_k, S_{k+1}]$. Let $T_k^L = T(S_k+1, S_k), T_k^R = T(S_k+1, S_{k+1})$, then

$$\begin{aligned}T(S_k+1, S_{k+1}) &\overset{\mathrm{d}}{=} \mathbb{1}_{T_k^L < T_k^R}(T_k^L + T'(S_k, S_{k+1})) + \mathbb{1}_{T_k^R < T_k^L} T_k^R \\ &= T_k^L \wedge T_k^R + \mathbb{1}_{T_k^L < T_k^R} T'(S_k, S_{k+1})\end{aligned}$$

where $T'(S_k, S_{k+1})$ is a copy of $T(S_k, S_{k+1})$, independent of $T(S_k+1, S_{k+1})$, T_k^L, and T_k^R.

Similarly, with $T_{k-1}^R = T(S_k-1, S_k)$ and $T_{k-1}^L = T(S_k-1, S_{k-1})$,

$$\begin{aligned}T(S_k-1, S_{k+1}) \overset{\mathrm{d}}{=}\ & T_{k-1}^L \wedge T_{k-1}^R \\ &+ \mathbb{1}_{T_{k-1}^L < T_{k-1}^R} T(S_{k-1}, S_k) \\ &+ T''(S_k, S_{k+1}).\end{aligned}$$

Denote $\mu_k = \mathrm{E}_\omega \mathbb{T}_k$. Using (2.1.1) and (2.1.2) we get

$$\begin{aligned}\mathrm{E}_\omega T(S_k+1, S_{k+1}) &= \mathrm{E}_\omega[T_k^L \wedge T_k^R] + \mathrm{P}_\omega[T_k^L < T_k^R]\mathrm{E}_\omega T(S_k, S_{k+1}) \\ &= \xi_{k+1} - 1 + \frac{\xi_{k+1}-1}{\xi_{k+1}}\mu_{k+1}, \\ \mathrm{E}_\omega T(S_k-1, S_{k+1}) &= \mathrm{E}_\omega[T_{k-1}^L \wedge T_{k-1}^R] + \mathrm{P}_\omega[T_{k-1}^L < T_{k-1}^R]\mathrm{E}_\omega T(S_{k-1}, S_k) + \mathrm{E}_\omega T(S_k, S_{k+1}) \\ &= \xi_k - 1 + \frac{1}{\xi_k}\mu_k + \mu_{k+1},\end{aligned}$$

which leads to

$$\mu_{k+1} = \lambda_k \left(\xi_{k+1} + \frac{\xi_{k+1}-1}{\xi_{k+1}}\mu_{k+1}\right) + (1-\lambda_k)\left(\xi_k + \frac{1}{\xi_k}\mu_k + \mu_{k+1}\right).$$

Assume we have verified that $\mu_k < \infty$ for every $k \in \mathbb{Z}$. Then the above formula may be rewritten as

$$\frac{\mu_{k+1}}{\xi_{k+1}} = \rho_k \frac{\mu_k}{\xi_k} + \rho_k \xi_k + \xi_{k+1}. \tag{2.3.3}$$

Iterating (2.3.3), we obtain

$$\frac{\mu_{k+1}}{\xi_{k+1}} = \sum_{i=-n}^{k} (\rho_i \xi_i + \xi_{i+1}) \Pi_{i+1,k} + \Pi_{-n,k} \frac{\mu_{-n}}{\xi_{-n}}, \tag{2.3.4}$$

which gives

$$\frac{\mu_{k+1}}{\xi_{k+1}} = \sum_{i \leqslant k} (\rho_i \xi_i + \xi_{i+1}) \Pi_{i+1,k} = \xi_{k+1} + 2W_k,$$

if we may verify that $\Pi_{-n,k}\mu_{-n}/\xi_{-n} \to 0$ as $n \to \infty$.

To show (2.3.1) formally, one may repeat the above calculation for the truncated times $\mathbb{T}_k \wedge M$, for $M > 0$. One then sees that a sequence $\mu_k^M = \mathrm{E}_\omega[\mathbb{T}_k \wedge M]$ satisfies

$$\frac{\mu_{k+1}^M}{\xi_{k+1}} \leqslant \rho_k \frac{\mu_k^M}{\xi_k} + \rho_k \xi_k + \xi_{k+1} \leqslant \sum_{i=-n}^{k} (\rho_i \xi_i + \xi_{i+1}) \Pi_{i+1,k} + \Pi_{-n,k} M.$$

Observe that the assumption $\mathrm{E} \log \rho < 0$ guarantees that $\Pi_{-n,k} \to 0$ P-almost surely. Therefore

$$\mu_{k+1} = \lim_{M \to \infty} \mu_{k+1}^M \leqslant \xi_{k+1}^2 + 2\xi_{k+1} W_k,$$

P-almost surely. In particular, $\mu_k < \infty$ P-a.s., for every $k \in \mathbb{Z}$, and (2.3.4) holds. This in turn implies

$$\mu_{k+1} \geqslant \xi_{k+1}^2 + 2\xi_{k+1} W_k$$

and ends the proof of (2.3.1).

Next, denote $\sigma_k = \mathrm{E}_\omega \mathbb{T}_k^2$. A similar calculation using Lemma 2.1.1 and relation (2.3.3) gives

$$\frac{\sigma_{k+1}}{\xi_{k+1}} = \rho_k \frac{\sigma_k}{\xi_k} + f_{k+1}, \tag{2.3.5}$$

where

$$\begin{aligned} f_{k+1} = &\frac{1}{3} \left(\xi_{k+1}^3 + \rho_k \xi_k^3 - 4\xi_{k+1} - 4\rho_k \xi_k \right) \\ &+ \frac{2}{3} \left(\frac{\mu_{k+1}}{\xi_{k+1}} \left(-\xi_{k+1}^2 + 1 \right) + \rho_k \frac{\mu_k}{\xi_k} \left(\xi_k^2 - 1 \right) \right) \\ &+ \frac{2\mu_{k+1}^2}{\xi_{k+1}}. \end{aligned}$$

Proceeding as before, we obtain (2.3.2). □

Having obtained the exact formulae for the quenched mean and variance of the hitting times, we may proceed as in the proof of Lemma 2.2.1 to show the following result:

Corollary 2.3.2. *If* $\mathrm{E}\rho^2 < 1$, $\mathrm{E}\xi^2\rho^2 < \infty$, *and* $\mathrm{E}\xi^4 < \infty$, *then* $\mathbb{E}\mathbb{T}_1 < \infty$ *and* $\mathrm{E}[\mathrm{Var}_\omega \mathbb{T}_1] < \infty$.

Observe that $\mathbb{T}_k$ can be decomposed into a sum of two parts: the time the particle, after reaching S_{k-1}, but before it hits S_k, spends to the left of S_{k-1}, and the time it spends to the right of S_{k-1}. For technical reasons that will become clear below, we divide the visits exactly at point S_{k-1} between these two sets depending on the direction from which the particle hit S_{k-1}. To be precise, we define

$$\mathbb{T}_k^l = \left|\left\{n \in (T_{S_{k-1}}, T_{S_k}] \,:\, X_n < S_{k-1} \text{ or } (X_{n-1}, X_n) = (S_{k-1}-1, S_{k-1})\right\}\right|, \tag{2.3.6}$$

i.e. $\mathbb{T}_k^l$ is the sum of the time the particle spends in $(-\infty, S_{k-1}-1]$ and the number of steps from $S_{k-1}-1$ to S_{k-1}. Similarly we define

$$\mathbb{T}_k^r = \left|\left\{n \in (T_{S_{k-1}}, T_{S_k}] \,:\, S_{k-1} < X_n \leqslant S_k \text{ or } (X_{n-1}, X_n) = (S_{k-1}+1, S_{k-1})\right\}\right|. \tag{2.3.7}$$

Thus we can write

$$\mathbb{T}_k = T_{S_k} - T_{S_{k-1}} = \mathbb{T}_k^l + \mathbb{T}_k^r.$$

Observe that, given ω, the random variables $\{\mathbb{T}_k\}_{k\in\mathbb{N}}$ are independent under P_ω, however for fixed k, $\mathbb{T}_k^l$ and $\mathbb{T}_k^r$ mutually depend on each other.

Lemma 2.3.3. *Assume that* $\mathrm{E}\log\rho < 0, \mathrm{E}\log\xi < \infty$. *Then for any* $k \in \mathbb{Z}$, P-*almost surely,*

$$\mathrm{E}_\omega \mathbb{T}_k^r = \xi_k^2, \tag{2.3.8}$$

$$\mathrm{Var}_\omega \mathbb{T}_k^r = \frac{2}{3}\left(\xi_k^4 - \xi_k^2\right), \tag{2.3.9}$$

$$\mathrm{E}_\omega \mathbb{T}_k^l = 2\xi_k W_{k-1}, \tag{2.3.10}$$

$$\begin{aligned}\mathrm{Var}_\omega \mathbb{T}_k^l = 8\xi_k \sum_{i<k} \left(\xi_i W_{i-1}^2 + \xi_i^2 W_{i-1} + \frac{1}{3}(\xi_i^3 - \xi_i)\right)\Pi_{i,k-1} \\ + 4\xi_k^2 W_{k-1}^2 + 4\xi_k W_{k-1}.\end{aligned} \tag{2.3.11}$$

Proof. Fix $k \in \mathbb{Z}$. Observe that, under P_ω, $\mathbb{T}_{k+1}^r$ equals in distribution to the time it takes a simple symmetric random walk on $[0, \xi_{k+1}]$ with a reflecting barrier placed in 0 to reach ξ_{k+1} for the first time when starting from 0. This is the reason we include into $\mathbb{T}_{k+1}^r$ the visits at S_k, but only those from S_k+1. Equivalently, it is the distribution of the time it takes a simple symmetric random walk starting from 0 to reach $-\xi_{k+1}$ or ξ_{k+1}. Therefore (2.3.8) and (2.3.9) follow from Lemma 2.1.1. Now, (2.3.10) follows from (2.3.1) and (2.3.8).

To examine the variance of $\mathbb{T}_{k+1}^l$, observe that we may express it as a sum of independent copies of a variable F_k, which denotes the length of a single excursion to the left from S_k. That is,

$$\mathbb{T}_{k+1}^l = \sum_{m=0}^{M_k} \sum_{j=1}^{N_m} F_k(j, m), \tag{2.3.12}$$

where M_k, N_m's and $F_k(j,m)$'s are independent under P_ω; M_k is the number of times the particle hit S_k from the right before it reached S_{k+1}, and N_m is the number of its excursions to the left between m'th and $m+1$'st step from S_k to S_k+1. Observe that M_k is geometrically distributed; by (2.1.1), its parameter is $1/\xi_{k+1}$. Moreover, N_m's are also geometrically distributed; since the probability of going left from S_k is $1-\lambda_k$, we have $N_m \sim Geo(\lambda_k)$.

Recall that if $S_N = \sum_{i=1}^N X_i$ for some random variable N and an i.i.d. sequence $(X_n)_{n\in\mathbb{N}}$ independent of N, then

$$\mathrm{Var} S_N = \mathrm{E}N \cdot \mathrm{Var} X_1 + \mathrm{Var} N \cdot (\mathrm{E}X_1)^2. \tag{2.3.13}$$

The above formula together with (2.3.12) easily entails

$$\begin{aligned}\mathrm{Var}_\omega \mathbb{T}^l_{k+1} &= \xi_{k+1}\rho_k \mathrm{Var}_\omega F_k + \left(\xi_{k+1}^2\rho_k^2 + \xi_{k+1}\rho_k\right)(\mathrm{E}_\omega F_k)^2\\ &= \xi_{k+1}\rho_k \mathrm{E}_\omega F_k^2 + \xi_{k+1}^2\rho_k^2(\mathrm{E}_\omega F_k)^2.\end{aligned}$$

From here on we may proceed as in the proof of Lemma 2.3.1. Since F_k is the time of a single excursion from S_k that begins with a step left, we have

$$F_k \stackrel{\mathrm{d}}{=} 1 + T(S_k - 1, S_k).$$

Decomposing $T(S_k-1, S_k)$ depending on the point by which the particle left the interval $[S_{k-1}, S_k]$, we obtain, with the help of Lemma 2.1.1,

$$\begin{aligned}\mathrm{E}_\omega T(S_k-1, S_k) &= \xi_k - 1 + \frac{\mu_k}{\xi_k},\\ \mathrm{E}_\omega T(S_k-1, S_k)^2 &= 1 - \frac{4}{3}\xi_k + \frac{1}{3}\xi_k^3 + \frac{2}{3}\left(\xi_k\mu_k - \frac{\mu_k}{\xi_k}\right) + \frac{\sigma_k}{\xi_k}.\end{aligned}$$

In particular,

$$\mathrm{E}_\omega F_k = 2(\xi_k + W_k). \tag{2.3.14}$$

Now, (2.3.11) may be obtained using Lemma 2.3.1 and relation (2.2.3).

□

Chapter 3

Limit theorems for a random walk in a sparse random environment

3.1 Limit theorems for a random walk in a sparse random environment: an overview

In this section we present an overview of known results concerning limiting behaviour of transient RWSRE. Since a walk in i.i.d. random environment is a special case of RWSRE, it is natural to first evoke limit theorems in this case.

3.1.1 Independent, identically distributed environment

Assume that $\mathrm{E}\log\rho < 0$ and let $r : [0,\infty) \to [0,\infty]$ be defined as

$$r(x) = \mathrm{E}\rho^x. \tag{3.1.1}$$

We will assume that r is finite in some neighbourhood of 0. Observe that the function r is convex. Moreover, $r(0) = 1$ and $r'(0) = \mathrm{E}\log\rho < 0$. This implies that if an $\alpha > 0$ satisfying

$$r(\alpha) = 1 \tag{3.1.2}$$

exists, then it is unique. Moreover, $r(x) < 1$ for $0 < x < \alpha$ and $r(x) > 1$ for $x > \alpha$. It may happen, however, that such $\alpha > 0$ does not exist, since r may jump to $+\infty$ before obtaining value 1 or decrease to 0 if $\rho \leqslant 1$ almost surely.

It turns out that the limit theorems for a walk in i.i.d. environment depend entirely on the function r. If $r(2) < 1$, then under the annealed measure a CLT holds for the sequence of hitting times, from which it is not difficult to deduce a CLT for the position of the walk (see [22, Theorem 3.8]). Different behaviour appears if (3.1.2) holds for $\alpha \in (0,2)$. It was shown by Kesten et al. in [18] that in this case, under the annealed measure, the sequence of hitting times lies in the domain of attraction of some α-stable variable L_α. The limit of $(X_n)_{n\in\mathbb{N}}$ is not Gaussian, but is closely related to this α-stable law. For example, if $\alpha \in (0,1)$, then

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{X_n}{n^\alpha} > x\right] = \mathbb{P}\left[L_\alpha < x^{-1/\alpha}\right]. \tag{3.1.3}$$

For the full statement, in particular the case $\alpha \in [1, 2)$, see the main Theorem in [18].

To describe the quenched limit theorems, observe that for any sequences a_n, b_n, possibly depending on ω,

$$\mu_{n,\omega}(\cdot) = \mathrm{P}_\omega \left[\frac{T_n - b_n}{a_n} \in \cdot \right]$$

is, under P, a random element of $\mathcal{M}_1(\mathbb{R})$, i.e. a random probability measure on $\mathbb{R}$. Therefore one can distinguish two types of limiting behaviour of $(\mu_n)_{n\in\mathbb{N}}$. We will say that a *strong* quenched limit theorem for T holds, if $\mu_n \to \mu$ almost surely in $\mathcal{M}_1(\mathbb{R})$, that is for P-a.e. ω the sequence of measures $(\mu_{n,\omega})_{n\in\mathbb{N}}$ converges to μ in the Prokhorov metric. We will say that a *weak* quenched limit theorem for T holds, if $\mu_n \Rightarrow \mu$ in $\mathcal{M}_1(\mathbb{R})$, that is for any bounded, continuous function $f : \mathcal{M}_1(\mathbb{R}) \to \mathbb{R}$ we have $\mathrm{E}f(\mu_n) \to \mathrm{E}f(\mu)$ as $n \to \infty$.

If $r(2) < 1$, then it may be shown that

$$\lim_{n\to\infty} \mathrm{P}_\omega \left[\frac{T_n - \mathrm{E}_\omega T_n}{\sigma\sqrt{n}} \leqslant t \right] \to \Phi(t) \quad \text{P-a.s.},$$

for some constant σ, where Φ is a cumulative distribution function of a standard normal variable. This fact is used in [15] by Goldsheid to deduce a strong quenched CLT for the position of the walk. However, the quenched counterpart of (3.1.3) is more complex. As seen from the results presented in [21, 24], in the case $\alpha < 2$ there is no strong quenched limit theorem for T. Indeed, it turns out that in this case one can find different strong quenched limits for the hitting times along different subsequences. This in turn leads to the analysis of T in the weak quenched setting. To this end, consider the mapping $H : \mathcal{M}_p((0, \infty)) \to \mathcal{M}_1(\mathbb{R})$ given as follows: for a point measure $\zeta = \sum_{i\geqslant 1} \delta_{x_i}$, where $(x_i)_{i\in\mathbb{N}_+}$ is an arbitrary enumeration of the points, define

$$H(\zeta)(\cdot) = \begin{cases} \mathbb{P}\left[\sum_{i\geqslant 1} x_i(\tau_i - 1) \in \cdot\,\right] & \text{if } \sum_{i\geqslant 1} x_i^2 < \infty, \\ \delta_0(\cdot) & \text{otherwise,} \end{cases}$$

where the probability is taken with respect to $(\tau_i)_{i\in\mathbb{N}}$, a sequence of i.i.d., mean one exponential random variables. Then the main result of [23] states that for $\alpha < 2$,

$$\mathrm{P}_\omega \left[\frac{T_n - \mathrm{E}_\omega T_n}{n^{1/\alpha}} \in \cdot \right] \Rightarrow H(N) \tag{3.1.4}$$

in $\mathcal{M}_1(\mathbb{R})$, where N is a Poisson point process on $(0, \infty)$ with intensity $c_N x^{-\alpha-1}\mathrm{d}x$ for some constant $c_N > 0$. From this follows a quenched version of (3.1.3); namely, for $\alpha \in (0, 1)$, for any $x \in \mathbb{R}$,

$$\mathrm{P}_\omega \left[\frac{X_n}{n^\alpha} < x \right] \Rightarrow H(N)(x^{-1/\alpha}, \infty). \tag{3.1.5}$$

The limiting variables in the case $\alpha \in [1, 2)$ are more complex; we refer the reader to [23, Corollary 1.8]. Observe that the convergence in (3.1.5) is given in terms of pointwise weak convergence of quenched cumulative distribution functions, therefore it is even weaker than a weak quenched limit theorem.

3.1.2 Sparse environment

In the setting of a sparse random environment, the asymptotic behaviour of the walk is driven by two ingredients. The first one is the drift, that is the distribution of λ. The second one is sparsity, that is the tail behaviour of ξ. Since RWSRE may be seen as a model in-between a walk in i.i.d. environment and a symmetric one, we may expect that, depending on the interplay between the drift and the sparsity, it should manifest behaviour resembling one or the other.

As it is in the case of i.i.d. environment, whenever it is the drift that determines the asymptotic behaviour of the walk, the shape of the limit depends on the function r defined in (3.1.1). In Section 3.2 we show how to adapt Goldsheid's result into the setting of RWSRE. That is, we prove that if $r(2) < 1$, then the main assumption that guarantees a strong quenched CLT for T is that $\mathrm{E}\xi^4 < \infty$. Next we obtain the limit theorem for the position of the walk, in the same fashion as it was done in [15].

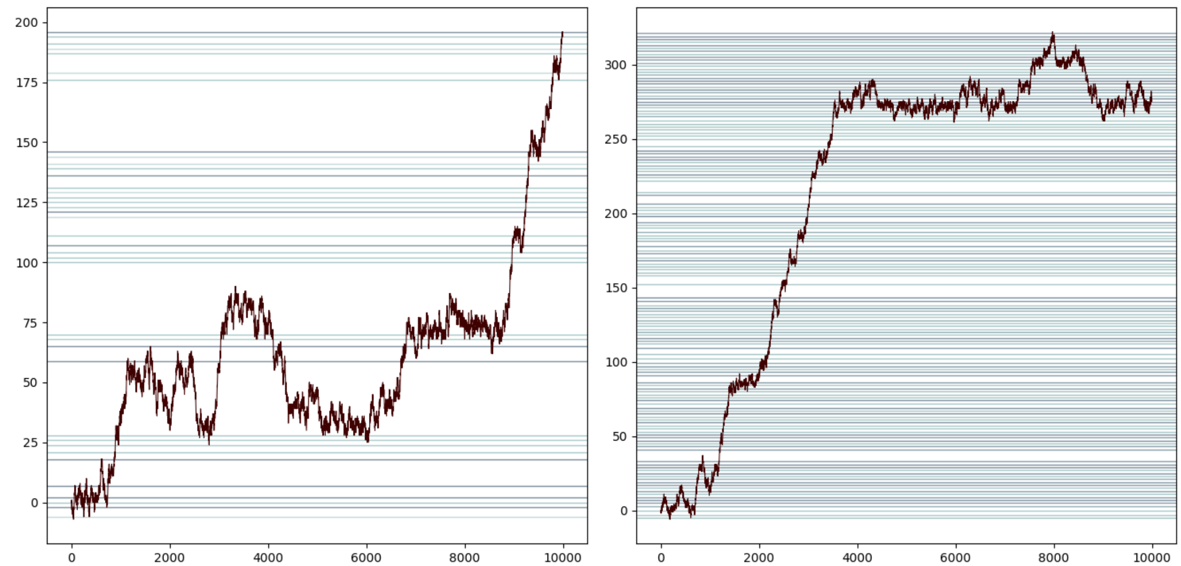
In their paper introducing the RWSRE model, Matzavinos et al. proved annealed limit theorems for a transient RWSRE that generalize results on RWRE described above (see [20, Theorem 3.8]). However, one of their assumptions was that ξ is bounded. More general results were proven by Buraczewski et al. in [7, 6]. It turns out that if $r(\alpha) = 1$ for some $\alpha \in (0, 2]$, then the key assumption under which the sequence of hitting times lies in the domain of attraction of an α-stable law is that

$$\mathrm{E}\xi^{2\alpha} < \infty. \tag{3.1.6}$$

In this case one may generalize the proof given by Kesten et al. and obtain an annealed limit theorem for T, with α-stable distribution in the limit, and then deduce the annealed limit theorems for X (see Corollary 2.4 in [7]). The limiting behaviour of the walk is determined mostly by the drift and the presence of blocks on which the movement is symmetric has little impact on the shape of the limit.

To describe the complementary case, in which it is the sparsity that plays the dominant role in the limiting behaviour of the walk, the authors consider ξ having regularly varying tails with parameter $-\beta$ such that $\beta \in (0, 4)$ and $r(\beta/2) < 1$. Observe that in this case, if $\alpha > 0$ satisfying (3.1.2) exists, then $\alpha > \beta/2$ and (3.1.6) does not hold. In [7], the authors show that if $\mathrm{E}\xi < \infty$, then, under the annealed measure, the sequence of hitting times lies in the domain of attraction of a $\beta/2$-stable law. If $\mathrm{E}\xi = \infty$, then the limiting distribution is more complex; we describe it briefly in Chapter 4.

The phenomenon described above may be explained heuristically with the help of Lemma 2.3.3. Equations (2.3.8) and (2.3.10) suggest that $\mathbb{T}_k^r$, which counts the time spent by the particle in k'th block when crossing it for the first time, should inherit tail behaviour from ξ_k^2, while $\mathbb{T}_k^l$, which is the duration of its excursions to the left, should have asymptotics similar to $\xi_k W_{k-1}$. If α given by (3.1.2) exists and (3.1.6) holds, then by Kesten-Goldie theorem [4, Theorem 2.4.4], W has regularly varying tails with index $-\alpha$, while the tails of ξ^2 are lighter. Therefore we may expect that large $\mathbb{T}_k$'s are obtained when the particle makes excursions to the left that are long because of unfavourable drift. In the complementary case it is the tail of ξ^2 that is heavier, and large $\mathbb{T}_k$'s occur when the particle crosses a particularly long block



(a) $\alpha \approx 1.75, \beta = 0.9$ (b) $\alpha \approx 1.19, \beta = 2.5$

Figure 3.1.1: Exemplary trajectories of RWSRE for ξ with regularly varying tails. Horizontal lines indicate marked points; the darker the line, the stronger the drift to $-\infty$.

for the first time.

Since the limit theorems for the hitting times under the annealed measure resemble those already known for the i.i.d. environment whenever condition (3.1.6) is satisfied, it is natural to expect the same similarity under the quenched measure. That is, obtaining a result analogous to (3.1.4) should require a modification of the techniques used in [23]. Therefore in Chapter 4 we focus on the complementary case, that is we present the quenched counterpart of results described above in the case of ξ having regularly varying tails in which it is the sparsity of the environment that drives the limiting behaviour of the walk. We prove the weak quenched limit theorems for the sequence of hitting times and show that the strong limit theorems do not hold. However, due to the sparsity of the environment, deducing the limit theorems for X seems to require some additional analysis and we limit ourselves to the sequence T.

3.2 Quenched central limit theorem for the position of the walk

In this section we show how the results and techniques concerning strong quenched limit theorems for the first passage times and the position of the walk in a non-sparse environment presented in [15] may be adapted into the setting of RWSRE.

Throughout this section, we assume the following:

$$\mathrm{E}\rho^{2+2\delta} < 1, \quad \mathrm{E}(\rho\xi)^{2+2\delta} < \infty, \quad \mathrm{E}\xi^{4+4\delta} < \infty \quad \text{for some } \delta > 0. \tag{3.2.1}$$

The first assumption is an exact analogue to the case of i.i.d. environment. The other two guarantee that the blocks in which the walk is symmetric are not large enough to influence its limiting behaviour significantly.

Recall that $\mu_k = \mathrm{E}_\omega \mathbb{T}_k$ is the quenched mean time the walker needs to reach S_k when starting from S_{k-1}. Note that the sequence $(\mu_k)_{k\in\mathbb{Z}}$ is stationary under P, and so is $(\mathrm{Var}_\omega \mathbb{T}_k)_{k\in\mathbb{Z}}$. Denote

$$\mu = \mathrm{E}\mu_1 = \mathbb{E}\mathbb{T}_1, \quad \sigma^2 = \mathrm{E}\left[\mathrm{Var}_\omega \mathbb{T}_1\right]$$

and let

$$b(n,\omega) = \frac{n}{\mu} - \frac{1}{\mu}\sum_{k=1}^{n/\mu}(\mu_k - \mu), \quad \tilde{\sigma} = \sigma\mu^{-3/2}\mathrm{E}\xi.$$

Observe that, by Corollary 2.3.2, μ and $\tilde{\sigma}$ are finite under assumptions (3.2.1), and $b(n)$ is finite for every $n \in \mathbb{N}$, P-almost surely.

Theorem 3.2.1. *Under assumptions* (3.2.1), *P-almost surely,*

$$\mathrm{P}_\omega\left[\frac{X_n - S_{b(n)}}{\tilde{\sigma}\sqrt{n}} < t\right] \to \Phi(t),$$

where Φ is the cumulative distribution function of a standard normal variable.

3.2.1 The sequence $(T_{S_n})_{n\in\mathbb{N}}$

Let us first derive a quenched central limit theorem for the sequence $(T_{S_n})_{n\in\mathbb{N}}$. Since under P_ω the variables $\mathbb{T}_k$ are independent, but not identically distributed, we will use Lindeberg-Feller theorem.

Theorem 3.2.2. *For* P*-almost every ω,*

$$\mathrm{P}_\omega\left[\frac{T_{S_n} - \mathrm{E}_\omega T_{S_n}}{\sigma\sqrt{n}} \leqslant t\right] \xrightarrow{n\to\infty} \Phi(t). \tag{3.2.2}$$

Proof. Since $(\mathrm{Var}_\omega \mathbb{T}_k)_{k\in\mathbb{N}}$ is a stationary sequence, the ergodic theorem implies that P-almost surely,

$$\sum_{k=1}^{n} \mathrm{E}_\omega\left(\frac{\mathbb{T}_k - \mathrm{E}_\omega \mathbb{T}_k}{\sqrt{n}}\right)^2 = \frac{1}{n}\sum_{k=1}^{n}\mathrm{Var}_\omega \mathbb{T}_k \xrightarrow{n\to\infty} \sigma^2.$$

Similarly, for every $\varepsilon > 0$ and $M < \infty$,

$$\begin{aligned}
&\limsup_{n\to\infty}\sum_{k=1}^{n}\mathrm{E}_\omega\left[\left(\frac{\mathbb{T}_k - \mathrm{E}_\omega \mathbb{T}_k}{\sqrt{n}}\right)^2 \mathbb{1}_{|\mathbb{T}_k - \mathrm{E}_\omega \mathbb{T}_k| > \varepsilon\sqrt{n}}\right] \\
&\leqslant \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{E}_\omega\left[(\mathbb{T}_k - \mathrm{E}_\omega \mathbb{T}_k)^2\,\mathbb{1}_{|\mathbb{T}_k - \mathrm{E}_\omega \mathbb{T}_k| > M}\right] \\
&= \mathrm{E}\left[\mathrm{E}_\omega\left[(\mathbb{T}_1 - \mathrm{E}_\omega \mathbb{T}_1)^2\,\mathbb{1}_{|\mathbb{T}_1 - \mathrm{E}_\omega \mathbb{T}_1| > M}\right]\right]
\end{aligned}$$

P-almost surely. Since $\mathrm{E}[\mathrm{Var}_\omega \mathbb{T}_1] < \infty$, the last expression can be made arbitrarily small by taking large M. Therefore for P-almost every ω, the sequence $(\mathbb{T}_k)_{k\in\mathbb{N}}$ satisfies the Lindeberg-Feller conditions under P_ω. □

Remark 3.2.3. Since Φ is a continuous function, the convergence (3.2.2) is uniform in t.

3.2.2 The sequence $(X_n)_{n\in\mathbb{N}}$

To derive limit theorem for X, let us first consider $X_n^* = \max\{X_k : k \leqslant n\}$. Then for any $j \in \mathbb{N}$,

$$\mathrm{P}_\omega\left[X_n^* < S_j\right] = \mathrm{P}_\omega\left[T_{S_j} > n\right] = \mathrm{P}_\omega\left[\frac{T_{S_j} - \mathrm{E}_\omega T_{S_j}}{\sigma\sqrt{j}} > \frac{n - \mathrm{E}_\omega T_{S_j}}{\sigma\sqrt{j}}\right].$$

Our aim is to find a sequence $j = j(n, \omega, t)$ such that $(n - \mathrm{E}_\omega T_{S_j})/\sigma\sqrt{j} \to -t$ P-almost surely. Note that

$$n - \mathrm{E}_\omega T_{S_j} = n - j\mu - \sum_{k=1}^{j}(\mu_k - \mu).$$

To eliminate the linear term, denote $j(n, \omega, t) = n/\mu + h(n, \omega, t)$, then

$$n - \mathrm{E}_\omega T_{S_j} = -\mu h - \sum_{k=1}^{n/\mu+h}(\mu_k - \mu).$$

Since we want the expression to tend to $-t$, it is natural to consider, for some constant c,

$$h(n, \omega, t) = ct\sqrt{n} - \frac{1}{\mu}\sum_{k=1}^{n/\mu}(\mu_k - \mu).$$

Then

$$\frac{j(n, \omega, t)}{n} = \frac{1}{\mu} + \frac{ct}{\sqrt{n}} - \frac{1}{n\mu}\sum_{k=1}^{n/\mu}(\mu_k - \mu) \to \frac{1}{\mu}$$

P-almost surely, by the ergodic theorem. Since we want the limit of

$$\frac{n - \mathrm{E}_\omega T_{S_j}}{\sigma\sqrt{j}} = -\frac{\mu ct\sqrt{n}}{\sigma\sqrt{j}} - \frac{\mu}{\sigma\sqrt{j}}\sum_{k=n/\mu+1}^{n/\mu+h}(\mu_k - \mu)$$

to be $-t$ a.s., we should put $c = \sigma\mu^{-3/2}$ and show that

$$\frac{1}{\sqrt{n}}\sum_{k=n/\mu+1}^{n/\mu+h}(\mu_k - \mu) \to 0. \tag{3.2.3}$$

As was mentioned, we follow the approach presented in [15]. In particular, instead of presenting all the details, we shall only give the essential part of the proof. Denote

$$\mathcal{H}(n, \omega) = \sum_{j=1}^{n}(\mu_j - \mu), \quad \mathcal{H}^*(n, \omega) = \max_{s\leqslant n}\mathcal{H}(s, \omega).$$

For $p \geqslant 1$, denote by $\|\cdot\|_p$ the L^p norm with respect to $\mathbb{P}$. It may be seen from the proofs of Lemmas 5 and 7 in [15] that the following result is sufficient for (3.2.3):

Lemma 3.2.4. *Under assumptions* (3.2.1) *there exists* $C > 0$ *such that*

$$||\mathcal{H}^*(n)||_{2+2\delta} \leqslant C\sqrt{n}.$$

Proof. Recall that by (2.3.1),

$$\mu_k - \mu = \xi_k^2 + 2\xi_k W_{k-1} - \mathrm{E}\xi^2 + 2\mathrm{E}\xi\mathrm{E}W$$

and

$$\mathrm{E}W = \mathrm{E}\left[\sum_{k=-\infty}^{0} \xi_k \Pi_{k,0}\right] = \mathrm{E}\rho\xi \sum_{k=0}^{\infty} r(1)^k,$$

therefore, for $c_0 = \mathrm{E}\xi\,\mathrm{E}\rho\xi < \infty$,

$$\mathcal{H}(n,\omega) = \sum_{k=1}^{n}\left(\xi_k^2 - \mathrm{E}\xi^2\right) + \sum_{k=1}^{n}\sum_{j=-\infty}^{k-1} 2\left(\xi_k\xi_j\Pi_{j,k-1} - c_0 r(1)^{k-j-1}\right). \tag{3.2.4}$$

By Marcinkiewicz-Zygmund inequality, since the variables $\xi_k^2 - \mathrm{E}\xi^2$ are i.i.d. with mean 0, for some constant C_1,

$$\begin{aligned}\left\|\sum_{k=1}^{n}\left(\xi_k^2 - \mathrm{E}\xi^2\right)\right\|_{2+2\delta} &\leqslant C_1 \left\|\sum_{k=1}^{n}\left(\xi_k^2 - \mathrm{E}\xi^2\right)^2\right\|_{1+\delta}^{1/2} \\ &\leqslant C_1\left(\sum_{k=1}^{n}\left\|(\xi_k^2 - \mathrm{E}\xi^2)^2\right\|_{1+\delta}\right)^{1/2} \\ &= C_1\left\|\left(\xi^2 - \mathrm{E}\xi^2\right)^2\right\|_{1+\delta}^{1/2}\sqrt{n},\end{aligned}$$

and since $\mathrm{E}\xi^{4+4\delta} < \infty$, all the above norms are finite. This estimate together with Doob's maximal inequality imply that for a constant C_2,

$$\left\|\max_{s\leqslant n}\sum_{k=1}^{s}\left(\xi_k^2 - \mathrm{E}\xi^2\right)\right\|_{2+2\delta} \leqslant \frac{2+2\delta}{1+\delta}\left\|\sum_{k=1}^{n}\left(\xi_k^2 - \mathrm{E}\xi^2\right)\right\|_{2+2\delta} \leqslant C_2\sqrt{n}. \tag{3.2.5}$$

To estimate the maxima of the second addend in (3.2.4), we first split the sum into blocks. To this end, denote

$$B(k,l) = 2\left(\xi_k\xi_{k-l}\Pi_{k-l,k-1} - c_0 r(1)^{l-1}\right),$$

and let $\gamma = r(2+2\delta)^{1/(2+2\delta)}$. By Jensen's inequality, $r(1) \leqslant \gamma < 1$. Moreover, for any $l > 0$,

$$\begin{aligned}\|B(k,l)\|_{2+2\delta} &\leqslant 2\left\|\xi_k\xi_{k-l}\Pi_{k-l,k-1}\right\|_{2+2\delta} + 2c_0 r(1)^{l-1} \\ &= 2\left\|\xi\right\|_{2+2\delta}\left\|\xi\rho\right\|_{2+2\delta}\gamma^{l-1} + 2c_0 r(1)^{l-1} \\ &\leqslant C_3\gamma^{l-1}\end{aligned} \tag{3.2.6}$$

for some constant C_3. We have

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j<k} 2\left(\xi_k\xi_j\Pi_{j,k-1} - c_0 r(1)^{k-j-1}\right) &= \sum_{k=1}^{n}\sum_{l=1}^{\infty} B(k,l) \\ &= \sum_{l=1}^{\sqrt{n}}\sum_{k=1}^{n} B(k,l) + \sum_{l>\sqrt{n}}\sum_{k=1}^{n} B(k,l),\end{aligned} \tag{3.2.7}$$

thus

$$\max_{s\leqslant n}\sum_{k=1}^{s}\sum_{l=1}^{\infty}B(k,l)\leqslant\sum_{l=1}^{\sqrt{n}}\max_{s\leqslant n}\sum_{k=1}^{s}B(k,l)+\sum_{l>\sqrt{n}}\sum_{k=1}^{n}|B(k,l)|.$$

Denote $B_n(l)=\sum_{k=1}^{n}B(k,l)$ and $B_n^*(l)=\max_{s\leqslant n}B_s(l)$. Then by (3.2.6) and (3.2.7),

$$\left\|\max_{s\leqslant n}\sum_{k=1}^{s}\sum_{l=1}^{\infty}B(k,l)\right\|_{2+2\delta}\leqslant\sum_{l=1}^{\sqrt{n}}\|B_n^*(l)\|_{2+2\delta}+C_3\sum_{l>\sqrt{n}}n\gamma^{l-1}.$$

It remains to give bounds on $\|B_n^*(l)\|_{2+2\delta}$. For fixed l, let

$$s_i=\left\lfloor\frac{n-i}{2l}\right\rfloor,\quad D_n(l,i)=\sum_{j=0}^{s_i}B(i+2lj,l),\quad D_n^*(l,i)=\max_{s\leqslant s_i}\left|\sum_{j=0}^{s}B(i+2lj,l)\right|,$$

so that $B_n(l)=\sum_{i=1}^{2l}D_n(l,i)$. Observe that each $D_n(l,i)$ is a sum of centered, i.i.d. variables. Therefore we may, as before, use the Doob's and Marcinkiewicz-Zygmund inequalities to obtain, for a constant C_4,

$$\begin{aligned}\|D_n^*(l,i)\|_{2+2\delta}&\leqslant C_4\left\|\sum_{j=0}^{s_i}B(i+2lj,l)^2\right\|_{1+\delta}^{1/2}\\&\leqslant C_4\left(\frac{n}{2l}\right)^{1/2}\|B(1,l)\|_{2+2\delta}\\&\leqslant C_4C_3\left(\frac{n}{2l}\right)^{1/2}\gamma^{l-1},\end{aligned}$$

where the last inequality follows from (3.2.6). Therefore, for some constant C_5,

$$\|B_n^*(l)\|_{2+2\delta}\leqslant\sum_{i=1}^{2l}\|D_n^*(l,i)\|_{2+2\delta}\leqslant C_5\sqrt{nl}\gamma^{l-1},$$

which together with (3.2.5) gives

$$\|\mathcal{H}^*(n)\|_{2+2\delta}\leqslant C_2\sqrt{n}+C_5\sqrt{n}\sum_{l=1}^{\sqrt{n}}\sqrt{l}\gamma^{l-1}+C_3n\sum_{l>\sqrt{n}}\gamma^{l-1}=O(\sqrt{n}).$$

□

Proof of Theorem 3.2.1. Since Lemma 3.2.4 implies (3.2.3), we have

$$\mathrm{P}_\omega\left[X_n^*<S_{j(n,\omega,t)}\right]=\mathrm{P}_\omega\left[\frac{T_{S_j}-\mathrm{E}_\omega T_{S_j}}{\sigma\sqrt{j}}>\frac{n-\mathrm{E}_\omega T_{S_j}}{\sigma\sqrt{j}}\right]\to 1-\Phi(-t)=\Phi(t).$$

Recall that $j(n,\omega,t) = b(n,\omega) + ct\sqrt{n}$ for $c = \sigma\mu^{3/2}$, thus

$$\begin{aligned}\{X_n^* < S_{j(n,\omega,t)}\} &= \left\{X_n^* < S_{b(n)} + \sum_{k=b(n)+1}^{b(n)+ct\sqrt{n}} \xi_k\right\}\\ &= \left\{X_n^* - S_{b(n)} < \sum_{k=b(n)+1}^{b(n)+ct\sqrt{n}} (\xi_k - \mathrm{E}\xi) + c\mathrm{E}\xi t\sqrt{n}\right\}\\ &= \left\{\frac{X_n^* - S_{b(n)}}{c\mathrm{E}\xi\sqrt{n}} - \frac{1}{c\mathrm{E}\xi\sqrt{n}} \sum_{k=b(n)+1}^{b(n)+ct\sqrt{n}} (\xi_k - \mathrm{E}\xi) < t\right\}\end{aligned}$$

We may repeat an argument used to show inequality (3.2.5) and obtain, for some $C > 0$,

$$\left\|\max_{s\leqslant n} \sum_{k=1}^{s} (\xi_k - \mathrm{E}\xi)\right\|_{2+2\delta} \leqslant C\sqrt{n}.$$

Since $b(n)/n \to 1/\mu$ a.s., we may invoke the Lemmas 5 and 7 from [15] once more to get

$$\sum_{k=b(n)+1}^{b(n)+ct\sqrt{n}} (\xi_k - \mathrm{E}\xi) \to 0$$

and thus

$$\mathrm{P}_\omega\left[\frac{X_n^* - S_{b(n)}}{c\mathrm{E}\xi\sqrt{n}} < t\right] \to \Phi(t).$$

Finally, it remains to show that we may replace X^* with X. To this end, denote

$$\nu_n = \inf\{k > 0 \,:\, S_k > n\}$$

and observe that

$$\frac{|X_n^* - X_n|}{\sqrt{n}} \leqslant \frac{n - T_{X_n^*}}{\sqrt{n}} \leqslant \frac{T_{\nu_{X_n^*}} - T_{X_n^*}}{\sqrt{n}} \leqslant \frac{\mathbb{T}_{\nu_{X_n^*}}}{\sqrt{n}}.$$

Since $\mathbb{E}\mathbb{T}_k^2 < \infty$, the ergodic theorem implies that $\mathbb{T}_k/\sqrt{k} \to 0$ $\mathbb{P}$-almost surely as $k \to \infty$, while by Proposition 2.2.4 and the law of large numbers, $\nu_{X_n^*}/n \to v/\mathrm{E}\xi$ $\mathbb{P}$-almost surely. Therefore $\mathbb{T}_{\nu_{X_n^*}}/\sqrt{n} \to 0$, which finishes the proof of the theorem. □

Chapter 4

Weak quenched limit theorems for the first passage times

In this chapter, we present the weak quenched limit theorems for first passage times of RWSRE in the case of dominating sparsity. The following is an extract from [8] with alterations done by the author to keep consistency with other chapters.

4.1 General setting

We will study the distribution of T in the weak quenched setting, which means that we will investigate the behaviour of a sequence of random measures

$$\mu_{n,\omega}(\cdot) = \mathrm{P}_\omega\left[(T_n - b_n)/a_n \in \ \cdot\ \right]$$

for suitable choices of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, possibly depending on ω. Throughout this chapter, we will consider ξ having a regularly varying tail with index $-\beta$ for $\beta \in (0, 4)$, and assume that $\mathrm{E}\rho^{\beta/2} < 1$. As was remarked in Section 3.1, if $\beta \in [1, 4)$ and $\mathrm{E}\xi < \infty$, then with respect to the annealed probability T lies in the domain of attraction of $\beta/2$-stable law, while for $\beta < 1$ one sees an interplay between the contribution of the sparse random environment and the random movement of the process in the unmarked sites. To state this result take ϑ to be a non-negative random variable with the Laplace transform

$$\mathbb{E}\left[e^{-s\vartheta}\right] = \frac{1}{\cosh(\sqrt{s})}, \quad s > 0. \tag{4.1.1}$$

Note that 2ϑ is equal in distribution to the exit time of the one-dimensional Brownian motion from the interval $[-1, 1]$, see [27, Proposition II.3.7]. Next consider a measure η on $\mathbb{K} = [0, \infty]^2 \setminus \{(0, 0)\}$ given via

$$\eta(\{(v, u) \in \mathbb{K} \,:\, u > x_1 \text{ or } v > x_2\}) = x_1^{-\beta} + \mathbb{E}[\vartheta^{\beta/2}]x_2^{-\beta/2} - \mathbb{E}[\min\{x_1^{-\beta}, \vartheta^{\beta/2}x_2^{-\beta/2}\}]$$

for $x_1, x_2 > 0$. Let $N = \sum_k \delta_{(t_k, \mathbf{j}_k)}$ be a Poisson point process on $[0, \infty) \times \mathbb{K}$ with intensity $\mathrm{LEB} \otimes \eta$, where LEB stands for the one-dimensional Lebesgue measure. Under mild integrability assumptions, see [6, Lemma 6.4], the integral

$$\mathbf{L}(t) = (L_1(t), L_2(t)) = \int_{[0,t]\times\mathbb{K}} \mathbf{j}\, N(\mathrm{d}s, \mathrm{d}\mathbf{j}), \qquad t \geqslant 0$$

converges and defines a two-dimensional non-stable Lévy process with Lévy measure η. Next consider the β-inverse subordinator

$$L_1^{\leftarrow}(t) = \inf\{s > 0 \,:\, L_1(s) > t\}, \qquad t \geqslant 0.$$

Finally, if $\beta \in (0,1)$, then under some additional mild integrability assumptions [6, Theorem 21], with respect to the annealed probability,

$$T_n/n^2 \Rightarrow 2L_2(L_1^{\leftarrow}(1)^-) + 2\vartheta(1 - L_1(L_1^{\leftarrow}(1)^-))^2. \tag{4.1.2}$$

Our goal is to present quenched version of these results. As we will see in our main theorem, the terms $L_2(L_1^{\leftarrow}(1)-)$ and $L_1(L_1^{\leftarrow}(1-))$ present in (4.1.2) can be viewed as the contribution of the environment, whereas ϑ reflects the contribution of the movement of the random walker in the unmarked sites that are close to n.

The chapter is organised as follows: in Section 4.2 we give a precise description of our set-up and main results. In Section 4.3 we provide a preliminary analysis of the environment. The essential parts of the proof of our main results are in Sections 4.4 and 4.5, where we prove weak quenched limits and the absence of the strong quenched limit, respectively.

4.2 Weak quenched limit theorems

In this section we will present our main results. We assume that

$$\mathrm{P}[\xi > t] \sim t^{-\beta}\ell(t) \tag{4.2.1}$$

for some $\beta \in (0,4)$ and slowly varying ℓ. We will focus on the case in which the asymptotic of the system is not determined solely by the drifts at marked sites and thus we will also assume that

$$\mathrm{E}[\rho^{2\gamma}] < 1, \qquad \mathrm{E}[\xi^{3\gamma}\rho^{\gamma}] < \infty, \qquad \mathrm{E}[\xi^{2\gamma}\rho^{2\gamma}] < \infty, \qquad \text{for some } \gamma > \beta/4. \tag{4.2.2}$$

Without loss of generality we will assume that $\gamma < \min\{1, \beta/2\}$, in particular $\mathrm{E}\xi^{2\gamma} < \infty$. As we will see later, the first condition in (4.2.2) guarantees that a significant part of the fluctuations of T_n comes from the time that the process spends in the unmarked sites. The next conditions are purely technical. Note that we do not assume that there exists $\alpha > 0$ for which (3.1.2) holds, however if it does exist, then necessarily $2\alpha > \beta$.

As it is the case for annealed limit theorem, one needs to distinguish between a moderately ($\mathrm{E}\xi < \infty$) and strongly ($\mathrm{E}\xi = \infty$) sparse random environment.

To describe the former take $(\vartheta_j)_{j\in\mathbb{N}}$ to be a sequence of i.i.d. copies of ϑ distributed according to (4.1.1) and let $G : \mathcal{M}_p((0,\infty)) \to \mathcal{M}_1(\mathbb{R})$ be given via

$$G(\zeta)(\cdot) = \begin{cases} \mathbb{P}\left[\sum_{i\geqslant 1} x_i(2\vartheta_i - 1) \in \cdot\,\right], & \text{if } \int x^2\zeta(\mathrm{d}x) < \infty, \\ \delta_0(\cdot) & \text{otherwise,} \end{cases} \tag{4.2.3}$$

for $\zeta = \sum_{i\geqslant 1} \delta_{x_i}$, where $(x_i)_{i\geqslant 1}$ is an arbitrary enumeration of the points and the probability is taken with respect to $(\vartheta_j)_{j\in\mathbb{N}}$. Take $(a_n)_{n\in\mathbb{N}}$ to be any non-decreasing sequence of positive real numbers such that

$$n\mathrm{P}[\xi > a_n] \to 1.$$

Then, since the tail of ξ is assumed to be regularly varying, the sequence $(a_n)_{n\in\mathbb{N}}$ is also regularly varying with index $1/\beta$. That is for some slowly varying function ℓ_1,

$$a_n = n^{1/\beta}\ell_1(n).$$

The sequence $(a_n)_{n\in\mathbb{N}}$ will play the role of the scaling factor in our results.

Theorem 4.2.1. *Assume* (2.2.6), (4.2.1) *and* (4.2.2). *If* $\mathrm{E}\xi < \infty$, *then*

$$\mathrm{P}_\omega\left[(T_n - \mathrm{E}_\omega T_n)/a_n^2 \in \cdot\,\right] \Rightarrow G(N)(\cdot)$$

in $\mathcal{M}_1(\mathbb{R})$, *where* N *is a Poisson point process on* $(0,\infty)$ *with intensity* $\beta x^{-\beta/2-1}\mathrm{d}x/2\mathrm{E}\xi$.

Before we introduce the notation necessary to formulate our results in the strongly sparse random environment, we will first treat the critical case which is relatively simple to state. Denote

$$m_n = n\mathrm{E}\left[\xi\,\mathbb{1}_{\xi\leqslant a_n}\right].$$

Note that by Karamata's theorem [3, Theorem 1.5.11] the sequence $(m_n)_{n\in\mathbb{N}}$ is regularly varying with index $1/\beta$. Furthermore $a_n = o(m_n)$ if $\beta = 1$ and $a_n \sim (1-\beta)m_n$ if $\beta < 1$. Next let $(c_n)_{n\in\mathbb{N}}$ be the asymptotic inverse of $(m_n)_{n\in\mathbb{N}}$, i.e. any increasing sequence of natural numbers such that

$$\lim_{n\to\infty} c_{m_n}/n = \lim_{n\to\infty} m_{c_n}/n = 1.$$

By the properties of an asymptotic inversion of regularly varying sequences [3, Theorem 1.5.12], c_n is well defined up to asymptotic equivalence and is regularly varying with index β. Finally, by the properties of the composition of regularly varying sequences, $(a_{c_n})_{n\in\mathbb{N}}$ is regularly varying with index 1 and $a_{c_n} = o(n)$ if $\beta = 1$.

Theorem 4.2.2. *Assume* (2.2.6), (4.2.1) *and* (4.2.2). *If* $\mathrm{E}\xi = \infty$ *and* $\beta = 1$, *then*

$$\mathrm{P}_\omega\left[(T_n - \mathrm{E}_\omega T_n)/a_{c_n}^2 \in \cdot\,\right] \Rightarrow G(N)(\cdot)$$

in $\mathcal{M}_1(\mathbb{R})$, *where* N *is a Poisson point process on* $(0,\infty)$ *with intensity* $x^{-3/2}\mathrm{d}x/2$.

The limiting random measures in Theorems 4.2.1 and 4.2.2 share some of the properties of their counterpart in the case of i.i.d. environment [23, Remark 1.5]. Namely, using the superposition and scaling properties of Poisson point processes, one can directly show that for each $n \in \mathbb{N}$ and $G, G_1, \ldots, G_n$ being i.i.d. copies of the limit random measure $G(N)$ in Theorem 4.2.1 or Theorem 4.2.2,

$$G_1 * G_2 * \ldots * G_n(\cdot) \stackrel{d}{=} G(\cdot/n^{2/\beta}). \tag{4.2.4}$$

The statement of our results in the strongly sparse case needs some additional notation. As it is the case for the annealed results, it is most convenient to work in the framework of non-decreasing càdlàg functions rather than point processes. Denote by $\mathbb{D}^{\uparrow}$ the class of non-decreasing càdlàg functions $\mathbb{R}_+ \to \mathbb{R}_+$ and for $h \in \mathbb{D}^{\uparrow}$ consider

$$\Upsilon(h) = \sup\{h(t) \,:\, t \in \mathbb{R}_+,\ h(t) \leqslant 1\}. \tag{4.2.5}$$

Note that if $h(t) = 1$ for some t, then necessarily $\Upsilon(h) = 1$. For $h \in \mathbb{D}^{\uparrow}$ denote by $(x_k(h), t_k(h))_{k\in\mathbb{N}}$ an arbitrary enumeration of jumps of h, that is $t_k = t_k(h) \in \mathbb{R}_+$ for $k \in \mathbb{N}$ are all points on the non-negative half-line such that h has a (left) discontinuity with jump of size $x_k(h) = h(t_k) - h(t_k^-) > 0$ at t_k. Note that the random series $\sum_{k: h(t_k)\leqslant 1} x_k(h)^2(2\vartheta_k - 1)$ is convergent since it has an expected value bounded by $h(1)\mathbb{E}|2\vartheta - 1|$. Finally let $F\colon \mathbb{D}^{\uparrow} \to \mathcal{M}_1(\mathbb{R})$ be given by

$$F(h)(\cdot) = \mathrm{P}\left[(1 - \Upsilon(h))^2(2\vartheta_0 - 1) + \sum_{k: h(t_k)\leqslant 1} x_k(h)^2(2\vartheta_k - 1) \in \ \cdot \ \right].$$

Theorem 4.2.3. *Assume* (2.2.6), (4.2.1) *and* (4.2.2). *If* $\beta \in (0,1)$, *then*

$$\mathrm{P}_\omega\left[(T_n - \mathrm{E}_\omega T_n)/n^2 \in \ \cdot \ \right] \Rightarrow F(L)(\cdot)$$

in $\mathcal{M}_1(\mathbb{R})$, *where* L *is a* β*-stable Lévy subordinator with Lévy measure* $\nu(x, +\infty) = x^{-\beta}$.

Interestingly the limit measure $F(L)$ does not enjoy a self-similarity property in the sense of (4.2.4). Namely, for any $a, b \in \mathbb{R}$, $b > 0$ the laws of

$$F_1 * F_2(\cdot) \quad \text{and} \quad F((\cdot - a)/b)$$

are different, where F, F_1 and F_2 are independent copies of the limiting random measure $F(L)$ in Theorem 4.2.3.

Finally, we prove that the weak convergence stated above cannot be improved to convergence in distribution. Therefore, as in the case of i.i.d. environment, the asymptotic quenched behaviour of T ought to be expressed in terms of weak quenched convergence. To keep the proof relatively short, we omit the boundary case of $\beta = 1, \mathrm{E}\xi = \infty$.

Theorem 4.2.4. *Assume* (2.2.6), (4.2.1), *and* (4.2.2) *and consider*

$$\kappa_n = \begin{cases} a_n^2 & \text{if } \mathrm{E}\xi < \infty, \\ n^2 & \text{if } \mathrm{E}\xi = \infty \text{ and } \beta < 1. \end{cases} \tag{4.2.6}$$

Then P*-a.s. the sequence of probability distributions*

$$\mathrm{P}_\omega[(T_n - \mathrm{E}_\omega T_n)/\kappa_n \in \ \cdot\ ] \tag{4.2.7}$$

has no limit in the Prokhorov metric.

4.3 Auxiliary results

We will now present a few lemmas that we will use in our proofs. We will discuss the asymptotic behaviour of the hitting times (2.2.4). It will allow us to understand the process X better and indicate its ingredients which play an essential role in the proof of our main results. We will first analyse the hitting times T along the marked sites S, that is

$$T_{S_k} = \inf\{n : \ X_n = S_k\}, \qquad k \geqslant 1.$$

Recall $\mathbb{T}_k^r$, $\mathbb{T}_k^l$ defined by (2.3.7) and (2.3.6). They give rise to the following decomposition that will be used repeatedly:

$$T_{S_k} = \sum_{j=1}^{k} \mathbb{T}_j = \sum_{j=1}^{k} \mathbb{T}_j^l + \sum_{j=1}^{k} \mathbb{T}_j^r =: T_{S_k}^l + T_{S_k}^r.$$

In Lemma 2.3.3, we calculated the quenched expected value and quenched variance of $\mathbb{T}_k^r, \mathbb{T}_k^l$. Below we prove that after hitting any chosen site S_k the consecutive excursions to the left are negligible. This entails that behaviour of T_{S_k} is determined mainly by $T_{S_k}^r$.

4.3.1 The sequence $(T_{S_n}^r)_{n\in\mathbb{N}}$

Recall that, under P_ω, $\mathbb{T}_k^r$ equals in distribution to the time it takes a simple symmetric random walk on $[0, \xi_k]$ with a reflecting barrier placed in 0 to reach ξ_k for the first time when starting from 0. Let $(Y_n)_{n\in\mathbb{N}}$ be a simple symmetric random walk on $\mathbb{Z}$ independent of the environment ω. Define

$$U_n = \inf\{m : |Y_m| = n\}, \tag{4.3.1}$$

i.e. U_n is the first time the reflected random walk hits n. Then for every $k > 0$, for fixed environment ω, $\mathbb{T}_k^r \stackrel{d}{=} U_{\xi_k}$. In what follows we investigate how the asymptotic properties of ξ_k affect those of $\mathbb{T}_k^r$. To do that, we will utilize the aforementioned equality in distribution and hence we first need to describe the asymptotic properties of U_n as n tends to infinity. The proof of the next lemma is omitted, since it follows from a standard application of Doob's optimal stopping theorem to martingales $Y_n^2 - n$, $Y_n^4 - 6nY_n^2 + 3n^2 + 2n$, $Y_n^6 - 15nY_n^4 + (45n^2 + 30n)Y_n^2 - (15n^3 + 30n^2 + 16n)$, and $\exp\{\pm tY_n\}\cosh(t)^{-n}$.

Lemma 4.3.1. *Let U_n, for $n \in \mathbb{N}$, be given in* (4.3.1)*. We have*

$$\mathrm{E}U_n = n^2, \quad \mathrm{E}U_n^2 = 5n^4/3 - 2n^2/3.$$

Moreover, as $n \to \infty$,

$$U_n/n^2 \Rightarrow 2\vartheta,$$

for ϑ *defined in* (4.1.1). *Furthermore the family of random variables* $\{n^{-4}U_n^2\}_{n\in\mathbb{N}}$ *is uniformly integrable.*

The sequence $T^r_{S_n} = \sum_{k=1}^n \mathbb{T}^r_k$ is a sum of P_ω-independent random variables. Since, by Lemma 4.3.1,

$$\mathrm{Var}_\omega \mathbb{T}^r_k = \frac{2}{3}\xi_k^4 - \frac{2}{3}\xi_k^2, \tag{4.3.2}$$

in our setting the variance $\mathrm{Var}_\omega T^r_{S_n}$ behaves asymptotically as $(2/3)\sum_{k=1}^n \xi_k^4$, thus obeys a stable limit theorem [14, Theorem 3.8.2]. Moreover, we can use precise large deviation results for sums of i.i.d. regularly varying random variables [10, Theorem 9.1] to describe the deviations of $\mathrm{Var}_\omega T^r_{S_n}$. That is for any sequence $(\alpha_n)_{n\in\mathbb{N}}$ that tends to infinity,

$$\mathrm{P}[\mathrm{Var}_\omega T^r_{S_n} \geqslant \alpha_n a_n^4] \sim (2/3)^{\beta/4} n \alpha_n^{-\beta/4} a_n^{-\beta} \ell(\alpha_n^{1/4} a_n).$$

We can now use Potter bounds [3, Theorem 1.5.6] to control $\ell(\alpha_n^{1/4} a_n)$ with $\ell(a_n)$. This in turn yields a large deviation result asymptotic on the logarithmic scale. We summarize this discussion in the following lemma.

Corollary 4.3.2. *The sequence* $(\mathrm{Var}_\omega T^r_{S_n}/a_n^4)_{n\in\mathbb{N}}$ *converges in distribution (with respect to* P*) to some stable random variable. Moreover for any sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *that tends to infinity,*

$$\log \mathrm{P}[\mathrm{Var}_\omega T^r_{S_n} \geqslant \alpha_n a_n^4] \sim -\beta \log(\alpha_n)/4.$$

4.3.2 The sequence $(T^l_{S_n})_{n\in\mathbb{N}}$

Recall the formulae for quenched mean and variance of variables $\mathbb{T}^l_k$ given in Lemma 2.3.3. The next lemma implies that in our setting, the left excursions of the process are negligible.

Lemma 4.3.3. *For every* $\varepsilon > 0$ *and* $\theta \geqslant 0$,

$$\mathrm{P}\big[\mathrm{Var}_\omega T^l_{S_n} \geqslant \varepsilon n^\theta a_n^4\big] \leqslant o(1)/n^{\theta\gamma}, \quad n \to \infty,$$

where γ *is a parameter satisfying* (4.2.2). *In particular,*

$$\frac{1}{a_n^4}\mathrm{Var}_\omega T^l_{S_n} \xrightarrow{\mathrm{P}} 0.$$

Proof. To prove the lemma one needs to deal with the formula for the variance (2.3.11). To avoid long and tedious arguments we will explain how to estimate two of the terms, i.e. we will prove

$$\mathrm{P}\Big[\sum_{k=1}^n \xi_k \cdot \sum_{j<k-1} \Pi_{j+1,k-1}\xi_{j+1}W_j^2 \geqslant \varepsilon n^\theta a_n^4\Big] \leqslant o(1)/n^{\theta\gamma}, \quad n \to \infty \tag{4.3.3}$$

and

$$\mathrm{P}\Big[\sum_{k=1}^{n}\xi_k\cdot\sum_{j<k-1}\Pi_{j+1,k-1}\xi_{j+1}^3\geqslant\varepsilon n^\theta a_n^4\Big]\leqslant o(1)/n^{\theta\gamma},\quad n\to\infty. \tag{4.3.4}$$

All the remaining terms can be treated using exactly the same arguments.

Recall that $\gamma\in(\beta/4,1)$ and $\mathrm{E}\rho^{2\gamma}<1$. The Markov inequality, subadditivity of the function $x\mapsto x^\gamma$, and independence of ξ_k, $\Pi_{j+2,k-1}$, $\rho_{j+1}\xi_{j+1}$ and W_j yield

$$\mathrm{P}\Big[\sum_{k=1}^{n}\xi_k\cdot\sum_{j<k-1}\Pi_{j+1,k-1}\xi_{j+1}W_j^2\geqslant\varepsilon n^\theta a_n^4\Big]\leqslant\frac{1}{\varepsilon^\gamma n^{\theta\gamma}a_n^{4\gamma}}\mathrm{E}\Big[\sum_{k=1}^{n}\xi_k\cdot\sum_{j<k-1}\Pi_{j+1,k-1}\xi_{j+1}W_j^2\Big]^\gamma$$
$$\leqslant\frac{1}{\varepsilon^\gamma n^{\theta\gamma}a_n^{4\gamma}}\sum_{k=1}^{n}\mathrm{E}\xi_k^\gamma\cdot\sum_{j<k-1}\mathrm{E}\Pi_{j+2,k-1}^\gamma\mathrm{E}[\rho_{j+1}^\gamma\xi_{j+1}^\gamma]\mathrm{E}W_j^{2\gamma}\leqslant\frac{Cn}{\varepsilon^\gamma n^{\theta\gamma}a_n^{4\gamma}}=\frac{o(1)}{n^{\theta\gamma}},$$

where the last inequality follows from our hypotheses (4.2.2) and Lemma 2.2.1. This proves (4.3.3). We proceed similarly with the second formula (4.3.4):

$$\mathrm{P}\Big[\sum_{k=1}^{n}\xi_k\cdot\sum_{j<k-1}\Pi_{j+1,k-1}\xi_j^3\geqslant\varepsilon n^\theta a_n^4\Big]\leqslant\frac{1}{\varepsilon^\gamma a_n^{4\gamma}}\mathrm{E}\Big[\sum_{k=1}^{n}\xi_k\cdot\sum_{j<k-1}\Pi_{j+1,k-1}\xi_j^3\Big]^\gamma$$
$$\leqslant\frac{1}{\varepsilon^\gamma n^{\theta\gamma}a_n^{4\gamma}}\sum_{k=1}^{n}\mathrm{E}\xi_k^\gamma\cdot\sum_{j<k-1}\mathrm{E}\Pi_{j+2,k-1}^\gamma\mathrm{E}[\rho_{j+1}^\gamma\xi_{j+1}^{3\gamma}]\leqslant\frac{Cn}{\varepsilon^\gamma n^{\theta\gamma}a_n^{4\gamma}}=\frac{o(1)}{n^{\theta\gamma}}.$$

Invoking the first part of the lemma with $\theta=0$ we conclude convergence of $\mathrm{Var}_\omega T_{S_n}^l/a_n^4$ to 0 in probability. □

4.4 Proofs of the weak quenched limit theorems

In this section we present a complete proof of our main results. We will begin by presenting a suitable coupling. Then we will treat the moderately sparse and strongly sparse case separately.

4.4.1 Coupling

In the first step we will prove our result along the marked sites. That is we analyse

$$\phi_{n,\omega}(\cdot)=\mathrm{P}_\omega\left[a_n^{-2}\left(T_{S_n}-\mathrm{E}_\omega T_{S_n}\right)\in\ \cdot\ \right]. \tag{4.4.1}$$

The main part of the argument concentrates on the limit law of $T_{S_n}^r=\sum_{k=1}^n\mathbb{T}_k^r$. Recall U_n defined in (4.3.1), which is the first time the reflected random walk hits n. For every $k>0$ and for fixed environment ω it holds that $\mathbb{T}_k^r\stackrel{d}{=}U_{\xi_k}$. By the merit of Lemma 4.3.1 and Skorokhod's representation theorem we may assume that our space holds random variables $U_n^{(k)}$ and ϑ_k such that:

- $\{U_n^{(k)}\}_n,\vartheta_k$ for $k\in\mathbb{N}$ are independent copies of $\{U_n\}_n,\vartheta$;

- $\{U_n^{(k)}, \vartheta_k : n, k \in \mathbb{N}\}$ and $\{\xi_k : k \in \mathbb{N}\}$ are independent;
- $U_n^{(k)}/n^2 \to 2\vartheta_k$ in L^2 as $n \to \infty$;
- for all ω, $U_{\xi_k}^{(k)}$ and $\mathbb{T}_k^r$ have the same distribution under P_ω.

Observe that the convergence in L^2 is secured by the convergence in distribution and uniform integrability provided in Lemma 4.3.1.

To simplify the notation we will write U_{ξ_k} instead of $U_{\xi_k}^{(k)}$.

Proposition 4.4.1. *Assume* (4.2.1). *Then as $n \to \infty$,*

$$a_n^{-4}\mathrm{Var}_\omega\left[T_{S_n}^r - \mathrm{E}_\omega T_{S_n}^r - \sum_{k=1}^n \xi_k^2(2\vartheta_k - 1)\right] \xrightarrow{\mathrm{P}} 0.$$

Proof. First, note that

$$\mathrm{Var}_\omega\left[T_{S_n}^r - \mathrm{E}_\omega T_{S_n}^r - \sum_{k=1}^n \xi_k^2(2\vartheta_k - 1)\right] \stackrel{\mathrm{d}}{=} \mathrm{Var}_\omega\left[\sum_{k=1}^n \left(U_{\xi_k} - 2\xi_k^2\vartheta_k\right)\right].$$

For $\varepsilon > 0$ let $I_n^1 = \{k \leqslant n : \xi_k > \varepsilon a_n\}$ and $I_n^2 = \{k \leqslant n : \xi_k \leqslant \varepsilon a_n\}$. Then for any $\delta > 0$,

$$\mathrm{P}\left[\mathrm{Var}_\omega\left[\sum_{k=1}^n \left(U_{\xi_k} - 2\xi_k^2\vartheta_k\right)\right] > \delta a_n^4\right] \leqslant$$
$$\mathrm{P}\left[\sum_{k\in I_n^1} \xi_k^4 \mathrm{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] > \frac{\delta a_n^4}{2}\right] + \mathrm{P}\left[\sum_{k\in I_n^2} \xi_k^4 \mathrm{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] > \frac{\delta a_n^4}{2}\right]. \quad (4.4.2)$$

Since $U_n^{(k)}, \vartheta_k$ are independent copies of U_n, ϑ such that $U_n/n^2 \to \vartheta$ in L^2, there exists $M > 0$ such that

$$\mathrm{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] < M \quad \text{for all } k, \omega,$$

and, for $N \in \mathbb{N}$ large enough,

$$\mathrm{Var}_\omega\left[\frac{U_N^{(k)}}{N^2} - 2\vartheta_k\right] < \varepsilon \quad \text{for all } k, \omega.$$

We can hence estimate, for n sufficiently large,

$$\mathrm{P}\left[\sum_{k\in I_n^1} \xi_k^4 \mathrm{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] > \frac{\delta a_n^4}{2}\right] \leqslant \mathrm{P}\left[\frac{\sum_{k=1}^n \xi_k^4}{a_n^4} > \frac{\delta}{2\varepsilon}\right].$$

Since the sequence $\sum_{k=1}^n \xi_k^4/a_n^4$ converges weakly (under P) to some $\beta/4$-stable variable $L_{\beta/4}$, the probability on the right hand side above converges to $\mathrm{P}[L_{\beta/4} > \delta/(2\varepsilon)]$. To estimate the second term in (4.4.2), note that

$$\mathrm{P}\left[\sum_{k\in I_n^2}\xi_k^4\mathrm{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2}-2\vartheta_k\right]>\frac{\delta a_n^4}{2}\right]\leqslant \mathrm{P}\left[\sum_{k=1}^n\xi_k^4\,\mathbb{1}_{\xi_k\leqslant\varepsilon a_n}>\frac{\delta a_n^4}{2M}\right]$$
$$\leqslant\frac{2M}{\delta}a_n^{-4}\mathrm{E}\left[\sum_{k=1}^n\xi_k^4\,\mathbb{1}_{\xi_k\leqslant\varepsilon a_n}\right]=\frac{2M}{\delta}na_n^{-4}\mathrm{E}\left[\xi^4\,\mathbb{1}_{\xi\leqslant\varepsilon a_n}\right].$$

By the Fubini theorem, we have

$$\mathrm{E}\left[\xi^4\,\mathbb{1}_{\xi\leqslant\varepsilon a_n}\right]\leqslant\int_0^{\varepsilon a_n}4t^3\mathrm{P}[\xi>t]dt$$

and the Karamata theorem [3, Theorem 1.5.11] entails that the expression on the right is asymptotically equivalent to $4\varepsilon^4a_n^4\mathrm{P}[\xi>\varepsilon a_n]\sim 4\varepsilon^{4-\beta}n^{-1}a_n^4$. Finally, we can conclude that for any $\varepsilon,\delta>0$,

$$\limsup_n \mathrm{P}\left[\sum_{k=1}^n\xi_k^4\mathrm{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2}-\vartheta_k\right]>\delta a_n^4\right]\leqslant\frac{8M}{\delta}\varepsilon^{4-\beta}+\mathrm{P}\left[L_{\beta/4}>\frac{\delta}{2\varepsilon}\right]$$

and passing with ε to 0 we conclude the desired result. □

We are now ready to determine the weak limit of the sequence $\phi_n(\omega)=\phi_{n,\omega}$ given by (4.4.1). Recall the map G defined in (4.2.3).

Lemma 4.4.2. *The map G is measurable.*

Remark 4.4.3. The proof of Lemma 4.4.2 is identical to that of Lemma 1.2 in [23] and therefore will be omitted. Part of the proof is showing that the map

$$G_2:\ell^2\ni(x_k)_{k\in\mathbb{N}}\mapsto\mathbb{P}\left[\sum_{k=1}^\infty x_k(2\vartheta_k-1)\in\ \cdot\ \right]\in\mathcal{M}_1(\mathbb{R})$$

is continuous.

Theorem 4.4.4. *Assume* (4.2.1) *and* (4.2.2)*. Then*

$$\phi_n\Rightarrow G(N_\infty)$$

in $\mathcal{M}_1(\mathbb{R})$, where N_∞ is a Poisson point process with intensity $\beta x^{-\beta/2-1}\mathrm{d}x/2$.

In the proof of this result we will use the following lemma.

Lemma 4.4.5 ([23, Remark 3.4])**.** *Let θ_n be a sequence of random probability measures on $\mathbb{R}^2$ defined on the same probability space. Let γ_n and γ_n' denote the marginals of θ_n. Suppose that*

$$\mathrm{E}_{\theta_n}(X-Y)\xrightarrow{\mathrm{P}}0\quad\textit{and}\quad\mathrm{Var}_{\theta_n}(X-Y)\xrightarrow{\mathrm{P}}0,$$

where X an Y are the coordinate variables in $\mathbb{R}^2$. If $\gamma_n\Rightarrow\gamma$, then $\gamma_n'\Rightarrow\gamma$.

Proof of Theorem 4.4.4. First, observe that the sequence of random point measures $N_n = \sum_{k=1}^n \delta_{\xi_k^2 a_n^{-2}}$ converges weakly to N_∞. Indeed, this follows by an appeal to [25, Proposition 3.21] and checking that

$$n\mathrm{P}[\xi^2/a_n^2 \in \cdot\,] \to \mu(\cdot) \quad \text{vaguely on } (0,\infty],$$

where $\mu(\mathrm{d}x) = \beta x^{-\beta/2-1}\mathrm{d}x/2$.

Since G is not continuous, we cannot simply apply the continuous mapping theorem and, similarly as in [23], we are forced to follow a more tedious argument. Define

$$G_\varepsilon : \mathcal{M}_p((0,\infty]) \ni \sum_{k=1}^\infty \delta_{x_k} \mapsto \mathbb{P}\left[\sum_{k=1}^\infty x_k(2\vartheta_k - 1)\,\mathbb{1}_{x_k>\varepsilon} \in \cdot\,\right] \in \mathcal{M}_1(\mathbb{R}).$$

Then for any $\varepsilon > 0$ the map G_ε is continuous on the set $\mathcal{M}_p^\varepsilon := \{\zeta \in \mathcal{M}_p : \zeta(\{\varepsilon,\infty\}) = 0\}$; indeed, take $\zeta_n, \zeta \in \mathcal{M}_p^\varepsilon$ such that $\zeta_n \to \zeta$ vaguely. Then, by [25, Proposition 3.13], since the set $[\varepsilon,\infty]$ is compact in $(0,\infty]$, there exists $p_\varepsilon < \infty$ and an enumeration of points of ζ and ζ_n (for n sufficiently large) such that

$$\zeta_n(\cdot \cap [\varepsilon,\infty]) = \sum_{k=1}^{p_\varepsilon} \delta_{x_k^n}, \quad \zeta(\cdot \cap [\varepsilon,\infty]) = \sum_{k=1}^{p_\varepsilon} \delta_{x_k}$$

and

$$(x_1^n, \dots, x_{p_\varepsilon}^n) \to (x_1, \dots, x_{p_\varepsilon}) \quad \text{as } n \to \infty.$$

Therefore

$$G_\varepsilon(\zeta_n)(\cdot) = \mathbb{P}\left[\sum_{k=1}^{p_\varepsilon} x_k^n(2\vartheta_k - 1) \in \cdot\,\right] \Rightarrow \mathbb{P}\left[\sum_{k=1}^{p_\varepsilon} x_k(2\vartheta_k - 1) \in \cdot\,\right] = G_\varepsilon(\zeta)(\cdot).$$

By [2, Theorem 3.2], to prove that $G(N_n) \Rightarrow G(N_\infty)$ it is enough to show

$$G_\varepsilon(N_n) \Rightarrow_n G_\varepsilon(N_\infty) \qquad \text{for all } \varepsilon > 0, \tag{4.4.3}$$

$$G_\varepsilon(N_\infty) \Rightarrow_\varepsilon G(N_\infty), \tag{4.4.4}$$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \mathrm{P}\left[\rho(G_\varepsilon(N_n), G(N_n)) > \delta\right] = 0 \qquad \text{for all } \delta > 0, \tag{4.4.5}$$

where ρ is the Prokhorov metric on $\mathcal{M}_1(\mathbb{R})$.

First, for any $\varepsilon > 0$, $N_\infty \in \mathcal{M}_p^\varepsilon$ almost surely. Thus (4.4.3) is satisfied by the continuous mapping theorem since G_ε is continuous.

For any sequence $\boldsymbol{x} = (x_k)_{k\in\mathbb{N}} \in \ell^2$ and $\varepsilon > 0$ define $\boldsymbol{x}^\varepsilon \in \ell^2$ by $x_k^\varepsilon = x_k\,\mathbb{1}_{x_k>\varepsilon}$. By the dominated convergence theorem, $\boldsymbol{x}^\varepsilon \to \boldsymbol{x}$ in ℓ^2 as $\varepsilon \to 0$. Hence, since the map G_2 defined in Remark 4.4.3 is continuous, also $G_2(\boldsymbol{x}^\varepsilon) \Rightarrow G_2(\boldsymbol{x})$. This means that for any point process $\zeta = \sum_k \delta_{x_k}$ such that $\boldsymbol{x} \in \ell^2$,

$$G_\varepsilon(\zeta) = G_2(\boldsymbol{x}^\varepsilon) \Rightarrow G_2(\boldsymbol{x}) = G(\zeta),$$

which gives (4.4.4).

Recall that if $\mathcal{L}_X, \mathcal{L}_Y$ are laws of random variables X, Y defined on the same probability space, then $\rho(\mathcal{L}_X, \mathcal{L}_Y)^3 < \mathrm{E}|X-Y|^2$ (c.f. [13, Theorem 11.3.5]). Thus

$$\begin{aligned}\mathrm{P}\left[\rho(G_\varepsilon(N_n), G(N_n)) > \delta\right] &\leqslant \mathrm{P}\left[\mathrm{E}_\omega \left| a_n^{-2} \sum_{k=1}^n \xi_k^2 \, \mathbb{1}_{\xi_k^2 \leqslant \varepsilon a_n^2} (2\vartheta_k - 1)\right|^2 > \delta^3\right] \\ &= \mathrm{P}\left[\mathbb{E}(2\vartheta_1 - 1)^2 a_n^{-4} \sum_{k=1}^n \xi_k^4 \, \mathbb{1}_{\xi_k^2 \leqslant \varepsilon a_n^2} > \delta^3\right],\end{aligned}$$

since $(2\vartheta_k - 1)_k$ is a sequence of mean 0 i.i.d. variables independent of the environment. Denote $C = \mathbb{E}(2\vartheta_1 - 1)^2$, then

$$\begin{aligned}\limsup_{n\to\infty} \mathrm{P}\left[\rho(G_\varepsilon(N_n), G(N_n)) > \delta\right] &\leqslant \limsup_{n\to\infty} \mathrm{P}\left[a_n^{-4} \sum_{k=1}^n \xi_k^4 \, \mathbb{1}_{\xi_k^2 \leqslant \varepsilon a_n^2} > \frac{\delta^3}{C}\right] \\ &\leqslant \frac{C}{\delta^3} \limsup_{n\to\infty} a_n^{-4} n \mathrm{E}\left[\xi^4 \, \mathbb{1}_{\xi \leqslant \varepsilon^{1/2} a_n}\right].\end{aligned}$$

As we have seen in the proof of Proposition 4.4.1, the expected value present above is dominated by an expression asymptotically equivalent to $4\varepsilon^{2-\beta/2} n^{-1} a_n^4$, thus

$$\limsup_{n\to\infty} \mathrm{P}\left[\rho(G_\varepsilon(N_n), G(N_n)) > \delta\right] \leqslant \frac{4C}{\delta^3} \varepsilon^{2-\beta/2},$$

which proves (4.4.5).

Therefore $G(N_n) \Rightarrow G(N_\infty)$. Now the claim of the theorem follows from Proposition 4.4.1 and Lemmas 4.4.5 and 4.3.3. □

4.4.2 Moderately sparse random environment

Proof of Theorem 4.2.1. Let $\mu_{n,\omega}$ denote the quenched law of $(T_n - \mathrm{E}_\omega T_n)/a_n^2$.

Since $\mathrm{E}\xi < \infty$, $\chi = (\mathrm{E}\xi)^{-1}$ is well defined. Let $N_\infty = \sum_n \delta_{x_n}$ be a Poisson point process as in Theorem 4.4.4 and let $N_\infty^\chi = \sum_n \delta_{\chi^{2/\beta} x_n}$. Then N_∞^χ is a Poisson point process with intensity $\beta\chi x^{-\beta/2-1}\mathrm{d}x/2$. Putting

$$\begin{aligned}\phi_n^\chi(\omega)(\cdot) = \phi_{n,\omega}^\chi(\cdot) &= \mathrm{P}_\omega\left[a_n^{-2}(T_{S_{\chi n}} - \mathrm{E}_\omega T_{S_{\chi n}}) \in \cdot\,\right] \\ &= \mathrm{P}_\omega\left[(a_{\chi n}/a_n)^2 a_{\chi n}^{-2}(T_{S_{\chi n}} - \mathrm{E}_\omega T_{S_{\chi n}}) \in \cdot\,\right],\end{aligned}$$

where $S_{\chi n} := S_{\lfloor \chi n \rfloor}$, it follows from Lemma 4.4.5, Theorem 4.4.4, and the convergence $a_{\chi n}/a_n \to \chi^{1/\beta}$ that $\phi_n^\chi \Rightarrow G(N_\infty^\chi)$.

It remains to show that

$$a_n^{-4}\mathrm{Var}_\omega\left[(T_{S_{\chi n}} - \mathrm{E}_\omega T_{S_{\chi n}}) - (T_n - \mathrm{E}_\omega T_n)\right] = a_n^{-4}\mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_n\right] \xrightarrow{\mathrm{P}} 0,$$

from which it follows, by Lemma 4.4.5, that $\mu_n \Rightarrow G(N_\infty^\chi)$.

Observe that on the event $\{n \leqslant S_{\chi n}\}$, for any k such that $S_k \leqslant n$,

$$\begin{aligned}\mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_n\right] &= \sum_{j=n+1}^{S_{\chi n}} \mathrm{Var}_\omega\left[T_j - T_{j-1}\right] \leqslant \sum_{j=S_k+1}^{S_{\chi n}} \mathrm{Var}_\omega\left[T_j - T_{j-1}\right] \\ &= \mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_{S_k}\right]\end{aligned}$$

and similarly on $\{S_{\chi n} \leqslant n\}$ for any k such that $S_k \geqslant n$,

$$\mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_n\right] \leqslant \mathrm{Var}_\omega\left[T_{S_k} - T_{S_{\chi n}}\right].$$

Therefore for any $\delta > 0$ and $\varepsilon > 0$,

$$\begin{aligned}\mathrm{P}\left[a_n^{-4}\mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_n\right] > \delta\right] &\leqslant \mathrm{P}\left[|S_{\chi n} - n| > \varepsilon n\right] \\ &\quad + \mathrm{P}\left[a_n^{-4}\mathrm{Var}_\omega\left[T_{S_{\lfloor \chi n\rfloor + \lfloor \varepsilon n\rfloor}} - T_{S_{\chi n}}\right] > \delta\right] \\ &\quad + \mathrm{P}\left[a_n^{-4}\mathrm{Var}_\omega\left[T_{S_{\chi n}} - T_{S_{\lfloor \chi n\rfloor - \lfloor \varepsilon n\rfloor}}\right] > \delta\right] \\ &= \mathrm{P}\left[\left|\frac{S_{\chi n}}{\chi n} - \frac{1}{\chi}\right| > \frac{\varepsilon}{\chi}\right] + 2\mathrm{P}\left[a_n^{-4}\mathrm{Var}_\omega\left[T_{S_{\varepsilon n}}\right] > \delta\right].\end{aligned}$$

The first term tends to 0 by the law of large numbers (recall $1/\chi = \mathrm{E}\xi$). To estimate the second, note that by Schwartz inequality,

$$\mathrm{Var}_\omega\left[T_{S_{\varepsilon n}}\right] = \mathrm{Var}_\omega\left[T^l_{S_{\varepsilon n}} + T^r_{S_{\varepsilon n}}\right] \leqslant 2\mathrm{Var}_\omega\left[T^l_{S_{\varepsilon n}}\right] + 2\mathrm{Var}_\omega\left[T^r_{S_{\varepsilon n}}\right].$$

By (4.3.2),

$$\mathrm{Var}_\omega\left[T^r_{S_{\varepsilon n}}\right] = \sum_{k=1}^{\varepsilon n} \mathrm{Var}_\omega \mathbb{T}^r_k = \sum_{k=1}^{\varepsilon n} \frac{2}{3}(\xi_k^4 - \xi_k^2) \leqslant \sum_{k=1}^{\varepsilon n} \xi_k^4,$$

and furthermore $a_n^{-4}\sum_{k=1}^{\varepsilon n}\xi_k^4 \Rightarrow \varepsilon^{-4/\beta}L_{\beta/4}$ with respect to P, while by Lemma 4.3.3 we have $a_n^{-4}\mathrm{Var}_\omega\left[T^l_{S_{\varepsilon n}}\right] \xrightarrow{\mathrm{P}} 0$. Therefore

$$\limsup_{n\to\infty} \mathrm{P}\left[a_n^{-4}\mathrm{Var}_\omega\left[T_{S_{\varepsilon n}}\right] > \delta\right] \leqslant \mathrm{P}\left[L_{\beta/4} > \frac{\delta}{2\varepsilon^{4/\beta}}\right].$$

The last expression can be made arbitrary small by taking sufficiently small ε. □

4.4.3 Strong sparsity: preliminaries

From now on we assume that $\mathrm{E}\xi = \infty$. This case is technically more involved, however the underlying principle remains the same. Denote the first passage time of S via

$$\nu_n = \inf\{k > 0 \,:\, S_k > n\}.$$

Recall that we write

$$m_n = n\mathrm{E}\left[\xi\, \mathbb{1}_{\xi \leqslant a_n}\right]$$

and we denote by $(c_n)_{n\in\mathbb{N}}$ the asymptotic inverse of $(m_n)_{n\in\mathbb{N}}$, i.e. any increasing sequence of real numbers such that

$$\lim_{n\to\infty} c_{m_n}/n = \lim_{n\to\infty} m_{c_n}/n = 1.$$

Let

$$d_n = 1/\mathrm{P}[\xi > n].$$

Lemma 4.4.6. *Assume* (4.2.1). *Under the introduced notation $a_{d_n}/n \to 1$.*

Proof. Since the sequence $(a_n)_{n\in\mathbb{N}}$ is asymptotically unique, we can take

$$a_n = \inf\{x \,:\, \mathrm{P}[\xi > x] \leqslant 1/n\}.$$

Then

$$a_{d_n} = \inf\{x \,:\, \mathrm{P}[\xi > x] \leqslant \mathrm{P}[\xi > n]\}.$$

In particular $n \geqslant a_{d_n}$. By the merit of regular variation of $\mathrm{P}[\xi > x]$ we have that for any $\varepsilon > 0$, $\mathrm{P}[\xi > (1-\varepsilon)n] > \mathrm{P}[\xi > n]$ for sufficiently large n. This secures $a_{d_n} \geqslant (1-\varepsilon)n$ for sufficiently large n and thus concludes the proof since $\varepsilon > 0$ is arbitrarily small. □

From the above lemma, by regular variation of a_n, we have that $a_{Cd_n} \sim C^{1/\beta} n$ for any constant $C > 0$.

As one may expect, S_n grows at a scale m_n and thus ν_n must grow at a scale c_n (in the sense of a limit theorem which we will soon make precise). For our purposes we need to justify that S_n/m_n and ν_n/c_n converge jointly with some other characteristics of the trajectory of S. For this reason we will need to use the setting of càdlàg functions. Recall that $\mathbb{D}^{\uparrow}$ stands for the space of non-decreasing right continuous functions that have a left limit at each point. For $h \in \mathbb{D}^{\uparrow}$ we define $h^{\leftarrow} \in \mathbb{D}$ via

$$h^{\leftarrow}(t) = \inf\{s \,:\, h(s) > t\}.$$

Consider $\mathcal{M} = \mathcal{M}_p((0,\infty] \times [0,\infty))$. Let $M \colon \mathbb{D}^{\uparrow} \to \mathcal{M}$ be given via

$$M(h) = \sum_k \delta_{x_k} \otimes \delta_{t_k},$$

where for $h \in \mathbb{D}^{\uparrow}$, $\{t_k\}_{k\in\mathbb{N}}$ are the discontinuity points of h and $x_k = h(t_k) - h(t_k^-)$ is the size of the jump at t_k.

Lemma 4.4.7. *The function $M \colon \mathbb{D}^{\uparrow} \to \mathcal{M}$ is continuous with respect to J_1 topology.*

Proof. Let $f_n, f \in \mathbb{D}^{\uparrow}$ be such that $f_n \to f$ in J_1 topology. For any nonnegative, continuous function $\varphi \colon (0,+\infty] \times [0,+\infty) \to \mathbb{R}$ with compact support we can find $\varepsilon > 0$ and $T > 0$ such that $\varphi(x,t) = 0$ if $x \leqslant \varepsilon$ or $t \geqslant T$. Since $f \in \mathbb{D}^{\uparrow}$, it has only finitely many jumps on the interval $[0,T]$ that are greater than ε, therefore

$$\int \varphi(x,t)\, Mf(\mathrm{d}x,\mathrm{d}t) = \sum_{k=1}^{N} \varphi(x_k,t_k)$$

for some N, $t_1 < \cdots < t_N < T$ and $x_k > \varepsilon$.

By the definition of J_1 topology, there exists a sequence of continuous increasing functions $\lambda_n : (0,\infty) \to (0,\infty)$ such that

$$\sup_{t\in[0,T]} |\lambda_n(t) - t| \to 0, \qquad \sup_{t\in[0,T]} |f_n(t) - f(\lambda_n(t))| \to 0. \tag{4.4.6}$$

For n sufficiently large, $\sup_{t\in[0,T]} |\lambda_n(t) - t| < T - t_N$, which means that $f \circ \lambda_n$ has exactly N jumps on the interval $[0,T)$, at times $\lambda_n^{-1}(t_k)$. Moreover, for large enough n, $\sup_{t\in[0,T]} |f_n(t) - f(\lambda_n(t))| < \varepsilon/3$, from which it follows that f_n cannot have jumps bigger than ε apart from the discontinuity points of $f \circ \lambda_n$.

Fix $k \in \{1, \dots N\}$. It follows from (4.4.6) that for n large enough f_n does have a jump at $\lambda_n^{-1}(t_k)$, denote it by x_k^n, and observe that $x_k^n \to x_k$ as $n \to \infty$; in particular $x_k^n > \varepsilon$ for large n. It also follows that $\lambda_n^{-1}(t_k) \to t_k$ as $n \to \infty$. This means that for n sufficiently large

$$\int \varphi(x,t)\, Mf_n(\mathrm{d}x, \mathrm{d}t) = \sum_{k=1}^{N} \varphi(x_k^n, \lambda_n^{-1}(t_k))$$

and the last expression tends to $\int \varphi(x,t)\, Mf(\mathrm{d}x,\mathrm{d}t)$ as $n \to \infty$, which gives $Mf_n \to Mf$. □

Consider a random element of $\mathcal{M}$ given by

$$\Lambda_n = \sum_{j=1}^{\infty} \delta_{\xi_j/a_n} \otimes \delta_{j/n}$$

and random elements of $\mathbb{D}^{\uparrow}$ defined via

$$L_n(t) = S_{\lfloor nt \rfloor}/a_n \text{ for } \beta < 1 \quad \text{and} \qquad \widetilde{L}_n(t) = S_{\lfloor nt \rfloor}/m_n \text{ for } \beta = 1. \tag{4.4.7}$$

Recall $\Upsilon \colon \mathbb{D}^{\uparrow} \to \mathbb{R}$ defined in (4.2.5).

Lemma 4.4.8. *If $\beta < 1$, then*

$$\left(L_n, \Lambda_n, \frac{\nu_n}{d_n}, \frac{S_{\nu_n - 1}}{n} \right) \Rightarrow (L, M(L), L^{\leftarrow}(1), \Upsilon(L)) \tag{4.4.8}$$

in $(\mathbb{D}, J_1) \times \mathcal{M} \times \mathbb{R} \times \mathbb{R}$, where $L = (L_t)_{t \geqslant 0}$ is a strictly increasing β-stable subordinator with Lévy measure given by $\nu(x, +\infty) = x^{-\beta}$.

If $\beta = 1$, then

$$\left(\widetilde{L}_n, \frac{\nu_n}{c_n}, \frac{S_{\nu_n - 1}}{n} \right) \Rightarrow (\mathrm{id}, 1, 1) \tag{4.4.9}$$

in $(\mathbb{D}, J_1) \times \mathbb{R} \times \mathbb{R}$, where $\mathrm{id} \colon \mathbb{R}_+ \to \mathbb{R}_+$ is the identity function.

Proof. Consider first $\beta < 1$. By an appeal to standard functional weak convergence to stable Lévy motion [26, Corollary 7.1],

$$L_n \Rightarrow L \quad \text{in } (\mathbb{D}, J_1).$$

Note that

$$\Lambda_n = M(L_n)$$

and the function M is J_1-continuous by Lemma 4.4.7. Moreover,

$$\frac{\nu_n}{d_n} = L^{\leftarrow}_{d_n}\left(\frac{n}{a_{d_n}}\right)$$

and the map $h \mapsto h^{\leftarrow}$ is continuous in M_1 topology by [30]. In what follows, we will use notation introduced in [29]. For $h \in \mathbb{D}$ let h^- be the lcrl (left-continuous, having right-hands limits) version of h, that is, $h^-(t) = \lim_{\varepsilon\to 0^+} h(t-\varepsilon)$ and $h^-(0) = 0$. Similarly, let h^+ denote rcll version of a lcrl path. Let $\Phi : \mathbb{D}^{\uparrow} \to \mathbb{D}$ be given by

$$\Phi(h) = (h^- \circ (h^{\leftarrow})^-)^+.$$

Finally, observe that for any $k \in \mathbb{N}$, $\Phi(L_{d_n})$ on the set $[S_k/a_{d_n}, S_{k+1}/a_{d_n})$ is constant and equal to S_k/a_{d_n}, therefore

$$\frac{S_{\nu_n-1}}{a_{d_n}} = \Phi\left(L_{d_n}\right)\left(\frac{n}{a_{d_n}}\right).$$

By [29], Φ is J_1-continuous on $\mathbb{D}^{\uparrow\uparrow} \subset \mathbb{D}$, the set of strictly increasing, unbounded functions. Since $L \in \mathbb{D}^{\uparrow\uparrow}$ almost surely, by the continuous mapping theorem we have joint convergence in distribution

$$(L_n, M(L_n), L_n^{\leftarrow}, \Phi(L_n)) \to (L, M(L), L^{\leftarrow}, \Phi(L))$$

in $(\mathbb{D}, J_1) \times \mathcal{M}_p((0,\infty] \times [0,\infty)) \times (\mathbb{D}, M_1) \times (\mathbb{D}, J_1)$. By Skorokhod's representation theorem we may assume that the above convergence holds almost surely.

Since the limiting processes admit no fixed discontinuities, Proposition 2.4 in [29] gives

$$\frac{\nu_n}{d_n} \to L^{\leftarrow}(1) \quad \text{and} \quad \frac{S_{\nu_n-1}}{a_{d_n}} \to \Phi(L)(1) = \Upsilon(L)$$

almost surely.

The case $\beta = 1$ is similar and follows from the fact that by [26, Corollary 7.1] and properties of J_1 topology,

$$\widetilde{L}_n \Rightarrow \mathrm{id} \quad \text{in } (\mathbb{D}, J_1).$$

One can combine this with

$$\frac{\nu_n}{c_n} = \widetilde{L}^{\leftarrow}_{c_n}\left(\frac{n}{m_{c_n}}\right), \qquad \frac{S_{\nu_n-1}}{m_{c_n}} = \Phi(\widetilde{L}_{m_n})\left(\frac{n}{m_{c_n}}\right),$$

and the arguments presented in the case $\beta < 1$ to get the desired claim. □

Remark 4.4.9. Observe that all information on the sequence $(\xi_k)_k$ is carried by the process Λ_n and therefore by L_n or, equivalently, $\widetilde{L}_n$. We may thus assume that our space holds random variables $U_n^{(k)}, \vartheta_k$ as described in Section 4.4.1 and at the same time the convergence given in Lemma 4.4.8 holds almost surely.

Lemma 4.4.10. *Assume that* (4.2.1) *holds true. If* $\beta < 1$, *then*

$$n^{-4}\mathrm{Var}_\omega\left[T^r_{S_{\nu_n-1}} - \mathrm{E}_\omega T^r_{S_{\nu_n-1}} - \sum_{k=1}^{\nu_n-1}\xi_k^2(2\vartheta_k-1)\right] \overset{\mathrm{P}}{\to} 0.$$

If $\beta = 1$ *and* $\mathrm{E}\xi = \infty$, *then*

$$a_{c_n}^{-4}\mathrm{Var}_\omega\left[T^r_{S_{\nu_n-1}} - \mathrm{E}_\omega T^r_{S_{\nu_n-1}} - \sum_{k=1}^{\nu_n-1}\xi_k^2(2\vartheta_k-1)\right] \overset{\mathrm{P}}{\to} 0.$$

Proof. One can use the same arguments as in the proof of Proposition 4.4.1. First consider $\beta \in (0,1)$. By tightness of $\{\nu_n/d_n\}_{n\in\mathbb{N}}$ we can choose $C > 0$ to make the probability $\mathrm{P}[\nu_n > Cd_n]$ arbitrarily small. Next, on the event $\{\nu_n \leqslant Cd_n\}$,

$$\mathrm{Var}_\omega\left[T^r_{S_{\nu_n-1}} - \mathrm{E}_\omega T^r_{S_{\nu_n-1}} - \sum_{k=1}^{\nu_n-1}\xi_k^2(2\vartheta_k-1)\right] \leqslant \mathrm{Var}_\omega\left[\sum_{k=1}^{Cd_n}\left(U_{\xi_k} - 2\xi_k^2\vartheta_k\right)\right].$$

From here, since $a_{Cd_n} \sim C^{1/\beta}n$, one argues as in the proof of Proposition 4.4.1 to show that

$$\sum_{k\in I_n^1}\frac{\xi_k^4}{n^4}\mathrm{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] \overset{\mathrm{P}}{\to} 0 \qquad \text{and} \qquad \sum_{k\in I_n^2}\frac{\xi_k^4}{n^4}\mathrm{Var}_\omega\left[\frac{U_{\xi_k}}{\xi_k^2} - 2\vartheta_k\right] \overset{\mathrm{P}}{\to} 0,$$

where $I_n^1 = \{k \leqslant Cd_n : \xi_k > \varepsilon n\}$, $I_n^2 = \{k \leqslant Cd_n : \xi_k \leqslant \varepsilon n\}$ with fixed $\varepsilon > 0$. In the case $\beta = 1$ and $\mathrm{E}\xi = \infty$ one can invoke the same arguments combined with the tightness of $\{\nu_n/c_n\}_{n\in\mathbb{N}}$. □

Lemma 4.4.11. *Assume that* (4.2.1) *holds true. If* $\beta \in (0,1)$, *then*

$$n^{-4}\mathrm{Var}_\omega T^l_{S_{\nu_n}} \overset{\mathrm{P}}{\to} 0.$$

If $\beta = 1$ *and* $\mathbb{E}\xi = \infty$, *then*

$$a_{c_n}^{-4}\mathrm{Var}_\omega T^l_{S_{\nu_n}} \overset{\mathrm{P}}{\to} 0.$$

Proof. Consider the case $\beta < 1$. Take any $C > 0$ and write

$$\mathrm{P}\left[n^{-4}\mathrm{Var}_\omega T^l_{S_{\nu_n}} \geqslant \varepsilon\right] \leqslant \mathrm{P}\left[\nu_n \geqslant Cd_n\right] + \mathrm{P}\left[\mathrm{Var}_\omega T^l_{S_{[Cd_n]}} \geqslant \varepsilon n^4\right].$$

Since $a_{Cd_n} \sim C^{1/\beta}n$, an appeal to Lemma 4.3.3 shows that the second term tends to 0 as $n \to \infty$. The first term can be made arbitrarily small by taking $C > 0$ sufficiently large. In the case $\beta = 1$ we can use an analogous argument with d_n replaced with c_n. □

For the purpose of the next lemma. let $(\{U_n^0\}_{n\in\mathbb{N}}, \vartheta_0)$ be, as before, a copy of $(\{U_n\}_{n\in\mathbb{N}}, \vartheta)$ given by the claim of Lemma 4.3.1 independent of the environment.

Lemma 4.4.12. *Assume that* (4.2.1) *and* (4.2.2) *hold true for* $\beta \leqslant 1$ *and* $\mathrm{E}\xi = \infty$. *Then*

$$\frac{U^0_{n-S_{\nu_n-1}} - \mathrm{E}_\omega U^0_{n-S_{\nu_n-1}}}{n^2} - (1-\Xi)^2(2\vartheta_0 - 1) \overset{\mathrm{P}}{\to} 0,$$

where $\Xi = \Upsilon(L)$ *for* $\beta < 1$ *and* $\Xi = 1$ *for* $\beta = 1$.

Proof. By the merit of Remark 4.4.9, $S_{\nu_n-1}/n \to \Xi$, P-almost surely. Secondly, by a standard application of the key renewal theorem [14, Theorem 2.6.12], the condition $\mathbb{E}\xi = \infty$ implies that $n - S_{\nu_n-1} \xrightarrow{\mathrm{P}} \infty$. The claim of the lemma follows from the fact that

$$\frac{U^0_{n-S_{\nu_n-1}} - \mathrm{E}_\omega U^0_{n-S_{\nu_n-1}}}{n^2} - (1-\Xi)^2(2\vartheta_0 - 1) = \\ -\left(1 - \frac{S_{\nu_n-1}}{n}\right)^2 + (1-\Xi)^2 + \left(1 - \frac{S_{\nu_n-1}}{n}\right)^2 \frac{U^0_{n-S_{\nu_n-1}}}{(n-S_{\nu_n-1})^2} - (1-\Xi)^2 2\vartheta_0$$

and Lemma 4.3.1. □

4.4.4 Strong sparsity: $\beta = 1$

We will now focus on the case when $\beta = 1$ and $\mathrm{E}\xi = \infty$. By Lemmas 4.4.5, 4.4.10, 4.4.11, and 4.4.12, it is sufficient to study the quenched behaviour of $\sum_{k=1}^{\nu_n-1} \xi_k^2(2\vartheta_k - 1)$.

Proof of Theorem 4.2.2. Fix $\varepsilon > 0$. On the set $\{|\nu_n - c_n| \leqslant \varepsilon c_n\}$,

$$\left(\sum_{k=c_n+1}^{\nu_n} \xi_k^2(2\vartheta_k - 1)\right)^2 \leqslant \max_{m:|m-c_n|<\varepsilon c_n} \left(\sum_{k=c_n+1}^{m} \xi_k^2(2\vartheta_k - 1)\right)^2 =^{st} \max_{m<\varepsilon c_n} \left(\sum_{k=1}^{m} \xi_k^2(2\vartheta_k - 1)\right)^2$$

and by Doob's maximal inequality,

$$\mathrm{P}_\omega\left[\max_{m<\varepsilon c_n}\left(\sum_{k=1}^{m} \xi_k^2(2\vartheta_k - 1)\right)^2 > \delta\right] \leqslant \delta^{-1}\mathrm{E}_\omega\left(\sum_{k=1}^{\varepsilon c_n} \xi_k^2(2\vartheta_k - 1)\right)^2 = \delta^{-1}\mathbb{E}(2\vartheta - 1)^2 \sum_{k=1}^{\varepsilon c_n} \xi_k^4.$$

Observe that

$$a_{c_n}^{-4}\sum_{k=1}^{\varepsilon c_n} \xi_k^4 = \varepsilon^4(1+o(1))\sum_{k=1}^{\varepsilon c_n} \xi_k^4/a^4_{\varepsilon c_n}.$$

Since the sequence on the right hand side is tight in n, it follows that

$$a_{c_n}^{-4}\mathrm{E}_\omega\left(\sum_{k=c_n+1}^{\nu_n} \xi_k^2(2\vartheta_k - 1)\right)^2 \xrightarrow{\mathrm{P}} 0.$$

In a similar fashion,

$$a_{c_n}^{-4}\mathrm{E}_\omega\left(\sum_{k=\nu_n+1}^{c_n} \xi_k^2(2\vartheta_k - 1)\right)^2 \xrightarrow{\mathrm{P}} 0.$$

Therefore the weak limit of the quenched law of $(T_n - \mathrm{E}_\omega T_n)/a_{c_n}^2$ will coincide with the limit of

$$\mathrm{P}_\omega\left[\sum_{k=1}^{c_n} \xi_k^2(2\vartheta_k - 1)/a_{c_n}^2 \in \cdot\right].$$

The weak limit of the latter is $G(N)$, which follows from the proof of Theorem 4.2.1. □

4.4.5 Strong sparsity: $\beta < 1$

Proof of Theorem 4.2.3. Let $\mu_{n,\omega}$ denote the quenched law of $(T_n - \mathrm{E}_\omega T_n)/n^2$. Then

$$\mu_{n,\omega}(\cdot) = \mathrm{P}_\omega\left[\frac{T_n - T_{S_{\nu_n-1}} - \mathrm{E}_\omega[T_n - T_{S_{\nu_n-1}}]}{n^2} + \frac{T_{S_{\nu_n-1}} - \mathrm{E}_\omega[T_{S_{\nu_n-1}}]}{n^2} \in \cdot\right]$$

To treat the second term under the probability we can, similarly as previously, decouple the times that the random walker spends between consecutive S_k's for $k \leqslant n$. The first part will be controlled with the help of Lemma 4.4.12. Let $(\{U_n^0\}, \vartheta_0)$ be, as before, a copy of $(\{U_n\}, \vartheta)$ given by the claim of Lemma 4.3.1 independent of the environment. Then $U^0_{n-S_{\nu_n-1}}$ has, under P_ω, the same distribution as the time the walk spends in $[S_{\nu_n-1}, n)$ after reaching S_{ν_n-1} and before reaching n. By Lemma 4.4.11 and Lemma 4.4.5 the weak limit of $\mu_{n,\omega}$ is the same as that of

$$\bar{\mu}_{n,\omega}(\cdot) = \mathrm{P}_\omega\left[\frac{U^0_{n-S_{\nu_n-1}} - \mathrm{E}_\omega U^0_{n-S_{\nu_n-1}}}{n^2} + \frac{T^r_{S_{\nu_n-1}} - \mathrm{E}_\omega[T^r_{S_{\nu_n-1}}]}{n^2} \in \cdot\right].$$

Recall the random functions L_n given in (4.4.7) and that for a càdlàg function h we denote by $(x_k(h), t_k(h))_k$ an arbitrary enumeration of its discontinuities, i.e. $x_k(h) = h(t_k) - h(t_k^-) > 0$, where $t_k(h) = t_k$. Note that, with Υ given in (4.2.5), one has by the merit of Lemmas 4.4.5, 4.4.10, 4.4.11, and 4.4.12 that the limit of $\bar{\mu}_{n,\omega}$ will coincide with the limit of

$$F^n(\cdot) = \mathrm{P}_\omega\left[\frac{a_{d_n}^2}{n^2}(1 - \Upsilon(L_{d_n}))^2(2\vartheta_0 - 1) + \frac{a_{d_n}^2}{n^2}\sum_k x_k(L_{d_n})^2(2\vartheta_k - 1)\,\mathbb{1}_{L_{d_n}(t_k) < n/a_{d_n}} \in \cdot\right].$$

It is enough to show that $F^n \Rightarrow F(L)$. To achieve that one uses the same approach as in the proof of Theorem 4.4.4. Namely by considering, for $\varepsilon > 0$,

$$F^n_\varepsilon(\cdot) = \mathrm{P}_\omega\Bigg[\frac{a_{d_n}^2}{n^2}(1 - \Upsilon(L_{d_n}))^2(2\vartheta_0 - 1) + \frac{a_{d_n}^2}{n^2}\sum_k x_k(L_{d_n})^2(2\vartheta_k - 1)\,\mathbb{1}_{x_k(L_{d_n}) > \varepsilon}\,\mathbb{1}_{L_{d_n}(t_k) < n/a_{d_n}} \in \cdot\Bigg].$$

For fixed $\varepsilon > 0$, $F^n_\varepsilon \to F^\infty_\varepsilon$, where

$$F^\infty_\varepsilon(\cdot) = \mathrm{P}_\omega\left[(1 - \Upsilon(L))^2(2\vartheta_0 - 1) + \sum_k x_k(L)^2(2\vartheta_k - 1)\,\mathbb{1}_{x_k(L) > \varepsilon}\,\mathbb{1}_{L(t_k) \leqslant 1} \in \cdot\right]$$

since associated point processes converge and $a_{d_n}/n \to 1$. Then we show that $F^\infty_\varepsilon \Rightarrow F(L)$ as $\varepsilon \to 0$. We finally prove that (4.4.5) also holds in this context, since

$$\limsup_{n\to\infty} \mathrm{P}\left[\rho(F^n_\varepsilon, F^n) > \delta\right] \leqslant \limsup_{n\to\infty} \mathrm{P}\left[\sum_k x_k(L_{d_n})\,\mathbb{1}_{L_{d_n}(t_k) < n/a_{d_n}} > \frac{\delta^3 n^4}{C\varepsilon^3 a_{d_n}^4}\right]$$

and the last expression tends to 0 as $\varepsilon \to 0$, because $n/a_{d_n} \to 1$ and $L_{d_n} \to L$ a.s. in J_1.

□

4.5 Absence of a strong limit

4.5.1 An auxiliary process $\overline{X}$ and scheme of the proof

Our aim now is to prove Theorem 4.2.4 stating that the strong limit in distribution does not exist, that is for κ_n given in (4.2.6) and for P-a.e. ω there is no random variable Y_ω such that

$$\frac{T_n - \mathrm{E}_\omega T_n}{\kappa_n} \Rightarrow Y_\omega, \qquad n \to \infty. \tag{4.5.1}$$

Before going into the details of the proof, let us explain its scheme. We will prove that there is a subset $\Omega_0 \subset \Omega$ of measure 1 such that for every $\omega \in \Omega_0$ one can find an infinite subsequence of integers $(k_m)_{m\in\mathbb{N}}$ (depending on ω) for which the values of ξ_{k_m+1} are exceptionally large. The time $T_{S_{k_m+1}} - T_{S_{k_m}}$ that the walk needs to move from S_{k_m} to S_{k_m+1} is then either much bigger or comparable with $T_{S_{k_m}}$ and must affect the limit Y_ω. As a consequence, the random variable Y_ω satisfies distributional equations which do not have any nontrivial solutions (see (4.5.19) and (4.5.20) below); this leads to the absence of the strong quenched limit.

Although the general idea is relatively easy to explain, since we have to deal with a.e. ω, the details are quite tedious. We start below with a general construction and then pass to a detailed proof for the case $\beta < 1$, keeping general notation for as long as possible. Finally we will study the other case.

For technical reasons, instead of the process X we need to consider a slightly different process $\overline{X} = (\overline{X}_k)_{k\in\mathbb{N}}$, whose trajectory contains independent pieces. We start by constructing a favourable environment of probability one. For this purpose consider two increasing sequences $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ diverging to $+\infty$ and satisfying

$$2p_n < q_n < p_{n+1}/2, \qquad p_n/q_n \to 0, \qquad \text{and} \qquad \frac{a_{q_n}}{a_{2p_n}} \geqslant n^\theta \tag{4.5.2}$$

for some $\theta > 1/\beta$. Notice that one may take e.g. $p_n = 2^{2^n}$, $q_n = p_{n+1}/4$.

The trajectory of the random walk X cannot be divided into independent pieces with respect to P, because the process can have large excursions to the left and the environment is not homogeneous. To remedy that we will censor the left excursions of X that become too large. We introduce a new process $\overline{X} = (\overline{X}_k)_{k\in\mathbb{N}}$. This process essentially behaves as the previous one and evolves in the same environment, with a small difference. Namely after $\overline{X}$ reaches S_{q_n} and before it reaches S_{2q_n}, we put a barrier at point S_{p_n}, i.e. the process cannot come back below S_{p_n}. However, this barrier is removed when $\overline{X}$ hits S_{2q_n}. Of course we can couple both processes on the same probability space by removing from X all left excursions from S_{p_n} that occur after hitting S_{q_n} and before reaching S_{2q_n}.

For any k, we define the random variables $\overline{T}_k$, $\overline{\mathbb{T}}_k$, $\overline{\mathbb{T}}^r_k$, $\overline{\mathbb{T}}^l_k$ in an obvious way, e.g.

$$\overline{T}_k = \inf\{j : \overline{X}_j = k\}, \quad \overline{\mathbb{T}}_k = \overline{T}_{S_k} - \overline{T}_{S_{k-1}}.$$

Then $\overline{\mathbb{T}}^r_k = \mathbb{T}^r_k$ for every k and $\overline{\mathbb{T}}^l_k = \mathbb{T}^l_k$ for $k \notin \bigcup_n (q_n, 2q_n]$. Note that for $k \in (q_n, 2q_n]$, $\mathbb{T}_k - \overline{\mathbb{T}}_k$ is the time that the process X spends below S_{p_n} after hitting S_{k-1} and before reaching S_k. The next lemma ensures that asymptotic properties of the processes X and $\overline{X}$ are comparable.

Lemma 4.5.1. *For any $\varepsilon \in (0,1)$ and* P*-a.e.* ω *there is* $N = N(\omega,\varepsilon)$ *such that*

$$\sum_{q_n < k \leqslant 2q_n} \mathrm{E}_\omega(\mathbb{T}_k - \overline{\mathbb{T}}_k) < \varepsilon^n \tag{4.5.3}$$

for $n > N$*. Moreover*

$$\mathbb{T}_n = \overline{\mathbb{T}}_n \text{ a.s. for large (random) } n.$$

Proof. Fix $k \in (q_n, 2q_n]$. To describe the quenched mean of $\mathbb{T}_k - \overline{\mathbb{T}}_k = \mathbb{T}_k^l - \overline{\mathbb{T}}_k^l$ we need to calculate the time the trajectory X, after it hits S_{k-1}, but before reaching S_k, spends below S_{p_n}. For this purpose we proceed as in the proof of Lemma 2.3.3, that is we decompose

$$\mathbb{T}_k - \overline{\mathbb{T}}_k = \sum_{m=1}^{M_k} \sum_{j=0}^{N_m} F_{p_n}(j,m), \tag{4.5.4}$$

where M_k denotes the number of times the walk visits S_{p_n} from the right in the time interval $(T_{S_{k-1}}, T_{S_k})$, N_m is the number of consecutive left excursions from S_{p_n} after hitting it from the right, and $F_{p_n}(j,m)$ is the length of the corresponding excursion. Note that N_m is geometrically distributed with mean ρ_{p_n} and variance $\rho_{p_n}(1+\rho_{p_n})$. Thus, by formula (2.3.14),

$$\mathrm{E}_\omega\Big[\sum_{j=0}^{N_m} F_{p_n}(j,m)\Big] = \rho_{p_n}\mathrm{E}_\omega F_{p_n} = 2W_{p_n} \tag{4.5.5}$$

Next, observe that for any $m > 0$, $\mathrm{P}_\omega[M_k = m] = rs^{m-1}(1-s)$, where

$$r = \mathrm{P}_\omega^{S_{k-1}}\left[T_{S_{p_n}} < T_{S_k}\right]$$

and, invoking once again the gambler's ruin problem,

$$s = \mathrm{P}_\omega^{S_{p_n}+1}\left[T_{S_{p_n}} < T_{S_k}\right] = 1 - \frac{1}{\xi_{p_n+1}}\mathrm{P}_\omega^{S_{p_n+1}}\left[T_{S_{p_n}} > T_{S_k}\right].$$

We may easily calculate the mean of M_k and use the formulae (2.2.5) to express it in terms of the environment. We get, after simplifying,

$$\mathrm{E}_\omega M_k = \frac{r}{1-s} = \xi_k \Pi_{p_n+1,k-1}, \tag{4.5.6}$$

Therefore, by (2.2.1), (4.5.5) and (4.5.6),

$$\mathrm{E}_\omega\left[\mathbb{T}_k - \overline{\mathbb{T}}_k\right] = 2\xi_k \Pi_{p_n+1,k-1} W_{p_n}. \tag{4.5.7}$$

Now we are ready to prove (4.5.3). We have

$$\mathrm{P}\Big[\sum_{q_n<k\leqslant 2q_n} \mathrm{E}_\omega(\mathbb{T}_k - \overline{\mathbb{T}}_k) \geqslant \varepsilon^n\Big] \leqslant \varepsilon^{-\gamma n} \sum_{q_n<k\leqslant 2q_n} \mathrm{E}\big[2\xi_k\Pi_{p_n+1,k-1}W_{p_n}\big]^\gamma \leqslant C\varepsilon^{-\gamma n}(\mathrm{E}\rho^\gamma)^{q_n-p_n},$$

where $\gamma \in (0,1)$ is as in (4.2.2). Then, by the Borel-Cantelli lemma,

$$\mathrm{P}\Big[\sum_{q_n<k\leqslant 2q_n} \mathrm{E}_\omega(\mathbb{T}_k - \overline{\mathbb{T}}_k) \geqslant \varepsilon^n \quad \text{i.o.}\Big] = 0,$$

which gives (4.5.3). Finally we write

$$\mathbb{P}[\mathbb{T}_k \neq \overline{\mathbb{T}}_k] = \mathrm{E}\left[\mathrm{P}_\omega[\mathbb{T}_k - \overline{\mathbb{T}}_k \geqslant 1]\right] \leqslant \mathrm{E}\left[\mathrm{E}_\omega(\mathbb{T}_k - \overline{\mathbb{T}}_k)\right] \leqslant C\big(\mathrm{E}\rho^\gamma\big)^{k-p_n}$$

to infer our final claim by yet another appeal to the Borel-Cantelli lemma. □

The advantage of introducing the new process $\overline{X}$ is that it behaves similarly to X and from the point of view of limit theorems this change is indistinguishable. However, here one can indicate independent pieces: $\{\overline{X}_k\}_{k\in(\overline{T}_{S_{q_n}},\overline{T}_{S_{2q_n}}]}$ are P-independent.

4.5.2 Proof of Theorem 4.2.4

From now on we assume that $\beta < 1$; in particular $E\xi = \infty$. We are ready to describe the required properties of the environment. The definition of the sets below depends on several parameters, but first of all it depends on our hypothesis on ξ (for $\beta > 1$ we will choose slightly different sets). Given $d < D$, $b < B$, and $\varepsilon > 0$ let

$$U_n(d,D,b,B,\varepsilon) = \left\{\exists k \in (q_n, 2q_n] \quad \frac{S_k - S_{2p_n}}{a_{k+1}} \in (d,D), \frac{E_\omega[\overline{G}_k^2]}{a_{k+1}} \leqslant \varepsilon, \frac{\xi_{k+1}}{a_{k+1}} \in (b,B)\right\},$$

where $\overline{G}_k$ is the length of the left excursion of $\overline{X}$ from S_k before hitting $S_k + 1$. Of course $\mathrm{E}_\omega\overline{G}_k \leqslant \mathrm{E}_\omega G_k$ and $\mathrm{Var}_\omega\overline{G}_k \leqslant \mathrm{E}_\omega G_k^2$. We want to consider environments which belong to infinitely many sets U_n. However, given ω, we want to have some freedom of choosing all the parameters. The lemma below justifies that the measure of these environments is one.

Lemma 4.5.2. *Assume that conditions* (4.2.1) *and* (4.2.2) *are satisfied. Then the event*

$$\mathcal{U} = \bigcap \left\{\limsup_n U_n(d,D,b,B,\varepsilon) \,:\, d,D,b,B \in \mathbb{Q}^+,\ d<D, b<B,\ \varepsilon > 0\right\}$$

has probability one.

Proof. Since in the above formula the intersections are essentially over a countable set of parameters (one can obviously restrict to the rational parameter ε), it is sufficient to prove that for fixed parameters $d < D$, $b < B$ and $\varepsilon > 0$,

$$\mathrm{P}\left[\limsup_n U_n\right] = 1,$$

for $U_n = U_n(d,D,b,B,\varepsilon)$. Observe that the events $\{U_n\}_{n\in\mathbb{N}}$ are independent, because U_n depends only on $\{\omega_j\}_{j\in[p_n,2q_n]}$ and thanks to (4.5.2) the sets $\{[p_n,2q_n]\}_{n\in\mathbb{N}}$ are pairwise disjoint. Thus, invoking the Borel-Cantelli Lemma, it is sufficient to prove that there is $\delta_0 > 0$ such that for large indices n,

$$\mathrm{P}[U_n] > \delta_0. \tag{4.5.8}$$

We need to estimate probabilities of all the events which appear in the definition of U_n. Denote

$$\begin{aligned} V_k^1 &= \{(S_k - S_{2p_n})/a_{k+1} \in (d,D)\}\,, \\ V_k^2 &= \left\{\mathrm{E}_\omega \overline{G}_k^2/a_{k+1} \leqslant \varepsilon\right\}, \\ V_{k+1}^3 &= \{\xi_{k+1}/a_{k+1} \in (b,B)\}\,. \end{aligned} \tag{4.5.9}$$

To estimate the probability of V_k^1, observe that thanks to (4.5.2) we have $a_{k-2p_n}/a_{k+1} \to 1$ for any $k \in (q_n, 2q_n]$. Therefore, since $\beta < 1$,

$$\mathrm{P}[V_k^1] = \mathrm{P}\left[\frac{\sum_{2p_n<j\leqslant k}\xi_k}{a_{k-2p_n}} \cdot \frac{a_{k-2p_n}}{a_{k+1}} \in (d,D)\right] \xrightarrow{n\to\infty} \delta \in (0,1).$$

Recalling that $\mathrm{E}_\omega \overline{G}_k^2 \leqslant \mathrm{E}_\omega G_k^2$, which is a stationary sequence, we obtain $\mathrm{E}_\omega \overline{G}_k^2/a_k \xrightarrow{\mathrm{P}} 0$, i.e. $\mathrm{P}[V_k^2] \to 1$. Next, observe that $j\mathrm{P}\left[\xi_j/a_j \in (b,B)\right] \to \delta' > 0$ as $j \to \infty$. Let us introduce an auxiliary family of sets

$$V_k^4 = \{\forall j \in (k, 2q_n)\ \xi_j/a_j \notin (b,B)\}.$$

For large n,

$$\mathrm{P}[V_k^4] = \prod_{j>k}^{2q_n} \mathrm{P}[\xi_j/a_j \notin (b,B)] \geqslant \prod_{j>k}^{2q_n}\left(1 - \frac{2\delta'}{j}\right) \geqslant \left(1 - \frac{2\delta'}{q_n}\right)^{2q_n-k} \geqslant e^{-3\delta'}.$$

Observe also that the sets $\{V_{k+1}^3 \cap V_{k+1}^4\}_{k\in(q_n,2q_n]}$ are pairwise disjoint. Therefore, for large n,

$$\begin{aligned} \mathrm{P}[U_n] &\geqslant \mathrm{P}\Bigg[\bigcup_{q_n<k\leqslant 2q_n} V_k^1 \cap V_k^2 \cap V_{k+1}^3 \cap V_{k+1}^4\Bigg] \\ &= \sum_{q_n<k\leqslant 2q_n} \mathrm{P}\left[V_k^1 \cap V_k^2\right] \mathrm{P}\left[V_{k+1}^3\right] \mathrm{P}\left[V_{k+1}^4\right] \\ &\geqslant \frac{\delta\delta' e^{-3\delta'}}{4} \sum_{q_n<k\leqslant 2q_n} \frac{1}{k} \sim \frac{\delta\delta' e^{-3\delta'}\log 2}{4}. \end{aligned}$$

In conclusion, the probabilities of U_n are bounded from below, which entails (4.5.8) and completes the proof. □

Proof of Theorem 4.2.4 for $\beta < 1$. In view of our hypothesis (4.5.2), the Borel-Cantelli lemma yields

$$\mathrm{P}\big[\exists\varepsilon > 0\ S_{2p_n} \geqslant a_{q_n}\varepsilon \ \text{ i.o.}\big] = 0.$$

Therefore, invoking Lemma 4.5.2, the set

$$\mathcal{U} \cap \left\{\exists\varepsilon > 0\ S_{2p_n} \geqslant a_{q_n}\varepsilon \text{ i.o.}\right\}^c \tag{4.5.10}$$

has probability 1. From now on we fix ω from the event above which also satisfies the claim of Lemma 4.5.1.

Assume that, for fixed ω,

$$\frac{T_n - \mathrm{E}_\omega T_n}{\kappa_n} \Rightarrow Y_\omega \qquad n \to \infty, \tag{4.5.11}$$

for some random variable Y_ω.

We fix parameters $d < D$, $b < B$ and $\varepsilon > 0$. Take two sequences $(n_m)_{m\in\mathbb{N}}$ and $k_m \in (q_{n_m}, 2q_{n_m}]$ such that

$$\omega \in V^1_{k_m} \cap V^2_{k_m} \cap V^3_{k_m+1},$$

where all the sets were defined in (4.5.9). We can additionally assume (removing a finite number of elements of the sequence if needed), that for all indices m

$$S_{2p_{n_m}} < a_{k_m}\varepsilon. \tag{4.5.12}$$

Lemma 4.5.1 says that, given ω, the difference $(T_n - \mathrm{E}_\omega T_n) - (\overline{T}_n - \mathrm{E}_\omega \overline{T}_n)$ remains a.s. bounded, hence (4.5.11) yields

$$\frac{\overline{T}_n - \mathrm{E}_\omega \overline{T}_n}{\kappa_n} \Rightarrow Y_\omega \qquad n \to \infty. \tag{4.5.13}$$

Consider the following decomposition:

$$\frac{\overline{T}_{S_{k_m+1}} - \mathrm{E}_\omega \overline{T}_{S_{k_m+1}}}{\kappa_{S_{k_m+1}}} = v_m \cdot V_m + w_m \cdot W_m + Z_m, \tag{4.5.14}$$

where

$$\begin{aligned} V_m &= \frac{\overline{T}_{S_{k_m}} - \mathrm{E}_\omega \overline{T}_{S_{k_m}}}{\kappa_{S_{k_m}}}, & v_m &= \frac{\kappa_{S_{k_m}}}{\kappa_{S_{k_m+1}}}, \\ W_m &= \frac{\overline{\mathbb{T}}^r_{k_m+1} - \mathrm{E}_\omega \overline{\mathbb{T}}^r_{k_m+1}}{\xi^2_{k_m+1}}, & w_m &= \frac{\xi^2_{k_m+1}}{\kappa_{S_{k_m+1}}}, \\ Z_m &= \frac{\overline{\mathbb{T}}^l_{k_m+1} - \mathrm{E}_\omega \overline{\mathbb{T}}^l_{k_m+1}}{\kappa_{S_{k_m+1}}}. \end{aligned} \tag{4.5.15}$$

Random variables V_m and (W_m, Z_m) are P_ω-independent. By (4.5.13), V_m converges in distribution to Y_ω, whereas W_m, by Lemma 4.3.1, converges to $2\vartheta - 1$. Therefore we need to understand the behaviour of both deterministic (given ω) sequences $(v_m)_{m\in\mathbb{N}}$, $(w_m)_{m\in\mathbb{N}}$ and of the sequence of random variables $(Z_m)_{m\in\mathbb{N}}$.

Let us start with estimates of v_n and w_n. In the case $\beta < 1$, $\kappa_n = n^2$, hence

$$b^2 a^2_{k_m+1} \leqslant \kappa_{S_{k_m+1}} \leqslant (D + B + \varepsilon)^2 a^2_{k_m+1}$$

Using the estimates in the definition of the event $U_n(d, D, b, B, \varepsilon)$ gives

$$(1-\delta) \cdot \left(\frac{d}{D + B + \varepsilon}\right)^2 \leqslant v_m \leqslant (1+\delta) \cdot \left(\frac{D+\varepsilon}{d+b}\right)^2 \tag{4.5.16}$$

and

$$(1-\delta)\cdot\left(\frac{b}{D+B+\varepsilon}\right)^2 \leqslant w_m \leqslant (1+\delta)\cdot\left(\frac{B}{b}\right)^2. \tag{4.5.17}$$

Now let us consider the sequence Z_m. We want to prove that it converges to 0 in probability. Since our argument will invoke the Chebyshev inequality, we need to bound the quenched variance of $\overline{\mathbb{T}}^l_{k_m+1}$. Note that on the considered event, recalling (2.3.13), we have

$$\mathrm{Var}_\omega \overline{\mathbb{T}}^l_{k_m+1} \leqslant \xi_{k_m+1}\mathrm{Var}_\omega \overline{G}_{k_m} + \xi^2_{k_m+1}(\mathrm{E}_\omega \overline{G}_{k_m})^2 \leqslant 2\varepsilon a^3_{k_m+1}B^2. \tag{4.5.18}$$

Observe that for any $\eta > 0$, using the Chebyshev inequality and (4.5.18), we have:

$$\mathrm{P}_\omega[|Z_m|>\eta] = \mathrm{P}_\omega\left[\left|\overline{\mathbb{T}}^l_{k_m+1} - \mathrm{E}_\omega\overline{\mathbb{T}}^l_{k_m+1}\right| > \eta\kappa_{S_{k_m+1}}\right] \leqslant \frac{\mathrm{Var}_\omega \overline{\mathbb{T}}^l_{k_m+1}}{\eta^2\kappa^2_{S_{k_m+1}}} \leqslant \frac{2\varepsilon B^2}{\eta^2 b^4} a^{-1}_{k_m+1}.$$

One can easily see that for any fixed d and D one can construct sequences $(b_m)_{m\in\mathbb{N}}$, $(B_m)_{m\in\mathbb{N}}$, $(k_m)_{m\in\mathbb{N}}$ such that $b_m, B_m \to \infty$, $b_m/B_m \to 1$ and inequalities (4.5.16) and (4.5.17) hold. Then $v_m \to 0$, $w_m \to 1$ and $Z_m \overset{\mathrm{P}_\omega}{\to} 0$. Since the sequence $(V_m)_m$ is tight, we have

$$\frac{\overline{T}_{S_{k_m+1}} - \mathrm{E}_\omega \overline{T}_{S_{k_m+1}}}{\kappa_{S_{k_m+1}}} = v_m\cdot V_m + w_m\cdot W_m + Z_m \Rightarrow 2\vartheta - 1.$$

So, if the limits (4.5.11), (4.5.13) exist, both must be equal to $Y_\omega = 2\vartheta - 1$.

Next, fixing all the parameters b, B, d, D observe that both sequences $(v_m)_m$, $(w_m)_m$ are bounded, therefore we can assume, possibly choosing their subsequences, that they are convergent to some strictly positive v and w, respectively. Since the families of random variables $\{V_m\}_m$ and $\{W_m\}_m$ are independent, we conclude

$$2\vartheta - 1 \overset{d}{=} Y_\omega \overset{d}{=} v(2\vartheta_v - 1) + w(2\vartheta_w - 1), \tag{4.5.19}$$

where ϑ_v, ϑ_w are independent copies of ϑ. However this equation cannot be satisfied e.g. by (4.1.1). That leads to a contradiction and proves that the limit (4.5.11) cannot exist. □

Proof of Theorem 4.2.4 for $\beta > 1$. We proceed similarly as in the previous case, but this time we need to redefine the sets U_n. Let

$$U_n(b,B,\varepsilon) = \left\{\exists k\in(q_n, 2q_n] \quad \frac{E_\omega[\overline{G}^2_k]}{a_{k+1}} \leqslant \varepsilon, \frac{\xi_{k+1}}{a_{k+1}} \in (b,B)\right\}.$$

Reasoning exactly as in the proof of Lemma (4.5.2) we prove that under conditions (4.2.1) and (4.2.2), the event

$$\mathcal{U} = \bigcap\left\{\limsup_n U_n(b,B,\varepsilon) \;:\; b,B\in\mathbb{Q}^+,\ b<B,\ \varepsilon>0\right\}$$

has probability one.

We consider the formula (4.5.14) and since in this case $\kappa_n = a_n^2$, by (4.5.15), we have

$$v_m \to 1 \qquad \text{and} \qquad b^2 \leqslant w_m \leqslant B^2,$$

because, by the strong law of large numbers, S_{m_k+1}/S_{m_k} converges to 1 a.s. Taking into account (4.5.18) and the calculations below we have

$$\mathrm{P}_\omega[|Z_m| > \eta] \leqslant \frac{\mathrm{Var}_\omega \overline{\mathbb{T}}^l_{k_m+1}}{\eta^2 \kappa^2_{S_{k_m+1}}} \leqslant \frac{2\varepsilon a^3_{k_m+1} B^2}{a^4_{S_{k_m+1}}} \to 0 \quad \text{a.s.}$$

for any B, because by the strong law of large numbers $a_n/a_{S_n} \to 1$ a.s. This proves $Z_m \overset{\mathrm{P}_\omega}{\to} 0$.

Now we can repeat the arguments from the previous proof. Fixing the parameters b and B we conclude that $(w_m)_{m\in\mathbb{N}}$ is bounded, therefore we can assume that it converges to some $w \neq 0$. Invoking (4.5.14), we obtain

$$Y_\omega \overset{d}{=} Y_\omega + w(2\vartheta_w - 1), \tag{4.5.20}$$

where Y_ω and ϑ_w are independent. That leads us once again to a contradiction. □

Chapter 5

Favourite sites of a random walk in moderately sparse random environment

In this chapter we present the limit theorems for maximal local times of RWSRE under the annealed measure. We consider only the case of moderately sparse environment, that is, we assume that $\mathrm{E}\xi < \infty$. The following is an extract from [19] with alterations done by the author to keep consistency with other chapters.

5.1 Introduction

Let, for $k \leqslant n$,

$$L_k(n) = |\{m \leqslant T_n \,:\, X_m = k\}| \tag{5.1.1}$$

be the *local time*, i.e. the number of times the walk visits k before reaching n. Our object of interest is the limiting behaviour of the maximal local time, that is the variable $\max_{k \leqslant n} L_k(n)$, as $n \to \infty$. We shall present two cases in which an annealed limit theorem holds for this sequence of variables, with Fréchet distribution in the limit.

As was described in Section 3.1 concerning limit theorems for hitting times, the shape of these theorems depends on the interplay between the drift and the sparsity. A similar dichotomy is seen for the local times, however the crucial assumption is no longer (3.1.6).

In the first case it is the drift that drives the limiting behaviour of local times. It may be seen as a generalization of results obtained by Dolgopyat and Goldsheid in [12, Theorem 4] in the setting of i.i.d. environment. However, the techniques used in [12] were different from those presented here. In this chapter we follow the method proposed by Kesten et al. in [18] when examining the hitting times, that is we rephrase the question posed for the walk into the setting of an associated branching process. This method proves useful both in the case of dominating drift and the complementary case, in which it is the tail behaviour of ξ that determines the shape of the limit.

Throughout this chapter we shall use a model slightly different from the one defined in (1.1.3). That is, we put

$$\omega_k = \begin{cases} \lambda_{n+1} & \text{if } k = S_n \text{ for some } n \in \mathbb{Z}, \\ 1/2 & \text{otherwise.} \end{cases}$$

Observe that in (1.1.3) we allowed for the dependence between the length of k'th block and the drift at S_k, which is its right end. In this chapter, when examining the case of dominating drift, we allow for dependence between the length of the block and the drift at its left end. In the case of dominating sparsity we shall assume ξ and λ to be independent, so that we only change enumeration of drifts. Definition (1.1.3) used so far is the one given by Matzavinos et al. in [20], while the setting used here is the same as in [6, 7]. This change of convention arises naturally from time reversal coming with the associated branching process which we introduce in Section 5.3.

The chapter is organised as follows: in Section 5.2 we present the statement of our results. Section 5.3 introduces the branching process associated with the walk and presents some of its properties. The proofs of the main theorems are given in Sections 5.4 and 5.5.

5.2 Annealed limit laws for the maximal local time

In this section we present our results. Relations (2.2.6) remain our standing assumptions, i.e. we examine the RWSRE which is transient to $+\infty$. We consider two sets of assumptions:

Assumptions (A)**:** For some $\alpha \in (0, 2)$,

- $\mathrm{E}\rho^\alpha = 1$;
- $\mathrm{E}\rho^\alpha \log^+ \rho < \infty$;
- the distributions of ρ and $\log \rho$ are non-arithmetic;
- $\mathrm{E}\xi^{(\alpha+\delta)\vee 1} < \infty$ for some $\delta > 0$;
- $\mathrm{E}\xi^\alpha \rho^\alpha < \infty$.

Note that without loss of generality we may assume that $\alpha + \delta \leqslant 2$. In this case the limiting behaviour of maxima is determined mostly by the parameter α, that is by properties of ρ; it is a generalization of the result known for the walk in i.i.d. environment. We shall prove the following:

Theorem 5.2.1. *Under assumptions* (A)*, there is a constant* $c_\alpha > 0$ *such that for all* $x > 0$*,*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\max_{k\leqslant n} L_k(n)}{n^{1/\alpha}} > x\right] = 1 - e^{-c_\alpha x^{-\alpha}}.$$

It turns out that the crucial assumption in this case is that $\mathrm{E}\xi^{\alpha+\delta} < \infty$. Different behaviour appears when ξ does not have high enough moments. Consider the following:

Assumptions (B)**:** For some $\beta \in [1, 2)$,

- $\mathrm{P}[\xi > x] \sim x^{-\beta}\ell(x)$ for some slowly varying ℓ;

- $\mathrm{E}\rho^{\beta+\delta} < 1$ for some $\delta > 0$;
- ξ and ρ are independent;
- if $\beta = 1$, assume $\mathrm{E}\xi < \infty$.

In this case we may also assume that $\beta + \delta \leqslant 2$. Observe that we do not assume that there exists α such that $\mathrm{E}\rho^\alpha = 1$. However, if it does exist, then $\alpha > \beta$ and $\mathrm{E}\xi^\alpha = \infty$. Since ξ has regularly varying tails, a good scaling for maxima of $(\xi_n)_{n\in\mathbb{N}}$ is a sequence $(a_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} n\mathrm{P}[\xi > a_n] = 1. \tag{5.2.1}$$

It turns out it is also a good scaling for maxima of L.

Theorem 5.2.2. *Under assumptions (B), there is a constant $c_\beta > 0$ such that for all $x > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\max_{k\leqslant n} L_k(n)}{a_n} > x\right] = 1 - e^{-c_\beta x^{-\beta}}.$$

The exact forms of constants c_α, c_β will be given during the proofs.

We conclude this section by remarking that in the moderately sparse environment it is enough to consider the sequence of maximal local times along the marked points. Note that $(a_n)_{n\in\mathbb{N}}$ given by (5.2.1) is regularly varying with index $1/\beta$.

Lemma 5.2.3. *Assume that $\mathrm{E}\xi < \infty$. If there exist constants $c > 0$, $\gamma > 0$ and a sequence $(b(n))_{n\in\mathbb{N}}$ which is regularly varying with index $1/\gamma$ such that for every $x > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\max_{k\leqslant S_n} L_k(S_n)}{b(n)} > x\right] = 1 - e^{-cx^{-\gamma}},$$

then for every $x > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\max_{k\leqslant n} L_k(n)}{b(n)} > x\right] = 1 - e^{-(c/\mathrm{E}\xi)x^{-\gamma}}.$$

Proof. Denote, for $n \in \mathbb{N}$,

$$\nu_n = \inf\{k > 0 \,:\, S_k > n\}.$$

Then the assumption $\mathrm{E}\xi < \infty$ and the law of large numbers guarantee that P-almost surely

$$\frac{\nu_n}{n} \xrightarrow{n\to\infty} \frac{1}{\mathrm{E}\xi}.$$

Denote, for $m \in \mathbb{N}$, $M(m) = \max_{k\leqslant S_m} L_k(S_m)$. Since $S_{\nu_n - 1} \leqslant n < S_{\nu_n}$, we have, for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\left[b(n)^{-1}\max_{0\leqslant k<n} L_k(n) > x\right] &\geqslant \mathbb{P}\left[b(n)^{-1}M(\nu_n - 1) > x\right]\\ &\geqslant \mathbb{P}\left[b(n)^{-1}M(n(1/\mathrm{E}\xi - \varepsilon) - 1) > x\right] - \mathbb{P}\left[|1/\mathrm{E}\xi - \nu_n/n| > \varepsilon\right]\\ &\xrightarrow{n\to\infty} 1 - \exp(-c(1/\mathrm{E}\xi - \varepsilon)x^{-\gamma}),\end{aligned}$$

where we used the fact that

$$\frac{b(n(1/\mathrm{E}\xi - \varepsilon) - 1)}{b(n)} \to (1/\mathrm{E}\xi - \varepsilon)^{1/\gamma}$$

since $b(n)$ is regularly varying. Similarly,

$$\begin{aligned}\mathbb{P}\left[b(n)^{-1} \max_{0 \leqslant k < n} L_k(n) > x\right] &\leqslant \mathbb{P}\left[b(n)^{-1} M(\nu_n) > x\right] \\ &\leqslant \mathbb{P}\left[b(n)^{-1} M(n(1/\mathrm{E}\xi + \varepsilon)) > x\right] + \mathbb{P}\left[|1/\mathrm{E}\xi - \nu_n/n| > \varepsilon\right] \\ &\xrightarrow{n\to\infty} 1 - \exp(-c(1/\mathrm{E}\xi + \varepsilon)x^{-\gamma}),\end{aligned}$$

which ends the proof since $\varepsilon > 0$ is arbitrary. □

5.3 Auxiliary results

Instead of examining the local times explicitly, we pass to a branching process associated with RWSRE. In this section we describe the construction of this process and prove auxiliary lemmas which we will use in both examined cases.

5.3.1 Associated branching process

An important property of a transient simple random walk on $\mathbb{Z}$ is its duality with a branching process. Consider a walk $(X_n)_{n\in\mathbb{N}}$ such that $X_0 = 0$ and $X_n \to \infty$ almost surely, evolving in an environment $\omega = (\omega_k)_{k\in\mathbb{Z}}$. Recall that, for $n \in \mathbb{N}$,

$$T_n = \inf\{k \in \mathbb{N} : X_k = n\}$$

is the first passage time and, for $k \leqslant n$,

$$L_k(n) = |\{m \leqslant T_n : X_m = k\}|$$

is the local time, i.e. the number of times the walk visits site k before reaching n. First of all, note that the transience of the walk implies that, almost surely, the walk spends only finite time on the negative half-axis. That is, for any sequence $b_n \to \infty$,

$$\frac{\max_{k<0} L_k(n)}{b_n} \to 0 \quad \mathbb{P}\text{-a.s.}$$

Therefore, when examining the limit theorems, we may restrict our analysis to the variables $L_k(n)$ for $k \geqslant 0$.

The visits to $k \geqslant 0$ counted by $L_k(n)$ may be split into visits from the left and from the right, that is,

$$\begin{aligned}L_k(n) &= |\{m \leqslant T_n : X_m = k\}| \\ &= |\{m \leqslant T_n : X_{m-1} = k-1, X_m = k\}| + |\{m \leqslant T_n : X_{m-1} = k+1, X_m = k\}|.\end{aligned}$$

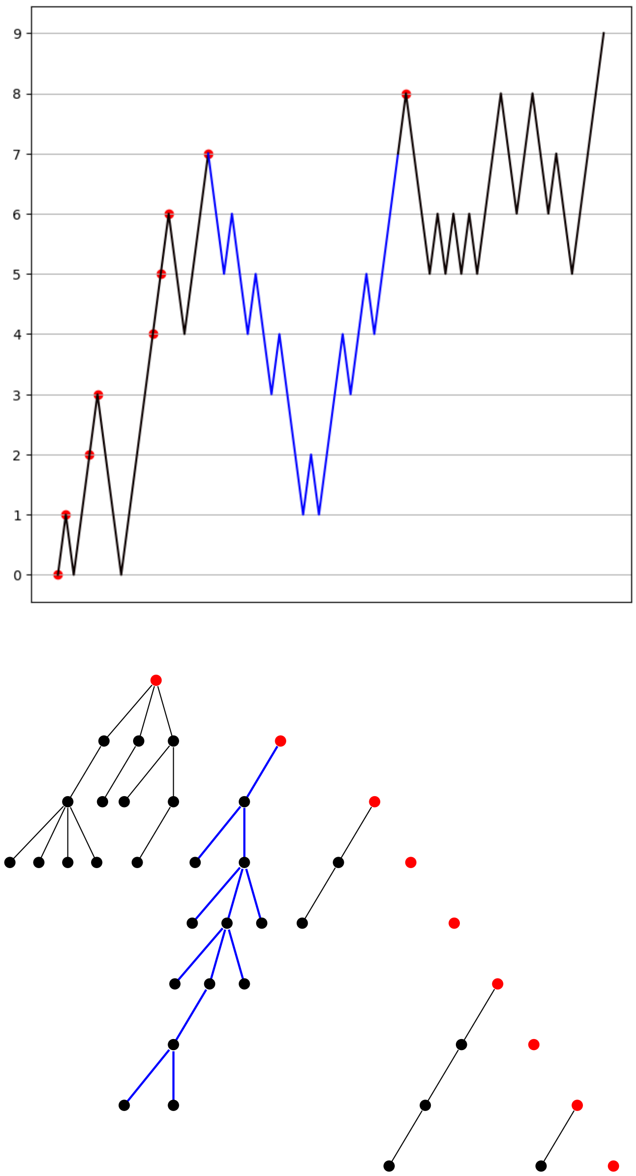


Figure 5.3.1: Exemplary path of a simple walk and corresponding realization of a branching process. Immigrants (marked in red) correspond to arrivals to new sites. The subtrees correspond to the excursions of the walk; the first excursion from 7 and its corresponding subtree were marked in blue.

Moreover, since the walk is simple, it makes a step from $k-1$ to k when it visits site k for the first time. After that, it may make some excursions to the left from k; such an excursion always begins with a step from k to $k-1$ and ends with a step from $k-1$ to k. Therefore, to count all the visits the walk makes to given sites, it is enough to count its steps to the left.

That is, for fixed $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n$,

$$\begin{aligned} L_k(n) &= 1 + |\{m \leqslant T_n \,:\, X_{m-1} = k,\, X_m = k-1\}| + |\{m \leqslant T_n \,:\, X_{m-1} = k+1,\, X_m = k\}| \\ &= 1 + \widetilde{Z}_{k-1} + \widetilde{Z}_k, \end{aligned}$$

where $\widetilde{Z}_k = |\{m \leqslant T_n \,:\, X_{m-1} = k+1,\, X_m = k\}|$ is the number of visits to point k from the right. The main observation is that the process given by $Z_k = \widetilde{Z}_{n-k}$ has a branching structure. Every step from $n-k$ to $n-k-1$ occurs either before the walk discovered the site $n-k+1$, or between consecutive steps from $n-k+1$ to $n-k$. That is,

$$Z_{k+1} \stackrel{\mathrm{d}}{=} \sum_{j=1}^{Z_k+1} G_{n,k}^{(j)},$$

where $G_{n,k}^{(j)}$, for $j \leqslant Z_k$, counts the number of steps from $n-k$ to $n-k-1$ between j'th and $j+1$'th step from $n-k+1$ to $n-k$, and $G_{n,k}^{(Z_k+1)}$ counts the number of steps from $n-k$ to $n-k-1$ before the first visit to $n-k+1$. Observe that, due to the strong Markov property of the walk, the variables $G_{n,k}^{(j)}$ are i.i.d., independent of Z_k, and have geometric distribution with parameter ω_{n-k}. Therefore, $Z = (Z_k)_{k\in\mathbb{N}}$ is a branching process in random environment with unit immigration; note that we do not count the immigrant, so that $Z_0 = 0$. Moreover, for any fixed $n \in \mathbb{N}$,

$$(L_k(n))_{0\leqslant k\leqslant n} \stackrel{\mathrm{d}}{=} (1 + Z_{n-k+1} + Z_{n-k})_{0\leqslant k\leqslant n}\,. \tag{5.3.1}$$

In particular, if X is a random walk in a sparse random environment, its associated branching process is a branching process in a sparse random environment (BPSRE). If in the above construction we consider the walk stopped upon reaching a marked point S_n, the branching process starts from one immigrant and evolves in the environment divided into blocks of lengths given by $(\xi_{n-k})_{k\in\mathbb{N}}$; within the blocks the reproduction is given by the law $Geo(1/2)$, while the particles in the k'th marked generation are born with the law $Geo(\lambda_{n-k})$. When examining the process Z, it is convenient – and valid, since the environment is given by an i.i.d. sequence – to reverse the enumeration, so that the block lengths are given by $(\xi_k)_{k\in\mathbb{N}}$ and reproduction law in k'th marked point is $Geo(\lambda_k)$. The process Z may be then defined formally as follows: for any fixed environment ω, under P_ω,

$$\begin{aligned} Z_0 &= 0, \\ Z_k &= \sum_{j=1}^{Z_{k-1}+1} G_k^{(j)}, \end{aligned}$$

where the variables $(G_k^{(j)})_{j\in\mathbb{N}}$ are independent of Z_{k-1} and each other, and

$$G_k^{(j)} \stackrel{\mathrm{d}}{=} Geo(\omega_k) \quad \text{for} \quad \omega_k = \begin{cases} \lambda_n & \text{if } k = S_n \text{ for some } n \in \mathbb{N}; \\ 1/2 & \text{otherwise.} \end{cases}$$

Whenever examining a BPSRE, we will distinct the population at marked generations with bold letters, that is, for example, $\mathbb{Z}_n = Z_{S_n}$.

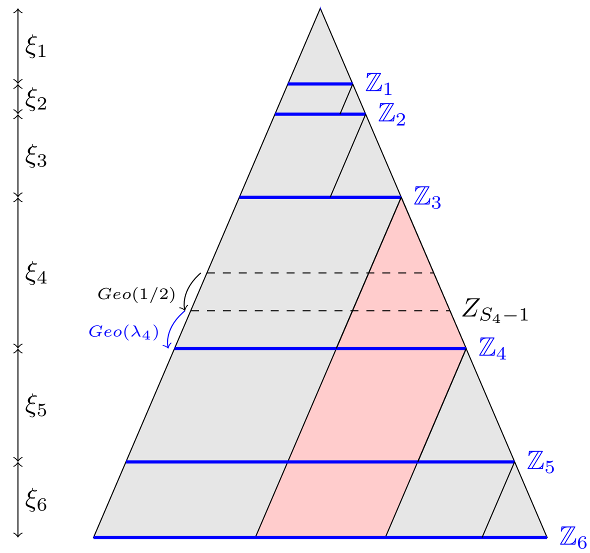


Figure 5.3.2: Schematic picture of the process Z. Horizontal blue lines represent marked generations. Within each block between marked generations, the triangular area represents progeny of immigrants that arrived in this block. The coloured region represents process Y^4.

Recall that by Lemma 5.2.3, it is enough to analyse the sequence of maximal local times along the marked points. Due to (5.3.1) we have, for any $n \in \mathbb{N}$,

$$\max_{0 \leqslant k \leqslant S_n} L_k(S_n) \stackrel{\mathrm{d}}{=} 1 + \max_{0 \leqslant k \leqslant S_n} (Z_k + Z_{k+1}). \tag{5.3.2}$$

Therefore, to prove Theorems 5.2.1 and 5.2.2, we will examine the maximal generations of the corresponding branching process.

For $k \in \mathbb{N}$, we will denote by Y^k the process counting the progeny of immigrants from k'th block, i.e. those arriving at times $S_{k-1}, S_{k-1}+1, \dots S_k - 1$. Let, for $j \geqslant 0$, Y_j^k denote the number of descendants of these immigrants present in generation $S_{k-1}+j$. Observe that the process Y^k starts with one immigrant at time $j = 0$; it evolves with unit immigration and $Geo(1/2)$ reproduction law up until time $j = \xi_k - 1$. The last immigrant arrives at this time, and the particles at time $j = \xi_k$ are born with the law $Geo(\lambda_k)$. From there on the process Y^k evolves without immigration (see Figure 5.3.2).

We will use the convention that $Y_j^k = 0$ for $j < 0$, so that

$$Z_n = \sum_{k \in \mathbb{N}} Y^k_{n - S_{k-1}}.$$

Observe that the processes Y^k are independent under P_ω and identically distributed under $\mathbb{P}$.

The branching process in a sparse random environment was studied in [7] for the purpose of proving annealed limit theorems for the first passage times. An important observation is that the transience of the walk implies quick extinctions of the branching process. Let

$$\tau_0 = 0, \quad \tau_n = \inf\{k > \tau_{n-1} \,:\, \mathbb{Z}_k = 0\}$$

be the extinction times (note that we only consider the extinctions at marked generations). Observe that when the extinction occurs, the process starts anew from one immigrant. Thus the sequence $(\tau_n - \tau_{n-1})_{n\geqslant 1}$ is i.i.d. under $\mathbb{P}$, and the extinction times split the process Z into independent epochs. The following is Lemma 4.1 from [7]; it implies that the extinctions occur rather often in the case of transient RWSRE.

Lemma 5.3.1. *Assume that* $\mathrm{E}\log\rho < 0$ *and* $\mathrm{E}\log\xi < \infty$. *Then* $\mathbb{E}\tau_1 < \infty$. *If additionally* $\mathrm{E}\rho^\varepsilon < \infty$ *and* $\mathrm{E}\xi^\varepsilon < \infty$ *for some* $\varepsilon > 0$, *then there exists* $c > 0$ *such that* $\mathbb{E}e^{c\tau_1} < \infty$.

5.3.2 Estimates of the processes related to the environment

Define

$$\bar{R}_n = 1 + \rho_n + \rho_n\rho_{n+1} + \cdots = \sum_{k=n-1}^{\infty} \Pi_{n,k}, \tag{5.3.3}$$

for $\Pi_{n,k}$ defined in (2.2.1). Then the following relation holds:

$$\bar{R}_n = 1 + \rho_n \bar{R}_{n+1}. \tag{5.3.4}$$

Moreover, the sequence $(\bar{R}_n)_{n\in\mathbb{Z}}$ is stationary under P. Observe that if $\mathrm{E}\rho^\gamma < 1$ for some $\gamma > 0$, then $\mathrm{E}\bar{R}_1^\gamma < \infty$ (see the proof of Lemma 2.3.1 in [4]), whereas under (A), the distribution of ρ satisfies the assumptions of Kesten-Goldie theorem (see [4, Theorem 2.4.4]), thus

$$\mathrm{P}[\bar{R}_1 > x] \sim c_\alpha x^{-\alpha}$$

for some constant c_α. Therefore

$$\mathrm{P}[\bar{R}_1 > x] \leqslant C_\gamma x^{-\gamma} \quad \text{for some } C_\gamma < \infty \text{ and all } x > 0, \tag{5.3.5}$$

whenever either $\mathrm{E}\rho^\gamma < 1$, or $\mathrm{E}\rho^\gamma = 1$ and Kesten-Goldie theorem holds for $\bar{R}_1$. As can be seen in the proofs of Lemma 6 in [18] and Lemma 5.6 in [7], in the case of dominating drift it is $\bar{R}_1$ from whom the total population of the process Z (which corresponds to first passage times of the walk) inherits its annealed tail behaviour.

Let, for $m \in \mathbb{N}$, the *potential* Ψ be defined as

$$\Psi_{m,k} = \Pi_{m,n} \quad \text{for } k \in [S_n, S_{n+1}). \tag{5.3.6}$$

As we will see, maxima of the potential determine the limiting behaviour of maximal generation of Z in the same way as $\bar{R}_1$ determines the asymptotics of the total population. Let

$$M_{\Psi,m} = \max_{k\geqslant S_m - 1} (\Psi_{m,k} + \Psi_{m,k+1}). \tag{5.3.7}$$

Then the sequence $(M_{\Psi,m})_{m\in\mathbb{N}}$ is stationary under P; denote by M_Ψ its generic element. Observe that

$$M_{\Psi,1} \leqslant 2 \max_{k\geqslant S_1 - 1} \Psi_{1,k} = 2\max_{n\geqslant 0} \Pi_{1,n} \leqslant 2\bar{R}_1,$$

thus

$$\mathrm{E}M_\Psi^\gamma < \infty \quad \text{whenever } \mathrm{E}\rho^\gamma < 1. \tag{5.3.8}$$

5.3.3 Auxiliary lemmas

The following lemma, concerning a classic Galton-Watson process, will be used repeatedly to estimate the growth of BPSRE in the unmarked generations.

Lemma 5.3.2. *Let $(X_n)_{n\geqslant 0}$ be a Galton-Watson process with $X_0 = x_0$, reproduction law Geo$(1/2)$, and no immigrants, and let $(\bar{X}_n)_{n\geqslant 0}$ be an analogous process with unit immigration. Then the following hold for any $N \in \mathbb{N}$:*

$$\mathbb{E}\left[\max_{k\leqslant N}(X_k - x_0)^2\right] \leqslant 8Nx_0, \tag{5.3.9}$$

$$\mathbb{E}\left[\max_{k\leqslant N}\bar{X}_k^2\right] \leqslant 16(N^2 + Nx_0 + x_0^2). \tag{5.3.10}$$

Proof. Since the process $(X_k)_{k\in\mathbb{N}}$ is a martingale with mean x_0, Doob's maximal inequality implies

$$\mathbb{E}\left[\max_{k\leqslant N}(X_k - x_0)^2\right] \leqslant 4\mathbb{E}(X_N - x_0)^2 = 4\mathrm{Var}X_N.$$

Now, a standard calculation gives

$$\mathrm{Var}X_N = 2Nx_0,$$

which implies (5.3.9).

Observe that $\bar{X}_n = X'_n + I_n$, where X' denotes the descendants of the initial x_0 particles, and I denotes the progeny of immigrants. The processes I and X' are independent, and X' has the same distribution as X. Moreover, the process $(\bar{X}_n)_{n\in\mathbb{N}}$ is a non-negative submartingale, thus by Doob's maximal inequality,

$$\mathbb{E}\left[\max_{k\leqslant N}\bar{X}_k^2\right] \leqslant 4\mathbb{E}\left[\bar{X}_N^2\right] = 4\left(\mathrm{Var}X'_N + \mathrm{Var}I_N + (\mathbb{E}X'_N + \mathbb{E}I_N)^2\right).$$

We have already examined the mean and variance of X'_N. To calculate moments of I_N, we may express I as a sum of independent copies of X. Alternatively, we may use the duality of I and a simple symmetric random walk. It implies that I_N equals in distribution to the number of times the walk hits 0 from the right when crossing the interval $[0, N+1]$ for the first time. By (2.1.1), the probability that the walk passes from 0 to $N+1$ without returning to 0 from the right, is $1/(N+1)$. Therefore $I_N \sim$ *Geo*$(1/(N+1))$, from which it follows that

$$\mathbb{E}I_N = N+1, \quad \mathrm{Var}I_N = N^2 + N.$$

Hence

$$\mathbb{E}\left[\bar{X}_N^2\right] = 2Nx_0 + N^2 + N + (x_0 + N + 1)^2 \leqslant 4(N^2 + Nx_0 + x_0^2),$$

which ends the proof of (5.3.10).

□

The next two lemmas will be of use to us both under assumptions (A) and (B). Therefore we shall consider the following set of assumptions:

Assumptions (Γ)**:** for some $\gamma \leqslant 2$,

- $\mathrm{E}\rho^\gamma \leqslant 1$ and (5.3.5) holds,
- $\mathrm{E}\xi^{\gamma/2} < \infty$,
- $\mathrm{E}\rho^\gamma \xi^{\gamma/2} < \infty$.

Let U_n be the progeny of the first immigrant residing in generation n, with the convention $U_0 = 1$, and denote $\mathbb{U}_n = U_{S_n}$. For fixed $N \in \mathbb{N}$, let U^k for $k = 1, \dots, N$ be copies of the process $U = (U_n)_{n\in\mathbb{N}}$, evolving in the same environment and independent under P_ω. That is, $(\sum_{k=1}^N U_n^k)_{n\in\mathbb{N}}$ is a BPSRE with N initial particles evolving without immigration. Although the first part of the following lemma is analogous to results presented in [18, Lemma 3] and [7, Lemma 5.6], we provide the full proof as it gives some insight into the properties of the process U.

Lemma 5.3.3. *Assume* (Γ)*. Then for some constant* C_1*,*

$$\mathbb{P}\left[\sum_{k=1}^N \sum_{n\geqslant 0} \mathbb{U}_n^k > x\right] \leqslant C_1 N^\gamma x^{-\gamma}, \tag{5.3.11}$$

$$\mathbb{P}\left[\sum_{n\geqslant 0}\left|\sum_{k=1}^N \mathbb{U}_n^k - N\Pi_{1,n}\right| > x\right] \leqslant C_1 N^{\gamma/2} x^{-\gamma}. \tag{5.3.12}$$

Moreover,

$$\mathbb{P}\left[\max_{n\geqslant 1}\sum_{k=1}^N U_n^k > x\right] \leqslant C_1 N^\gamma x^{-\gamma}, \tag{5.3.13}$$

$$\mathbb{P}\left[\sum_{n\geqslant 1}\sum_{k=1}^N \max_{S_{n-1}\leqslant j<S_n} |U_j^k - \mathbb{U}_{n-1}^k| > x\right] \leqslant C_1 N^{\gamma/2} x^{-\gamma}. \tag{5.3.14}$$

Proof. For fixed $n \geqslant 1$, under P_ω,

$$\mathbb{U}_n \overset{\mathrm{d}}{=} \sum_{k=1}^{U_{S_n-1}} G_k^{(n)},$$

where $G_k^{(n)}$ are random variables with law $Geo(\lambda_n)$, independent of U_{S_n-1} and each other. In particular,

$$\mathrm{E}_\omega G_k^{(n)} = \rho_n, \quad \mathrm{Var}_\omega G_k^{(n)} = \rho_n + \rho_n^2.$$

Since in generations $S_{n-1}+1, \dots S_n-1$ the process evolves with offspring distribution $Geo(1/2)$, standard calculation gives

$$\mathrm{E}_\omega[U_{S_n-1}|\mathbb{U}_{n-1}] = \mathbb{U}_{n-1} \quad \text{and} \quad \mathrm{Var}_\omega(U_{S_n-1}|\mathbb{U}_{n-1}) = 2(\xi_n - 1)\mathbb{U}_{n-1}.$$

This in turn implies

$$\begin{aligned} \mathrm{E}_\omega[\mathbb{U}_n|\mathbb{U}_{n-1}] &= \rho_n \mathbb{U}_{n-1}, \\ \mathrm{E}_\omega[(\mathbb{U}_n - \rho_n\mathbb{U}_{n-1})^2|\mathbb{U}_{n-1}] &= (\rho_n - \rho_n^2 + 2\rho_n^2\xi_n)\mathbb{U}_{n-1}. \end{aligned} \tag{5.3.15}$$

In particular $\mathrm{E}_\omega \mathbb{U}_n = \Pi_{1,n}$.

Observe that the processes U^k evolve without immigration and the extinction time of each U^k is stochastically dominated by τ_1, which is finite $\mathbb{P}$-a.s. by Lemma 5.3.1. In particular, with probability 1 the series

$$\sum_{k=1}^N \sum_{n\geqslant 0} \mathbb{U}_n^k$$

is indeed a finite sum. Recall the sequence $\bar{R}$ defined in (5.3.3) and observe that, by (5.3.4),

$$\begin{aligned} \sum_{k=1}^N \sum_{n\geqslant 0} \mathbb{U}_n^k &= \sum_{k=1}^N \sum_{n\geqslant 0} \mathbb{U}_n^k (\bar{R}_{n+1} - \rho_{n+1}\bar{R}_{n+2}) \\ &= \sum_{n\geqslant 1} \left(\sum_{k=1}^N (\mathbb{U}_n^k - \rho_n \mathbb{U}_{n-1}^k) \right) \bar{R}_{n+1} + N\bar{R}_1 \end{aligned}$$

and thus

$$\sum_{n\geqslant 0} \left(\sum_{k=1}^N \mathbb{U}_n^k - N\Pi_{1,n} \right) = \sum_{n\geqslant 1} \left(\sum_{k=1}^N (\mathbb{U}_n^k - \rho_n \mathbb{U}_{n-1}^k) \right) \bar{R}_{n+1}.$$

Therefore

$$\mathbb{P}\left[\sum_{n\geqslant 0} \left| \sum_{k=1}^N \mathbb{U}_n^k - N\Pi_{1,n} \right| > x \right] \leqslant \mathbb{P}\left[\sum_{n\geqslant 1} \left| \sum_{k=1}^N \mathbb{U}_n^k - \rho_n \mathbb{U}_{n-1}^k \right| \bar{R}_{n+1} > x \right]$$

and

$$\mathbb{P}\left[\sum_{k=1}^N \sum_{n\geqslant 1} \mathbb{U}_n^k > x \right] \leqslant \mathbb{P}\left[\sum_{n\geqslant 1} \left| \sum_{k=1}^N (\mathbb{U}_n^k - \rho_n \mathbb{U}_{n-1}^k) \right| \bar{R}_{n+1} > x/2 \right] + \mathbb{P}[N\bar{R}_1 > x/2].$$

Observe that for any $n \geqslant 1$, $\bar{R}_{n+1}$ is independent of $(\mathbb{U}_n^k - \rho_n \mathbb{U}_{n-1}^k)$. Thus for any $x > 0$,

$$\begin{aligned} \mathbb{P}\left[\sum_{n\geqslant 1} \left| \sum_{k=1}^N (\mathbb{U}_n^k - \rho_n \mathbb{U}_{n-1}^k) \right| \bar{R}_{n+1} > x \right] &\leqslant \sum_{n\geqslant 1} \mathbb{P}\left[\left| \sum_{k=1}^N (\mathbb{U}_n^k - \rho_n \mathbb{U}_{n-1}^k) \right| \bar{R}_{n+1} > x/2n^2 \right] \\ &= \sum_{n\geqslant 1} \int_{[0,\infty)} \mathrm{P}[\bar{R}_{n+1} > x/2tn^2] \mathbb{P}\left[\left| \sum_{k=1}^N (\mathbb{U}_n^k - \rho_n \mathbb{U}_{n-1}^k) \right| \in dt \right] \\ &\leqslant C_\gamma \sum_{n\geqslant 1} \int_{[0,\infty)} (x/2tn^2)^{-\gamma} \mathbb{P}\left[\left| \sum_{k=1}^N (\mathbb{U}_n^k - \rho_n \mathbb{U}_{n-1}^k) \right| \in dt \right] \\ &= 2^\gamma C_\gamma x^{-\gamma} \sum_{n\geqslant 1} n^{2\gamma} \mathbb{E} \left| \sum_{k=1}^N (\mathbb{U}_n^k - \rho_n \mathbb{U}_{n-1}^k) \right|^\gamma, \end{aligned}$$

where the second inequality follows from (5.3.5).

The relations (5.3.15) imply that for any fixed n, under P_ω, $\sum_{k=1}^N(\mathbb{U}_n^k-\rho_n\mathbb{U}_{n-1}^k)$ is a sum of independent centered variables; in particular, using formulae (5.3.15), we obtain

$$\begin{aligned}\mathrm{E}_\omega\left(\sum_{k=1}^N(\mathbb{U}_n^k-\rho_n\mathbb{U}_{n-1}^k)\right)^2 &= N\mathrm{E}_\omega(\mathbb{U}_n-\rho_n\mathbb{U}_{n-1})^2\\ &= N(\rho_n+2\rho_n^2\xi_n-\rho_n^2)\mathrm{E}_\omega\mathbb{U}_{n-1}\\ &= N(\rho_n+2\rho_n^2\xi_n-\rho_n^2)\Pi_{1,n-1}.\end{aligned}$$

Therefore, conditional Jensen's inequality and subadditivity of the function $x\mapsto x^{\gamma/2}$ (recall $\gamma\leqslant 2$) give

$$\begin{aligned}\sum_{n\geqslant 1}n^{2\gamma}\mathbb{E}\left|\sum_{k=1}^N(\mathbb{U}_n^k-\rho_n\mathbb{U}_{n-1}^k)\right|^\gamma &\leqslant \sum_{n\geqslant 1}n^{2\gamma}\mathrm{E}\left(\mathrm{E}_\omega\left(\sum_{k=1}^N(\mathbb{U}_n^k-\rho_n\mathbb{U}_{n-1}^k)\right)^2\right)^{\gamma/2}\\ &= N^{\gamma/2}\sum_{n\geqslant 1}n^{2\gamma}\mathrm{E}((\rho_n+2\rho_n^2\xi_n-\rho_n^2)\Pi_{1,n-1})^{\gamma/2}\\ &\leqslant N^{\gamma/2}\sum_{n\geqslant 1}n^{2\gamma}(\mathrm{E}\rho^{\gamma/2}+2\mathrm{E}\rho^\gamma\xi^{\gamma/2})(\mathrm{E}\rho^{\gamma/2})^{n-1}.\end{aligned}$$

The assumptions of the lemma guarantee that the series is convergent and thus for some constant $C>0$,

$$\begin{aligned}\mathbb{P}\left[\sum_{n\geqslant 1}\left|\sum_{k=1}^N(\mathbb{U}_n^k-\rho_n\mathbb{U}_{n-1}^k)\right|\bar{R}_{n+1}>x\right] &\leqslant 2^\gamma C_\gamma\, x^{-\gamma}\sum_{n\geqslant 1}n^{2\gamma}\mathbb{E}\left|\sum_{k=1}^N(\mathbb{U}_n^k-\rho_n\mathbb{U}_{n-1}^k)\right|^\gamma\\ &\leqslant CN^{\gamma/2}x^{-\gamma},\end{aligned}$$

which proves (5.3.12). Invoking (5.3.5) once again, we conclude that

$$\begin{aligned}\mathbb{P}\left[\sum_{k=1}^N\sum_{n\geqslant 1}\mathbb{U}_n^k>x\right] &\leqslant \mathbb{P}\left[\sum_{n\geqslant 1}\left(\sum_{k=1}^N(\mathbb{U}_n^k-\rho_n\mathbb{U}_{n-1}^k)\right)\bar{R}_{n+1}>x/2\right]+\mathbb{P}[N\bar{R}_1>x/2]\\ &\leqslant CN^{\gamma/2}(x/2)^{-\gamma}+C_\gamma N^\gamma(x/2)^{-\gamma},\end{aligned}$$

which proves (5.3.11).

To show (5.3.13), decompose

$$\begin{aligned}\mathbb{P}\left[\max_{j\geqslant 0}\sum_{k=1}^N U_n^k>x\right] &= \mathbb{P}\left[\max_{n\geqslant 0}\max_{S_n\leqslant j<S_{n+1}}\sum_{k=1}^N U_j^k>x\right]\\ &\leqslant \mathbb{P}\left[\sum_{n\geqslant 0}\sum_{k=1}^N\max_{S_n\leqslant j<S_{n+1}}U_j^k>x\right]\\ &\leqslant \mathbb{P}\left[\sum_{n\geqslant 0}\sum_{k=1}^N\left(\mathbb{U}_n^k+\max_{S_n\leqslant j<S_{n+1}}|U_j^k-\mathbb{U}_n^k|\right)>x\right]\\ &\leqslant \mathbb{P}\left[\sum_{k=1}^N\sum_{n\geqslant 0}\mathbb{U}_n^k>x/2\right]+\mathbb{P}\left[\sum_{n\geqslant 1}\sum_{k=1}^N\max_{S_{n-1}\leqslant j<S_n}|U_j^k-\mathbb{U}_{n-1}^k|>x/2\right],\end{aligned}$$

which means that (5.3.13) follows from (5.3.11) and (5.3.14). To show (5.3.14), note that, by Lemma 5.3.2,

$$\mathrm{E}_\omega\left[\max_{S_{n-1}\leqslant j<S_n}|U_j-\mathbb{U}_{n-1}|^2\right]\leqslant 8\xi_n\mathrm{E}_\omega\mathbb{U}^k_{n-1}=8\xi_n\Pi_{1,n-1}.$$

Therefore

$$\begin{aligned}\mathbb{P}\left[\sum_{n\geqslant1}\sum_{k=1}^N\max_{S_{n-1}\leqslant j<S_n}|U^k_j-\mathbb{U}^k_{n-1}|>x/2\right]&\leqslant\sum_{n\geqslant1}\mathbb{P}\left[\sum_{k=1}^N\max_{S_{n-1}\leqslant j<S_n}|U^k_j-\mathbb{U}^k_{n-1}|>x/4n^2\right]\\&\leqslant\sum_{n\geqslant1}(x/4n^2)^{-\gamma}N^{\gamma/2}\mathrm{E}\left(\mathrm{E}_\omega\max_{S_{n-1}\leqslant j<S_n}|U_j-\mathbb{U}_{n-1}|^2\right)^{\gamma/2}\\&\leqslant N^{\gamma/2}x^{-\gamma}\sum_{n\geqslant1}(4n)^{2\gamma}8^{\gamma/2}\mathrm{E}\xi^{\gamma/2}(\mathrm{E}\rho^{\gamma/2})^{n-1}\\&=C'N^{\gamma/2}x^{-\gamma},\end{aligned}$$

for some constant $C'>0$, which proves (5.3.13) and (5.3.14). □

Let $Y=(Y_n)_{n\in\mathbb{N}}$ be a copy of the process $(Y^1_n)_{n\in\mathbb{N}}$. That is, Y starts with one immigrant in generation 0 and for the next ξ_1-1 generations evolves as a Galton-Watson process with unit immigration and reproduction law *Geo*$(1/2)$. The last immigrant arrives in generation ξ_1-1; particles there reproduce with distribution *Geo*(λ_1), giving birth to the first marked generation $\mathbb{Y}_1=Y_{S_1}$. From there on the process evolves without immigration, with particles in each marked generation $\mathbb{Y}_n=Y_{S_n}$ being born with *Geo*(λ_n) distribution, and *Geo*$(1/2)$ in consecutive blocks of lengths given by ξ_n-1 for $n\geqslant2$.

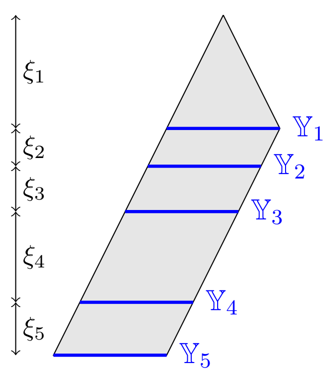


Figure 5.3.3: Schematic picture of the process Y. Horizontal blue lines represent marked generations. The immigrants arrive only in the first block.

Lemma 5.3.4. *Assume* (Γ). *Then for some constant* C_2,

$$\mathbb{P}\left[\max_{n\geqslant1}Y_n>x\right]\leqslant C_2x^{-\gamma}\left(\mathrm{E}\left(\mathrm{E}_\omega Y^2_{\xi_1-1}\right)^{\gamma/2}+\mathrm{E}\mathbb{Y}^\gamma_1\right).\tag{5.3.16}$$

If additionally $\mathrm{E}\xi^\gamma<\infty$ *and* $\mathrm{E}\xi^\gamma\rho^\gamma<\infty$, *then for some constant* C_3,

$$\mathbb{P}\left[\max_{n\geqslant1}Y_n>x\right]\leqslant C_3x^{-\gamma}.\tag{5.3.17}$$

Proof. We have

$$\mathbb{P}\left[\max_{n\geqslant 1} Y_n > x\right] \leqslant \mathbb{P}\left[\max_{n<S_1} Y_n > x\right] + \mathbb{P}\left[\max_{n\geqslant S_1} Y_n > x\right]. \tag{5.3.18}$$

For the first $\xi_1 - 1$ generations Y evolves as a Galton-Watson process with unit immigration and reproduction law $Geo(1/2)$, therefore $(Y_n^2)_{n<S_1}$ is a submartingale with respect to P_ω. Using first Markov's, then Jensen's, and finally Doob's maximal inequality, we obtain

$$\mathbb{P}\left[\max_{j<S_1} Y_j > x\right] \leqslant x^{-\gamma}\mathbb{E}\left(\max_{j<S_1} Y_j\right)^{\gamma} \leqslant x^{-\gamma}\mathrm{E}\left(\mathrm{E}_\omega \max_{n<\xi_1} Y_n^2\right)^{\gamma/2} \leqslant x^{-\gamma}\mathrm{E}\left(4\mathrm{E}_\omega Y_{\xi_1-1}^2\right)^{\gamma/2}.$$

If additionally $\mathrm{E}\xi^\gamma < \infty$, then by Lemma 5.3.2,

$$\mathrm{E}_\omega \max_{n<\xi_1} Y_n^2 \leqslant 16\xi_1^2,$$

thus

$$\mathbb{P}\left[\max_{j<S_1} Y_j > x\right] \leqslant 16^{\gamma/2}\mathrm{E}\xi^\gamma x^{-\gamma}.$$

To estimate the second term in (5.3.18), observe that

$$(Y_{S_1+j})_{j\in\mathbb{N}} \stackrel{\mathrm{d}}{=} \left(\sum_{k=1}^{\mathbb{Y}_1} U_j^k\right)_{j\in\mathbb{N}},$$

where U^k's are (independent under P_ω) copies of the process U, independent of $\mathbb{Y}_1$ under $\mathbb{P}$. By Lemma 5.3.3,

$$\mathbb{P}\left[\max_{n\geqslant S_1} Y_n > x\right] \leqslant C_1\mathbb{E}\mathbb{Y}_1^\gamma x^{-\gamma},$$

which concludes the proof of the first part of the lemma. If $\mathrm{E}\xi^\gamma\rho^\gamma < \infty$, we may estimate $\mathbb{E}\mathbb{Y}_1^\gamma$. Under P_ω,

$$\mathbb{Y}_1 \stackrel{\mathrm{d}}{=} \sum_{k=1}^{Y_{\xi_1-1}+1} G_k,$$

where $G_k \sim Geo(\lambda_1)$ are independent of Y_{ξ_1-1} and each other. Moreover, as was explained in the proof of Lemma 5.3.2, $Y_{\xi_1-1} \sim Geo(1/\xi_1)$ under P_ω. Therefore

$$\mathrm{E}_\omega\mathbb{Y}_1^2 = \mathrm{E}_\omega\left[(Y_{\xi_1-1}+1)(2\rho_1^2+\rho_1) + (Y_{\xi_1-1}^2 + Y_{\xi_1-1})\rho_1^2\right] = 2\xi_1^2\rho_1^2 + \xi_1\rho_1.$$

Jensen's inequality and subadditivity of function $x \mapsto x^{\gamma/2}$ give

$$\mathbb{E}\mathbb{Y}_1^\gamma \leqslant \mathrm{E}\left(\mathrm{E}_\omega\mathbb{Y}_1^2\right)^{\gamma/2} \leqslant 2^{\gamma/2}\mathrm{E}\xi^\gamma\rho^\gamma + \mathrm{E}\xi^{\gamma/2}\rho^{\gamma/2} < \infty,$$

which proves (5.3.17).

□

5.4 Proof of Theorem 5.2.1

In the proof of Theorem 5.2.1 we will use the fact that the extinctions divide process Z into independent epochs. That is, we first determine tail asymptotics of the maximum up to time S_{τ_1}.

For any $A > 0$ denote $\sigma = \sigma(A) = \inf\{n : \mathbb{Z}_n \geqslant A\}$. The next lemma is an analogue of Lemma 4 in [18] and can be proved the very same way, that is by examining $\mathrm{E}_\omega[\mathbb{Z}_k^\alpha|\mathbb{Z}_{k-1}]$ using methods we've seen in previous proofs.

Lemma 5.4.1. *For any fixed $A > 0$, $0 < \mathbb{E}[\mathbb{Z}_\sigma^\alpha \mathbb{1}_{\sigma<\tau_1}] < \infty$.*

The main proof strategy is as follows: we choose sufficiently big A and argue that neither the particles living before time S_σ, nor the descendants of the immigrants arriving after this time contribute significantly to the examined maximum. Therefore its behavior is determined by $\mathbb{Z}_\sigma$ particles in the generation S_σ and their progeny.

Let us first take care of the particles alive before time S_σ.

Lemma 5.4.2. *For any fixed A,*

$$\mathbb{P}\left[\max_{n<S_\sigma\wedge S_{\tau_1}} Z_n > x\right] = o(x^{-\alpha}).$$

Proof. Fix A and let $x > A$. The only generations before time S_σ in which the population size may exceed x are the unmarked ones. However, since $\mathbb{Z}_k < A$ for $k < \sigma$, the maximum of Z in generations $S_{k-1}+1, \dots S_k - 1$ is stochastically dominated by M_k^A, the maximum of Galton-Watson process with $Geo(1/2)$ offspring distribution, unit immigration and A initial particles, evolving for time ξ_k. Observe that

$$\begin{aligned}\mathbb{P}\left[\max_{n<S_\sigma\wedge S_{\tau_1}} Z_n > x\right] &\leqslant \mathbb{P}\left[\max_{k<x^{\delta/2}} M_k^A > x\right] + \mathbb{P}\left[\tau_1 > x^{\delta/2}\right] \\ &\leqslant x^{\delta/2}\mathbb{P}\left[M_1^A > x\right] + \mathbb{P}\left[\tau_1 > x^{\delta/2}\right].\end{aligned}$$

Since $\alpha + \delta \leqslant 2$, by Markov's and Jensen's inequalities,

$$\mathbb{P}\left[M_1^A > x\right] \leqslant x^{-\alpha-\delta}\mathrm{E}\left(\mathrm{E}_\omega(M_1^A)^2\right)^{(\alpha+\delta)/2}.$$

Lemma 5.3.2 implies that

$$\mathrm{E}_\omega(M_1^A)^2 \leqslant 16(\xi_1^2 + A\xi_1 + A^2)$$

and thus, since $x \mapsto x^{(\alpha+\delta)/2}$ is subadditive,

$$x^{\delta/2}\mathbb{P}\left[M_1^A > x\right] \leqslant x^{-\alpha-\delta/2}16^{(\alpha+\delta)/2}\left(\mathrm{E}\xi^{\alpha+\delta} + A^{(\alpha+\delta)/2}\mathrm{E}\xi^{(\alpha+\delta)/2} + A^{\alpha+\delta}\right) = o(x^{-\alpha}).$$

The second term may be bounded using Lemma 5.3.1, that is

$$\mathbb{P}\left[\tau_1 > x^{\delta/2}\right] \leqslant e^{-cx^{\delta/2}}\mathbb{E}e^{c\tau_1} = o(x^{-\alpha}),$$

which ends the proof. □

The next lemma assures that the contribution of progeny of immigrants arriving after S_σ is negligible. Recall that Y^k counts the progeny of immigrants arriving in k'th block, that is in generations $S_{k-1}, S_{k-1}+1, \dots S_k - 1$.

Lemma 5.4.3. *Fix $\varepsilon > 0$. There exists $A_1(\varepsilon)$ such that for $A > A_1(\varepsilon)$,*

$$\mathbb{P}\left[\sum_{k=\sigma+1}^{\tau_1} \max_{n\geqslant 1} Y_n^k > \varepsilon x\right] \leqslant \varepsilon x^{-\alpha}. \tag{5.4.1}$$

Proof. We have

$$\begin{aligned}\mathbb{P}\left[\sum_{k=\sigma+1}^{\tau_1} \max_{n\geqslant 1} Y_n^k > \varepsilon x\right] &= \mathbb{P}\left[\sum_{k=1}^{\infty} \mathbb{1}_{\sigma\leqslant k<\tau_1} \max_{n\geqslant 1} Y_n^{k+1} > \varepsilon x\right] \\ &\leqslant \sum_{k=1}^{\infty} \mathbb{P}\left[\sigma \leqslant k < \tau_1, \max_{n\geqslant 1} Y_n^{k+1} > \varepsilon x/2k^2\right].\end{aligned}$$

Observe that the event $\{\sigma \leqslant k < \tau_1\}$ is defined in terms of $Z_1, \dots Z_{S_k}$, while the process Y^{k+1} evolves in the environment given by (ξ_j, ρ_j) for $j \geqslant k+1$, hence is independent of $Z_1, \dots Z_{S_k}$. Moreover, the second part of Lemma 5.3.4 applied with $\gamma = \alpha$ gives tail bounds on the maximum of Y^{k+1}. That is,

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbb{P}\left[\sigma \leqslant k < \tau_1, \max_{n\geqslant 1} Y_n^{k+1} > \varepsilon x/2k^2\right] &= \sum_{k=1}^{\infty} \mathbb{P}\left[\sigma \leqslant k < \tau_1\right] \mathbb{P}\left[\max_{n\geqslant 1} Y_n^{k+1} > \varepsilon x/2k^2\right] \\ &\leqslant C_3 \sum_{k=1}^{\infty} \mathbb{P}\left[\sigma \leqslant k < \tau_1\right] (\varepsilon x/2k^2)^{-\alpha} \\ &= C_3 2^\alpha (\varepsilon x)^{-\alpha} \sum_{k=1}^{\infty} k^{2\alpha} \mathbb{P}\left[\tau_1 \mathbb{1}_{\sigma<\tau_1} > k\right] \\ &= C_3 2^\alpha (2\alpha+1)^{-1} \varepsilon^{-\alpha} x^{-\alpha} \mathbb{E}\left[\tau_1^{2\alpha+1} \mathbb{1}_{\sigma<\tau_1}\right]\end{aligned}$$

Since $\mathbb{E}\tau_1^{2\alpha+1} < \infty$ and $\sigma(A) \xrightarrow{\mathbb{P}} \infty$ as $A \to \infty$, one may find $A_1(\varepsilon)$ such that for $A > A_1(\varepsilon)$ (5.4.1) holds.

□

We already gave bounds on the generations sizes of particles alive before time S_σ and those coming from immigrants arriving after that time. What is left is investigating behaviour of the particles residing exactly in generation S_σ and their progeny.

For $k \geqslant S_\sigma$ let $V_{\sigma,k}$ be the number of progeny of the particles from generation S_σ residing in generation k and let $\mathbb{V}_{\sigma,n} = V_{\sigma,S_n}$; in particular, $\mathbb{Z}_\sigma = \mathbb{V}_{\sigma,\sigma}$. Recall the variables $\Psi_{m,k}$ defined in (5.3.6).

Lemma 5.4.4. *For any $\varepsilon > 0$ there exists $A_2(\varepsilon)$ such that for $A > A_2(\varepsilon)$,*

$$\mathbb{P}\left[\left|\max_{k\geqslant S_\sigma}(V_{\sigma,k} + V_{\sigma,k+1}) - \mathbb{Z}_\sigma \max_{k\geqslant S_\sigma}(\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1})\right| > \varepsilon x, \sigma < \tau_1\right] \leqslant \varepsilon x^{-\alpha} \mathbb{E}\left[\mathbb{Z}_\sigma^\alpha \mathbb{1}_{\sigma<\tau_1}\right].$$

Proof. We begin by estimating the difference of maxima within one block. Observe that the potential Ψ is constant within each block, therefore for any $n \in \mathbb{N}$,

$$\begin{aligned}&\left|\max_{S_n \leqslant k < S_{n+1}} (V_{\sigma,k} + V_{\sigma,k+1}) - \mathbb{Z}_\sigma \max_{S_n \leqslant k < S_{n+1}} (\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1})\right| \\ &\leqslant \left|\max_{S_n \leqslant k < S_{n+1}-1} (V_{\sigma,k} + V_{\sigma,k+1}) - 2\mathbb{Z}_\sigma \Pi_{\sigma+1,n}\right| \\ &+ |V_{\sigma,S_{n+1}-1} + V_{\sigma,S_{n+1}} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n+1}|\end{aligned}$$

Let us estimate the first ingredient. Since

$$\max_{S_n \leqslant k < S_{n+1}-1} (V_{\sigma,k} + V_{\sigma,k+1}) = 2\mathbb{V}_{\sigma,n} + \max_{S_n \leqslant k < S_{n+1}-1} (V_{\sigma,k} + V_{\sigma,k+1} - 2\mathbb{V}_{\sigma,n}),$$

we have

$$\left|\max_{S_n \leqslant k < S_{n+1}-1} (V_{\sigma,k} + V_{\sigma,k+1}) - 2\mathbb{Z}_\sigma \Pi_{\sigma+1,n}\right| \leqslant 2\left(|\mathbb{V}_{\sigma,n} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n}| + \max_{S_n \leqslant k < S_{n+1}} |V_{\sigma,k} - \mathbb{V}_{\sigma,n}|\right).$$

The second ingredient may be estimated simply by

$$\begin{aligned}&|V_{\sigma,S_{n+1}-1} + V_{\sigma,S_{n+1}} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n+1}| \\ &\leqslant |\mathbb{V}_{\sigma,n+1} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n+1}| + |\mathbb{V}_{\sigma,n} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n}| + |V_{\sigma,S_{n+1}-1} - \mathbb{V}_{\sigma,n}|,\end{aligned}$$

which gives

$$\begin{aligned}&\left|\max_{S_n \leqslant k < S_{n+1}} (V_{\sigma,k} + V_{\sigma,k+1}) - \mathbb{Z}_\sigma \max_{S_n \leqslant k < S_{n+1}} (\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1})\right| \\ &\leqslant 3|\mathbb{V}_{\sigma,n} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n}| + 3 \max_{S_n \leqslant k < S_{n+1}} |V_{\sigma,k} - \mathbb{V}_{\sigma,n}| + |\mathbb{V}_{\sigma,n+1} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n+1}|.\end{aligned}$$

Next, in view of

$$\begin{aligned}&\left|\max_{k \geqslant S_\sigma} (V_{\sigma,k} + V_{\sigma,k+1}) - \mathbb{Z}_\sigma \max_{k \geqslant S_\sigma} (\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1})\right| \\ &= \left|\max_{n \geqslant \sigma} \max_{S_n \leqslant k < S_{n+1}} (V_{\sigma,k} + V_{\sigma,k+1}) - \max_{n \geqslant \sigma} \mathbb{Z}_\sigma \max_{S_n \leqslant k < S_{n+1}} (\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1})\right| \\ &\leqslant \sum_{n \geqslant \sigma} \left|\max_{S_n \leqslant k < S_{n+1}} (V_{\sigma,k} + V_{\sigma,k+1}) - \mathbb{Z}_\sigma \max_{S_n \leqslant k < S_{n+1}} (\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1})\right|,\end{aligned}$$

the above estimations give

$$\begin{aligned}&\mathbb{P}\left[\left|\max_{k \geqslant S_\sigma} (V_{\sigma,k} + V_{\sigma,k+1}) - \mathbb{Z}_\sigma \max_{k \geqslant S_\sigma} (\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1})\right| > \varepsilon x, \sigma < \tau_1\right] \\ &\leqslant \mathbb{P}\left[4 \sum_{n \geqslant \sigma} |\mathbb{V}_{\sigma,n} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n}| > \varepsilon x/2, \sigma < \tau_1\right] \\ &+ \mathbb{P}\left[3 \sum_{n \geqslant \sigma} \max_{S_n \leqslant k < S_{n+1}} |V_{\sigma,k} - \mathbb{V}_{\sigma,n}| > \varepsilon x/2, \sigma < \tau_1\right].\end{aligned}$$

Both ingredients can be estimated by Lemma 5.3.3 applied with $\gamma = \alpha$. Conditioned on $(\sigma, Z_1, \dots Z_{S_\sigma})$, the process $(V_{\sigma,n})_{n \geqslant S_\sigma}$ is a sum of $\mathbb{Z}_\sigma$ independent copies of the process U. We have, on the set $\{\sigma < \tau_1\}$,

$$\mathbb{P}\left[4 \sum_{n \geqslant \sigma} |\mathbb{V}_{\sigma,n} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n}| > \varepsilon x/2 \,\middle|\, \sigma, Z_1, \dots Z_{S_\sigma}\right] \leqslant C_1 (\varepsilon x/8)^{-\alpha} \mathbb{Z}_\sigma^{\alpha/2},$$

which gives

$$\mathbb{P}\left[4 \sum_{n \geqslant \sigma} |\mathbb{V}_{\sigma,n} - \mathbb{Z}_\sigma \Pi_{\sigma+1,n}| > \varepsilon x/2, \sigma < \tau_1\right] \leqslant C_1 8^\alpha (\varepsilon x)^{-\alpha} \mathbb{E}\left[\mathbb{Z}_\sigma^{\alpha/2} \mathbb{1}_{\sigma < \tau_1}\right].$$

Similarly,

$$\mathbb{P}\left[3 \sum_{n \geqslant \sigma} \max_{S_n < k < S_{n+1}} |V_{\sigma,k} - \mathbb{V}_{\sigma,n}| > \varepsilon x/2, \sigma < \tau_1\right] \leqslant C_1 6^\alpha (\varepsilon x)^{-\alpha} \mathbb{E}\left[\mathbb{Z}_\sigma^{\alpha/2} \mathbb{1}_{\sigma < \tau_1}\right].$$

Therefore, for some constant C_2,

$$\mathbb{P}\left[\left|\max_{k \geqslant S_\sigma} (V_{\sigma,k} + V_{\sigma,k+1}) - \mathbb{Z}_\sigma \max_{k \geqslant S_\sigma} (\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1})\right| > \varepsilon x, \sigma < \tau_1\right] \\ \leqslant C_2 (\varepsilon x)^{-\alpha} \mathbb{E}\left[\mathbb{Z}_\sigma^{\alpha/2} \mathbb{1}_{\sigma < \tau_1}\right].$$

Finally, for any fixed $\varepsilon > 0$, since $\mathbb{Z}_\sigma \geqslant A$, we have

$$\mathbb{E}\left[\mathbb{Z}_\sigma^{\alpha/2} \mathbb{1}_{\sigma < \tau_1}\right] \leqslant A^{-\alpha/2} \mathbb{E}\left[\mathbb{Z}_\sigma^{\alpha} \mathbb{1}_{\sigma < \tau_1}\right]$$

and one may choose $A_2(\varepsilon)$ large enough for the claim to hold.

□

Lemma 5.4.5. *There exists $c_\Psi \in (0, \infty)$ such that for any fixed $A > 0$,*

$$\mathbb{P}\left[\mathbb{Z}_\sigma \max_{k \geqslant S_\sigma} (\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1}) > x, \sigma < \tau_1\right] \sim c_\Psi \mathbb{E}\left[\mathbb{Z}_\sigma^\alpha \mathbb{1}_{\sigma < \tau_1}\right] x^{-\alpha}. \tag{5.4.2}$$

Proof. Since the sequence $\Psi_{\sigma+1,k}$ is constant on the blocks between marked points, we have

$$\max_{k \geqslant S_\sigma} (\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1}) = \max_{n \geqslant \sigma} \left(2 \, \mathbb{1}_{\xi_{n+1} > 1} \vee (1 + \rho_{n+1})\right) \Pi_{\sigma+1,n}.$$

Observe that

$$\log\left(\left(2 \, \mathbb{1}_{\xi_{n+1} > 1} \vee (1 + \rho_{n+1})\right) \Pi_{1,n}\right) = \sum_{k=1}^{n} \log(\rho_k) + \log(2 \, \mathbb{1}_{\xi_{n+1} > 1} \vee (1 + \rho_{n+1}))$$

is a perturbed random walk. By Theorem 1.3.8 in [17], assumptions (A) guarantee that

$$\mathbb{P}\left[\max_{n \geqslant 0} (2 \, \mathbb{1}_{\xi_{n+1} > 1} \vee (1 + \rho_{n+1})) \Pi_{1,n} > x\right] \sim c_\Psi x^{-\alpha}$$

for a constant $c_\Psi \in (0,\infty)$ given by

$$c_\Psi = \mathrm{E}(2^\alpha \mathbb{1}_{\xi_1>1} \vee (1+\rho_1)^\alpha - \max_{n\geqslant 2}(2^\alpha \mathbb{1}_{\xi_{n+1}>1} \vee (1+\rho_{n+1})^\alpha)\Pi_{1,n}^\alpha)_+.$$

Note that the variables $\mathbb{Z}_\sigma \mathbb{1}_{\sigma<\tau_1}$ and $\max_{n\geqslant\sigma}(2\,\mathbb{1}_{\xi_{n+1}>1} \vee (1+\rho_{n+1}))\Pi_{\sigma+1,n}$ are independent under $\mathbb{P}$. Therefore, by Breiman's lemma,

$$\mathbb{P}\left[\mathbb{Z}_\sigma \max_{k\geqslant S_\sigma}(\Psi_{\sigma+1,k} + \Psi_{\sigma+1,k+1}) > x, \sigma < \tau_1\right]$$
$$= \mathbb{P}\left[\mathbb{Z}_\sigma \mathbb{1}_{\sigma<\tau_1} \cdot \max_{n\geqslant\sigma}(2\,\mathbb{1}_{\xi_{n+1}>1} \vee (1+\rho_{n+1}))\Pi_{\sigma+1,n} > x\right] \sim \mathbb{E}\left[\mathbb{Z}_\sigma^\alpha \mathbb{1}_{\sigma<\tau_1}\right] c_\Psi x^{-\alpha}.$$

□

The rest of the proof is standard. First, all the lemmas proven so far allow us to determine the asymptotics of the maximum in time $[0, S_{\tau_1})$. Then we use the fact that the extinctions divide our process into independent pieces.

Proposition 5.4.6. *For some constant $c_M > 0$,*

$$\mathbb{P}\left[\max_{0\leqslant n<S_{\tau_1}}(Z_n + Z_{n+1}) > x\right] \sim c_M x^{-\alpha}.$$

Proof. Fix $\varepsilon > 0$ and take $A > A(\varepsilon) := \max\{A_1(\varepsilon), A_2(\varepsilon)\}$. First, observe that

$$\mathbb{P}\left[\max_{S_\sigma\leqslant n<S_{\tau_1}}(Z_n + Z_{n+1}) > x, \sigma < \tau_1\right] \leqslant \mathbb{P}\left[\max_{0\leqslant n<S_{\tau_1}}(Z_n + Z_{n+1}) > x\right]$$
$$\leqslant \mathbb{P}\left[\max_{S_\sigma\leqslant n<S_{\tau_1}}(Z_n + Z_{n+1}) > x, \sigma < \tau_1\right] + \mathbb{P}\left[\max_{n<S_\sigma\wedge S_{\tau_1}}(Z_n + Z_{n+1}) > x\right].$$

Lemma 5.4.2 ensures that for large enough x,

$$\mathbb{P}\left[\max_{n<S_\sigma\wedge S_{\tau_1}}(Z_n + Z_{n+1}) > x\right] \leqslant \mathbb{P}\left[2\max_{n<S_\sigma\wedge S_{\tau_1}} Z_n > x\right] \leqslant \varepsilon x^{-\alpha}.$$

Recall that by $Y^k = (Y_j^k)_{j\in\mathbb{Z}}$ we denoted the process counting the progeny of immigrants arriving in k'th block, with the convention $Y_j^k = 0$ for $j < 0$. For $n \geqslant S_\sigma$,

$$Z_n = V_{\sigma,n} + \sum_{k=\sigma+1}^{\tau_1} Y_{n-S_{k-1}}^k,$$

thus

$$\mathbb{P}\left[\max_{S_\sigma\leqslant n<S_{\tau_1}}(V_{\sigma,n} + V_{\sigma,n+1}) > x, \sigma < \tau_1\right] \leqslant \mathbb{P}\left[\max_{S_\sigma\leqslant n<S_{\tau_1}}(Z_n + Z_{n+1}) > x, \sigma < \tau_1\right]$$
$$\leqslant \mathbb{P}\left[\max_{S_\sigma\leqslant n<S_{\tau_1}}(V_{\sigma,n} + V_{\sigma,n+1}) > (1-\varepsilon)x, \sigma < \tau_1\right] + \mathbb{P}\left[2\sum_{k=\sigma+1}^{\tau_1}\max_{n\geqslant 1} Y_n^k > \varepsilon x\right]$$

and (5.4.1) ensures that

$$\mathbb{P}\left[2\sum_{k=\sigma+1}^{\tau_1}\max_{n\geqslant 1}Y_n^k > \varepsilon x\right] \leqslant \varepsilon x^{-\alpha}.$$

Finally,

$$\begin{aligned}
&\mathbb{P}\left[\mathbb{Z}_\sigma \max_{k\geqslant S_\sigma}(\Psi_{\sigma,k}+\Psi_{\sigma,k+1}) > (1+\varepsilon)x, \sigma<\tau_1\right] \\
&\quad - \mathbb{P}\left[\left|\max_{k\geqslant S_\sigma}(V_{\sigma,k}+V_{\sigma,k+1}) - \mathbb{Z}_\sigma \max_{k\geqslant S_\sigma}(\Psi_{\sigma,k}+\Psi_{\sigma,k+1})\right| > \varepsilon x, \sigma<\tau_1\right] \\
&\quad \leqslant \mathbb{P}\left[\max_{S_\sigma\leqslant n<S_{\tau_1}}(V_{\sigma,n}+V_{\sigma,n+1}) > x, \sigma<\tau_1\right] \\
&\quad \leqslant \mathbb{P}\left[\mathbb{Z}_\sigma \max_{k\geqslant S_\sigma}(\Psi_{\sigma,k}+\Psi_{\sigma,k+1}) > (1-\varepsilon)x, \sigma<\tau_1\right] \\
&\quad + \mathbb{P}\left[\left|\max_{k\geqslant S_\sigma}(V_{\sigma,k}+V_{\sigma,k+1}) - \mathbb{Z}_\sigma \max_{k\geqslant S_\sigma}(\Psi_{\sigma,k}+\Psi_{\sigma,k+1})\right| > \varepsilon x, \sigma<\tau_1\right],
\end{aligned}$$

and by Lemma 5.4.4,

$$\mathbb{P}\left[\left|\max_{k\geqslant S_\sigma}(V_{\sigma,k}+V_{\sigma,k+1}) - \mathbb{Z}_\sigma \max_{k\geqslant S_\sigma}(\Psi_{\sigma,k}+\Psi_{\sigma,k+1})\right| > \varepsilon x, \sigma<\tau_1\right] \leqslant \varepsilon x^{-\alpha}\mathbb{E}\left[\mathbb{Z}_\sigma^\alpha \mathbb{1}_{\sigma<\tau}\right].$$

Putting things together and invoking Lemma 5.4.5 we get that for any $\varepsilon > 0$ such that $\varepsilon(1-\varepsilon)^\alpha < c_\Psi$ and for any $A > A(\varepsilon)$,

$$\begin{aligned}
0 &< ((1+\varepsilon)^{-\alpha}c_\Psi - \varepsilon)\mathbb{E}\left[\mathbb{Z}_\sigma^\alpha \mathbb{1}_{\sigma<\tau_1}\right] \\
&\leqslant \liminf_{x\to\infty} x^\alpha \mathbb{P}\left[\max_{0\leqslant n<S_{\tau_1}}(Z_n+Z_{n+1}) > x\right] \leqslant \limsup_{x\to\infty} x^\alpha \mathbb{P}\left[\max_{0\leqslant n<S_{\tau_1}}(Z_n+Z_{n+1}) > x\right] \\
&\leqslant ((1-2\varepsilon)^{-\alpha}c_\Psi + \varepsilon)\mathbb{E}\left[\mathbb{Z}_\sigma^\alpha \mathbb{1}_{\sigma<\tau_1}\right] + 2\varepsilon < \infty.
\end{aligned}$$

Observe that this relation implies that both the limits

$$\lim_{x\to\infty} x^\alpha \mathbb{P}\left[\max_{0\leqslant n<S_{\tau_1}}(Z_n+Z_{n+1}) > x\right] \quad \text{and} \quad \lim_{A\to\infty}\mathbb{E}\left[\mathbb{Z}_{\sigma(A)}^\alpha \mathbb{1}_{\sigma(A)<\tau_1}\right]$$

exist, are positive and satisfy

$$\lim_{x\to\infty} x^\alpha \mathbb{P}\left[\max_{0\leqslant n<S_{\tau_1}}(Z_n+Z_{n+1}) > x\right] = c_\Psi \lim_{A\to\infty}\mathbb{E}\left[\mathbb{Z}_{\sigma(A)}^\alpha \mathbb{1}_{\sigma(A)<\tau_1}\right] =: c_M.$$

□

Due to Lemma 5.2.3 and the relation (5.3.2), the next result implies Theorem 5.2.1.

Theorem 5.4.7. *Under assumptions* (*A*),

$$\mathbb{P}\left[n^{-1/\alpha}\max_{0\leqslant k<S_n}(Z_k+Z_{k+1}) > x\right] \xrightarrow{n\to\infty} 1-\exp\left(-\frac{c_M}{\mathbb{E}\tau_1}x^{-\alpha}\right).$$

Proof. Since the extinctions divide the process Z into independent epochs, an immediate corollary of Proposition 5.4.6 is that

$$\mathbb{P}\left[n^{-1/\alpha}\max_{0\leqslant k<S_{\tau_n}}(Z_k+Z_{k+1})>x\right]\xrightarrow{n\to\infty}1-\exp(-c_M x^{-\alpha}).$$

Lemma 5.3.1 implies that $\mathbb{E}\tau_1<\infty$. Therefore passing from the maximum up to time S_{τ_n} to the maximum up to S_n may be done exactly as in the proof of Lemma 5.2.3. □

5.5 Proof of Theorem 5.2.2

As we have seen in the proof of Theorem 5.2.1, the limiting behaviour of maxima in case (A) comes from the tail asymptotics of variable M_Ψ defined in (5.3.7). The assumption $\mathrm{E}\xi^{\alpha+\delta}<\infty$ implies that for every k, $\max_{j<\xi_k}Y_j^k$ is negligible. In terms of the random walk, this means that the time the walker spends in a block when crossing it for the first time is negligible. As we will see, under assumptions (B) it is not; the maximal local time is obtained when the walker crosses a particularly long block for the first time, by their visits to sites within this block and potentially excursions to the left.

Consider a simple symmetric random walk on $\mathbb{Z}$ and denote by $\bar{L}_k(n)$ the number of times the walk visits site k before reaching n. Consider $(\bar{L}_s(n))_{s\in[0,n]}$ being a piecewise linear interpolation of $(\bar{L}_k(n))_{0\leqslant k\leqslant n}$. The Ray-Knight theorem (see [11, Theorem 2.15]) states that

$$\left(\frac{1}{n}\bar{L}_{n(1-t)}(n)\right)_{t\in[0,1]}\Rightarrow(B_t)_{t\in[0,1]}$$

in $C[0,1]$ as $n\to\infty$, where B is a squared Bessel process which may be defined as

$$B_t=||W(t)||^2, \tag{5.5.1}$$

for $W(t)=(W_1(t),W_2(t))$ being a standard two-dimensional Brownian motion with $W(0)=0$. By the continuous mapping theorem,

$$\left(\frac{1}{n}\max_{k\leqslant n}\bar{L}_k(n),\frac{1}{n}\bar{L}_0(n)\right)\Rightarrow(M_B,B(1)), \tag{5.5.2}$$

where $M_B=\sup\{B_t:t\in[0,1]\}$.

With this at hand, we may inspect the maximal local time that the RWSRE obtains when crossing a (long) block between marked points for the first time. To this end, consider a walk starting at 0 in the environment that has marked points only on the non-positive half-line, and stop it when it reaches point N. By Ray-Knight theorem, the limit of maximal local time in the interval $[1,N]$, where the walk is symmetric, scaled by N, is M_B. As we have seen in the proof of Theorem 5.2.1, the number of visits in the negative half-line should be controlled by the number of visits to 1 and the maxima of the potential Ψ.

In the associated branching process, the steps of the walk during its first crossing of a block between marked points are counted by the process Y. Therefore our goal is to understand the growth of maximal generation in the process Y as the size of the first block – in which the immigrants arrive – tends to infinity. To this end, for any $N \in \mathbb{N}$ let $Y^{(N)}$ be a BPSRE evolving in an environment with fixed $\xi_1 = N$ and such that the immigrants arrive only in generations up to $N-1$'th.

Lemma 5.5.1. *Under assumptions (B),*

$$\frac{1}{N}\max_{k\geqslant 0}(Y_k^{(N)} + Y_{k+1}^{(N)}) \Rightarrow M_\infty \quad \text{as } N\to\infty, \tag{5.5.3}$$

where $M_\infty \stackrel{d}{=} \max(M_B, B(1)M_\Psi/2)$ *and* M_Ψ *is a copy of the variable defined in* (5.3.7) *independent of the Bessel process* B.

Proof. To simplify the notation we shall write Y instead of $Y^{(N)}$. Observe that (5.5.2) and the duality between branching process and random walk imply

$$\left(\frac{1}{N}\max_{k\leqslant N-2}(Y_k+Y_{k+1}), \frac{1}{N}(Y_{N-1}+Y_{N-2})\right) \Rightarrow (M_B, B(1)).$$

However, since the particles in generation $N-1$ are children of those from $N-2$'th and an immigrant, born with distribution $Geo(1/2)$, we have

$$\mathbb{E}\,(Y_{N-1}-Y_{N-2}-1)^2 = \mathbb{E}(Y_{N-1}-\mathbb{E}\,[Y_{N-1}\,|\,Y_{N-2}])^2 = 2(\mathbb{E}Y_{N-2}+1) = 2(N-1),$$

which, together with Chebyshev's inequality, implies that $(Y_{N-1}-Y_{N-2})/N \xrightarrow{\mathbb{P}} 0$ and thus

$$\left(\frac{1}{N}\max_{k\leqslant N-2}(Y_k+Y_{k+1}), \frac{Y_{N-1}}{N}\right) \Rightarrow (M_B, B(1)/2).$$

Moreover, the variables Y_k for $k\leqslant N-1$ are independent of the environment, in particular of $\Psi_{1,n}, n\geqslant 0$.

From here on we proceed as in the proof of Lemma 5.4.4, to show that the maximum in generations after $N-1$'th is comparable with $Y_{N-1}M_\Psi$. That is, we use Lemma 5.3.3 applied with $\gamma=\beta$ to obtain, for some constant $C>0$,

$$\mathbb{P}\left[\left|\max_{k\geqslant N}(Y_k+Y_{k+1}) - \mathbb{Y}_1\max_{k\geqslant N}(\Psi_{2,k}+\Psi_{2,k+1})\right| > x\right] \leqslant Cx^{-\beta}\mathbb{E}\mathbb{Y}_1^{\beta/2} \tag{5.5.4}$$

for any $x>0$. The particles in the first marked generation $S_1=N$ are born with distribution $Geo(\lambda_1)$ from those counted by Y_{N-1} and an immigrant. Therefore we have $\mathrm{E}_\omega\mathbb{Y}_1 = N\rho_1$, and by Jensen's inequality,

$$\mathbb{E}\mathbb{Y}_1^{\beta/2} \leqslant N^{\beta/2}\mathrm{E}\rho^{\beta/2}.$$

Moreover, we may calculate quenched moments of $\mathbb{Y}_1$ conditioned on Y_{N-1} to get an analogue of (5.3.15). We obtain

$$\begin{aligned}\mathbb{E}\,|\mathbb{Y}_1-\rho_1Y_{N-1}|^\beta &\leqslant \mathrm{E}\left(\mathrm{E}_\omega(\mathbb{Y}_1-\rho_1Y_{N-1})^2\right)^{\beta/2} \\ &= \mathrm{E}\left((\mathrm{E}_\omega Y_{N-1}(\rho_1^2+\rho_1)+2\rho_1^2+\rho_1\right)^{\beta/2} \\ &\leqslant (N^{\beta/2}+1)(2^{\beta/2}\mathrm{E}\rho^\beta+\mathrm{E}\rho^{\beta/2}),\end{aligned} \tag{5.5.5}$$

where the last inequality follows from subadditivity of $x \mapsto x^{\beta/2}$ and the fact that $\mathrm{E}_\omega Y_{N-1} = N-1$. Observe that $\max_{k\geqslant N}(\Psi_{2,k}+\Psi_{2,k+1}) \leqslant 2+M_{\Psi,2}$ and by (5.3.8), $\mathrm{E}M_\Psi^\beta < \infty$. Therefore, since $(Y_{N-1}, \mathbb{Y}_1, \rho_1)$ is independent of $(\rho_j)_{j\geqslant 2}$, we have

$$\begin{aligned} &\mathbb{P}\left[\left|\mathbb{Y}_1 \max_{k\geqslant N}(\Psi_{2,k}+\Psi_{2,k+1}) - \rho_1 Y_{N-1}\max_{k\geqslant N}(\Psi_{2,k}+\Psi_{2,k+1})\right| > x\right] \\ &\quad\leqslant x^{-\beta}\mathrm{E}(2+M_\Psi)^\beta \mathbb{E}|\mathbb{Y}_1 - \rho_1 Y_{N-1}|^\beta \leqslant C' x^{-\beta}(N^{\beta/2}+1) \end{aligned} \tag{5.5.6}$$

for some constant $C' > 0$ and any $x > 0$.

Observe that (5.5.4) and (5.5.6) imply that for any fixed $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{P}\left[\left|\max_{k\geqslant N}(Y_k+Y_{k+1}) - Y_{N-1}\max_{k\geqslant N}(\Psi_{1,k}+\Psi_{1,k+1})\right| > \varepsilon N\right] \\ &\leqslant \mathbb{P}\left[\left|\max_{k\geqslant N}(Y_k+Y_{k+1}) - \mathbb{Y}_1\max_{k\geqslant N}(\Psi_{2,k}+\Psi_{2,k+1})\right| > \varepsilon N/2\right] \\ &+ \mathbb{P}\left[\left|\mathbb{Y}_1\max_{k\geqslant N}(\Psi_{2,k}+\Psi_{2,k+1}) - \rho_1 Y_{N-1}\max_{k\geqslant N}(\Psi_{2,k}+\Psi_{2,k+1})\right| > \varepsilon N/2\right] \\ &\leqslant (\varepsilon N/2)^{-\beta}\left(CN^{\beta/2}\mathrm{E}\rho^{\beta/2} + C'(N^{\beta/2}+1)\right) = O(N^{-\beta/2}). \end{aligned}$$

Finally, by (5.5.5), for any $\varepsilon > 0$,

$$\mathbb{P}\left[|\mathbb{Y}_1 - \rho_1 Y_{N-1}| > \varepsilon N\right] \leqslant \varepsilon^{-\beta}(N^{-\beta/2}+N^{-\beta})(2^{\beta/2}\mathrm{E}\rho^\beta + \mathrm{E}\rho^{\beta/2}) = O(N^{-\beta/2}),$$

therefore the weak limit of

$$\frac{1}{N}\max_{k\geqslant 0}(Y_k+Y_{k+1}) = \frac{1}{N}\max\left(\max_{k\leqslant N-2}(Y_k+Y_{k+1}), Y_{N-1}+\mathbb{Y}_1, \max_{k\geqslant N}(Y_k+Y_{k+1})\right)$$

is the same as that of

$$\begin{aligned} &\frac{1}{N}\max\left(\max_{k\leqslant N-2}(Y_k+Y_{k+1}), Y_{N-1}(1+\rho_1), Y_{N-1}\max_{k\geqslant N}(\Psi_{1,k}+\Psi_{1,k-1})\right) \\ &\qquad\qquad = \frac{1}{N}\max\left(\max_{k\leqslant N-2}(Y_k+Y_{k+1}), Y_{N-1}M_{\Psi,1}\right) \end{aligned}$$

which is $\max(M_B, B(1)M_\Psi/2)$ by the continuous mapping theorem. □

Remark 5.5.2. Under assumptions (B), $\mathbb{E}M_\infty^{\beta+\delta} < \infty$. Indeed, by (5.5.1),

$$M_B^2 = \sup\left\{\left(W_1(t)^2 + W_2(t)^2\right)^2 \,:\, t\in[0,1]\right\},$$

where W_1, W_2 are independent one-dimensional Brownian motions. Doob's maximal inequality applied to W_1, W_2 implies that $\mathbb{E}M_B^2 < \infty$. Since $\beta+\delta \leqslant 2$, it follows that $\mathbb{E}M_B^{\beta+\delta} < \infty$. Moreover, by (5.3.8), $\mathbb{E}M_\Psi^{\beta+\delta} < \infty$, and since M_Ψ and B are independent, we have

$$\mathbb{E}M_\infty^{\beta+\delta} \leqslant \mathbb{E}M_B^{\beta+\delta}\mathbb{E}(1+M_\Psi/2)^{\beta+\delta} < \infty.$$

Recall that the process Y^k counts the progeny of immigrants arriving in k'th block. Since Lemma 5.5.1 suggests that the maximum of process Y^k should be comparable with $\xi_k M_\infty$ when ξ_k is large, we begin the proof of Theorem 5.2.2 by distinguishing large blocks in the environment. Recall the sequence $(a_n)_{n\in\mathbb{N}}$ defined in (5.2.1). Fix $\varepsilon > 0$ and let

$$I_{n,\varepsilon} = \{k \leqslant n \,:\, \xi_k > \varepsilon a_n\}, \quad I_{n,\varepsilon}^c = \{k \leqslant n \,:\, \xi_k \leqslant \varepsilon a_n\}.$$

For fixed n and $k \leqslant n$, we will say that k'th block is large if $k \in I_{n,\varepsilon}$, and small otherwise.

It follows from the definition of the sequence $(a_n)_{n\in\mathbb{N}}$ and regular variation of the tails of ξ that for any $x > 0$,

$$n\mathrm{P}[\xi > xa_n] \to x^{-\beta}, \quad n \to \infty. \tag{5.5.7}$$

Therefore, by Proposition 3.21 in [25],

$$\sum_{k=1}^{n} \delta_{(\xi_k/a_n, k/n)} \Rightarrow P_\mu, \tag{5.5.8}$$

where P_μ is a Poisson point process on $(0,\infty] \times [0,\infty)$ with intensity measure $d\mu(x,t) = \beta x^{-\beta-1}dxdt$. In particular, as $n \to \infty$, the sequence of variables $|I_{n,\varepsilon}|$, which count the number of large blocks, converges weakly to Poisson distribution with parameter $\varepsilon^{-\beta}$.

We begin by showing that all the progeny of immigrants arriving in small blocks is negligible.

Proposition 5.5.3. *There is a constant C_5 such that for any $\varepsilon > 0$ and $\bar{\varepsilon} > 0$,*

$$\limsup_{n\to\infty} \mathbb{P}\left[\max_{j\geqslant 1} \sum_{k\in I_{n,\varepsilon}^c} Y_{j-S_{k-1}}^k > \bar{\varepsilon} a_n\right] \leqslant C_5 \bar{\varepsilon}^{-\beta-\delta} \varepsilon^{\delta}.$$

Proof. We will use the fact that the extinction times divide our process into i.i.d. pieces. Let

$$\eta_n = \inf\{k > 0 : \tau_k > n\}.$$

Since $\mathbb{E}\tau_1 < \infty$ by Lemma 5.3.1, the strong law of large numbers implies $\eta_n/n \to \eta := 1/\mathbb{E}\tau$ as $n \to \infty$, $\mathbb{P}$-a.s. We have

$$\mathbb{P}\left[\max_{j\geqslant 1} \sum_{k\in I_{n,\varepsilon}^c} Y_{j-S_{k-1}}^k > \bar{\varepsilon} a_n\right] \leqslant \mathbb{P}\left[\max_{j\geqslant 1} \sum_{k\leqslant \tau_{2n\eta}} Y_{j-S_{k-1}}^k \,\mathbb{1}_{\xi_k \leqslant \varepsilon a_n} > \bar{\varepsilon} a_n\right] + \mathbb{P}\left[|\eta - \eta_n/n| > \eta\right].$$

The second term tends to 0 as $n \to \infty$. Since the extinctions divide our process into i.i.d.

pieces, we have

$$\begin{aligned}\mathbb{P}\left[\max_{j\geqslant 1}\sum_{k\leqslant\tau_{2n\eta}} Y^k_{j-S_{k-1}}\,\mathbb{1}_{\xi_k\leqslant\varepsilon a_n} > \bar\varepsilon a_n\right] &\leqslant \sum_{m=1}^{2n\eta}\mathbb{P}\left[\max_{j\geqslant 1}\sum_{k=\tau_{m-1}}^{\tau_m} Y^k_{j-S_{k-1}}\,\mathbb{1}_{\xi_k\leqslant\varepsilon a_n} > \bar\varepsilon a_n\right]\\ &= 2n\eta\,\mathbb{P}\left[\max_{j\geqslant 1}\sum_{k=0}^{\tau_1} Y^k_{j-S_{k-1}}\,\mathbb{1}_{\xi_k\leqslant\varepsilon a_n} > \bar\varepsilon a_n\right]\\ &\leqslant 2n\eta\,\mathbb{P}\left[\sum_{k=0}^{\tau_1}\max_{j\geqslant 1} Y^k_j\,\mathbb{1}_{\xi_k\leqslant\varepsilon a_n} > \bar\varepsilon a_n\right]\\ &= 2n\eta\,\mathbb{P}\left[\sum_{k=1}^{\infty}\mathbb{1}_{k\leqslant\tau_1}\max_{j\geqslant 1} Y^k_j\,\mathbb{1}_{\xi_k\leqslant\varepsilon a_n} > \bar\varepsilon a_n\right]\\ &\leqslant 2n\eta\sum_{k=1}^{\infty}\mathbb{P}\left[\tau_1\geqslant k\right]\mathbb{P}\left[\max_{j\geqslant 1} Y^k_j\,\mathbb{1}_{\xi_k\leqslant\varepsilon a_n} > \bar\varepsilon a_n/2k^2\right],\end{aligned}$$

where in the last line we used the fact that $\{\tau_1\geqslant k\}$ and the process Y^k are independent.

Since the environment is given by an i.i.d. sequence, it is enough to estimate the tails of the maximum of the process $(Y_j\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n})_{j\geqslant 1}$. By Lemma 5.3.4 applied with $\gamma=\beta+\delta$,

$$\mathbb{P}\left[\max_{j\geqslant 1} Y_j\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n} > x\right] \leqslant C_2 x^{-\gamma}\left(\mathrm{E}\left(\mathrm{E}_\omega Y^2_{\xi_1-1}\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n}\right)^{\gamma/2} + \mathbb{E}\mathbb{Y}_1^\gamma\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n}\right).$$

As we have calculated in the proof of Lemma 5.3.4,

$$\mathrm{E}_\omega Y^2_{\xi_1-1}\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n} = \xi_1(\xi_1-1)\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n},\quad \mathrm{E}_\omega\mathbb{Y}_1^2 = (2\xi_1^2\rho_1^2+\xi_1\rho_1)\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n},$$

therefore

$$\mathrm{E}\left(\mathrm{E}_\omega Y^2_{\xi_1-1}\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n}\right)^{\gamma/2} \leqslant \mathrm{E}\xi^\gamma\,\mathbb{1}_{\xi\leqslant\varepsilon a_n}$$

and

$$\mathbb{E}\mathbb{Y}_1^\gamma\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n} \leqslant \mathrm{E}\left(\mathrm{E}_\omega\mathbb{Y}_1^2\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n}\right)^{\gamma/2} \leqslant \left(2^{\gamma/2}\mathrm{E}\rho^\gamma + \mathrm{E}\rho^{\gamma/2}\right)\mathrm{E}\xi^\gamma\,\mathbb{1}_{\xi\leqslant\varepsilon a_n}.$$

Putting things together, for some constant $C>0$ and any $x>0$,

$$\mathbb{P}\left[\max_{j\geqslant 1} Y_j\,\mathbb{1}_{\xi_1\leqslant\varepsilon a_n} > x\right] \leqslant Cx^{-\gamma}\mathrm{E}\xi^\gamma\,\mathbb{1}_{\xi\leqslant\varepsilon a_n} \leqslant Cx^{-\gamma}\int_0^{\varepsilon a_n} t^{\gamma-1}\mathrm{P}[\xi>t]dt.$$

By Karamata's theorem ([3], Theorem 1.5.11) and (5.5.7),

$$\int_0^{\varepsilon a_n} t^{\gamma-1}\mathrm{P}[\xi>t]dt \sim \frac{1}{\gamma+\beta}(\varepsilon a_n)^\gamma\mathrm{P}[\xi>\varepsilon a_n] \sim \frac{1}{\gamma+\beta}\varepsilon^{\gamma-\beta}a_n^\gamma n^{-1}.$$

Using those estimates, we obtain, for some constants $C, C'>0$,

$$\begin{aligned}\mathbb{P}\left[\max_{j\geqslant 1}\sum_{k\leqslant\tau_{2n\eta}} Y^k_{j-S_{k-1}}\,\mathbb{1}_{\{\xi_k\leqslant\varepsilon a_n\}} > \bar\varepsilon a_n\right] &\leqslant Cn\sum_{k=1}^{\infty}\mathbb{P}[\tau_1\geqslant k]\left(\bar\varepsilon a_n/2k^2\right)^{-\gamma}\varepsilon^{\gamma-\beta}a_n^\gamma n^{-1}\\ &\leqslant C'\bar\varepsilon^{-\gamma}\varepsilon^{\gamma-\beta}\mathbb{E}\tau_1^{2\gamma+1},\end{aligned}$$

which finishes the proof since $\gamma=\beta+\delta$ and $\mathbb{E}\tau_1^{2\gamma+1}<\infty$ by Lemma 5.3.1. □

The next step is to investigate the maximal generations among the progeny of immigrants from large blocks. Although it may happen that the descendants of particles from several large blocks coexist in one generation of the process Z, we will show later that it is unlikely, so that we may begin by investigating the maxima of $|I_{n,\varepsilon}|$ *independent* processes, each representing progeny of immigrants from a large block. To this end, assume that our probability space contains variables $\left\{(Y_k^{j,(N)})_{k\in\mathbb{N}} \,:\, j,N\in\mathbb{N}\right\}$ such that

- the processes $(Y_k^{j,(N)})_{k\in\mathbb{N}}$ are i.i.d. copies of $(Y_k^{(N)})_{k\in\mathbb{N}}$,
- the family $\left\{(Y_k^{j,(N)})_{k\in\mathbb{N}} \,:\, j,N\in\mathbb{N}\right\}$ is independent of the environment $\{(\xi_k,\lambda_k)\}_{k\in\mathbb{Z}}$.

For any $j,N\in\mathbb{N}$ denote $M_N^j = \max_{k\geqslant 0}(Y_k^{j,(N)}+Y_{k+1}^{j,(N)})$ and let $\bar{D}_{j,n} = \{\mathbb{Y}_{\sqrt{n}}^{j,(\xi_j)} = 0\}$. Observe that the event $\bar{D}_{j,n}$ means that the process $Y^{j,(\xi_j)}$ went extinct at most at its $\sqrt{n}$'th marked generation.

Proposition 5.5.4. *Fix* $\varepsilon>0$ *and let* $A_n\in\sigma(I_{n,\varepsilon})$ *be such that* $\mathrm{P}[A_n]\to 1$ *as* $n\to\infty$*. For any* $x>0$*,*

$$\lim_{n\to\infty}\mathbb{P}\left[\max_{j\in I_{n,\varepsilon}} M_{\xi_j}^j\,\mathbb{1}_{\bar{D}_{j,n}} > xa_n, A_n\right] = 1-\exp\left(-x^{-\beta}\mathbb{E}M_\infty^\beta\,\mathbb{1}_{M_\infty<x/\varepsilon} - \varepsilon^{-\beta}\mathbb{P}[M_\infty\geqslant x/\varepsilon]\right).$$

Proof. Observe that due to our assumptions the event $\bar{D}_{j,n}$ depends only on ξ_j and the process $Y^{j,(\xi_j)}$. Therefore we investigate a maximum of variables which are i.i.d. under $\mathbb{P}$.

Recall that $|I_{n,\varepsilon}|$ converges in distribution to $Poiss(\varepsilon^{-\beta})$. Moreover, conditioning on $|I_{n,\varepsilon}|=k$, the examined maximum is a maximum of k independent variables with distribution given by

$$\mathbb{P}\left[M_\xi\,\mathbb{1}_{\bar{D}_n}\in\cdot\,\middle|\,\xi>\varepsilon a_n\right],$$

for ξ independent of $\{Y^{(N)}, M_N \,:\, N\in\mathbb{N}\}$ and $\bar{D}_n=\{\mathbb{Y}_{\sqrt{n}}^{(\xi)}=0\}$. In particular,

$$\begin{aligned}\mathbb{P}\left[\max_{k\in I_{n,\varepsilon}} M_{\xi_k}^k\,\mathbb{1}_{\bar{D}_{k,n}} > xa_n\right] &= 1-\mathrm{E}\left[(1-\mathbb{P}\left[M_\xi>xa_n,\,\bar{D}_n\,|\,\xi>\varepsilon a_n\right])^{|I_{n,\varepsilon}|}\,\mathbb{1}_{A_n}\right]\\ &= 1-\mathrm{E}\left[(1-\mathbb{P}\left[M_\xi>xa_n,\,\bar{D}_n\,|\,\xi>\varepsilon a_n\right])^{|I_{n,\varepsilon}|}\right]+o(1),\end{aligned} \tag{5.5.9}$$

where the second equality follows from

$$\mathrm{E}\left[(1-\mathbb{P}\left[M_\xi>xa_n,\,\bar{D}_n\,|\,\xi>\varepsilon a_n\right])^{|I_{n,\varepsilon}|}\,\mathbb{1}_{A_n^c}\right]\leqslant \mathrm{P}[A_n^c].$$

Note that, since the extinction time of the process $Y^{(\xi)}$ is dominated by τ_1, Lemma 5.3.1 implies

$$\mathbb{P}[\bar{D}_n^c]\leqslant\mathbb{P}[\tau_1\geqslant\sqrt{n}]\leqslant e^{-c\sqrt{n}}\mathbb{E}e^{c\tau_1},$$

and by (5.5.7),

$$\mathbb{P}[\bar{D}_n^c\,|\,\xi>\varepsilon a_n]\leqslant e^{-c\sqrt{n}}\mathbb{E}e^{c\tau_1}\mathrm{P}[\xi>\varepsilon a_n]^{-1}\sim\mathbb{E}e^{c\tau_1}\varepsilon^\beta n e^{-c\sqrt{n}}\to 0$$

as $n \to \infty$. Therefore, for any fixed $\bar{\varepsilon} > 0$, for n large enough,

$$\mathbb{P}\left[M_\xi > xa_n,\, \bar{D}_n^c \,|\, \xi > \varepsilon a_n\right] \leqslant \bar{\varepsilon}. \tag{5.5.10}$$

By Lemma 5.5.1, $M_N/N \Rightarrow M_\infty$ as $N \to \infty$. Observe that the distribution of M_∞ is continuous and thus appropriate cumulative distribution functions converge uniformly; in particular, for large enough n,

$$\sup_{y>0} |\mathbb{P}[M_\xi > y \,|\, \xi > \varepsilon a_n] - \mathbb{P}[M_\infty > y/\xi \,|\, \xi > \varepsilon a_n]| < \bar{\varepsilon} \tag{5.5.11}$$

for M_∞ independent of ξ. Observe that

$$\begin{aligned}\mathbb{P}[M_\infty > xa_n/\xi \,|\, \xi > \varepsilon a_n] &= \frac{\mathbb{P}[\xi M_\infty > xa_n, \xi > \varepsilon a_n]}{\mathrm{P}[\xi > \varepsilon a_n]} \\ &= \frac{1}{\mathrm{P}[\xi > \varepsilon a_n]}\left(\int_{[0,x/\varepsilon)} \mathrm{P}[\xi > xa_n/t]\mathbb{P}[M_\infty \in dt] + \int_{[x/\varepsilon,\infty)} \mathrm{P}[\xi > \varepsilon a_n]\mathbb{P}[M_\infty \in dt]\right) \\ &= \int_{[0,x/\varepsilon)} \frac{\mathrm{P}[\xi > xa_n/t]}{\mathrm{P}[\xi > \varepsilon a_n]}\mathbb{P}[M_\infty \in dt] + \mathbb{P}[M_\infty \geqslant x/\varepsilon].\end{aligned}$$

By uniform convergence theorem for regularly varying functions (see (B.1.2) in [4]), for n large enough,

$$\sup_{c\geqslant 1}\left|\frac{\mathrm{P}[\xi > c\varepsilon a_n]}{\mathrm{P}[\xi > \varepsilon a_n]} - c^{-\beta}\right| < \bar{\varepsilon},$$

which means that

$$\left|\int_{[0,x/\varepsilon)} \frac{\mathrm{P}[\xi > xa_n/t]}{\mathrm{P}[\xi > \varepsilon a_n]} - \left(\frac{x}{t\varepsilon}\right)^{-\beta} \mathbb{P}[M_\infty \in dt]\right| < \bar{\varepsilon}.$$

Hence

$$\left|\mathbb{P}[M_\infty > xa_n/\xi \,|\, \xi > \varepsilon a_n] - \left(x^{-\beta}\varepsilon^{\beta}\mathbb{E}M_\infty^{\beta}\, \mathbb{1}_{M_\infty < x/\varepsilon} + \mathbb{P}[M_\infty \geqslant x/\varepsilon]\right)\right| < \bar{\varepsilon},$$

which together with (5.5.11) implies

$$\left|\mathbb{P}[M_\xi > xa_n \,|\, \xi > \varepsilon a_n] - \left(x^{-\beta}\varepsilon^{\beta}\mathbb{E}M_\infty^{\beta}\, \mathbb{1}_{M_\infty < x/\varepsilon} + \mathbb{P}[M_\infty \geqslant x/\varepsilon]\right)\right| < 2\bar{\varepsilon}. \tag{5.5.12}$$

Putting the estimates (5.5.10) and (5.5.12) to (5.5.9) and using the fact that $|I_{n,\varepsilon}| \Rightarrow Poiss(\varepsilon^{-\beta})$, we obtain

$$\begin{aligned} &1 - \exp\left(-\varepsilon^{-\beta}\left(x^{-\beta}\varepsilon^{\beta}\mathbb{E}M_\infty^{\beta}\, \mathbb{1}_{M_\infty < x/\varepsilon} + \mathbb{P}[M_\infty \geqslant x/\varepsilon] - 3\bar{\varepsilon}\right)\right) \\ &\quad \leqslant \liminf_{n\to\infty} \mathbb{P}\left[\max_{k\leqslant n} M_{\xi_k}^k\, \mathbb{1}_{\xi_k > \varepsilon a_n} > xa_n\right] \leqslant \limsup_{n\to\infty} \mathbb{P}\left[\max_{k\leqslant n} M_{\xi_k}^k\, \mathbb{1}_{\xi_k > \varepsilon a_n} > xa_n\right] \\ &\quad \leqslant 1 - \exp\left(-\varepsilon^{-\beta}\left(x^{-\beta}\varepsilon^{\beta}\mathbb{E}M_\infty^{\beta}\, \mathbb{1}_{M_\infty < x/\varepsilon} + \mathbb{P}[M_\infty \geqslant x/\varepsilon] + 3\bar{\varepsilon}\right)\right),\end{aligned}$$

which finishes the proof since $\bar{\varepsilon}$ is arbitrary. □

We are now ready to prove Theorem 5.2.2, rephrased into the setting of the associated branching process.

Theorem 5.5.5. *Under assumptions* (B),

$$\mathbb{P}\left[a_n^{-1}\max_{0\leqslant k<S_n}(Z_k+Z_{k+1})>x\right]\xrightarrow{n\to\infty}1-\exp\left(-\mathbb{E}M_\infty^\beta x^{-\beta}\right).$$

Proof. Fix $\varepsilon>0$. For any $\bar\varepsilon>0$,

$$\begin{aligned}\mathbb{P}\left[\max_{j\geqslant 1}\sum_{k\in I_{n,\varepsilon}}(Y^k_{j-S_{k-1}}+Y^k_{j-S_{k-1}+1})>xa_n\right]&\leqslant\mathbb{P}\left[\max_{j<S_n}(Z_j+Z_{j+1})>xa_n\right]\\ \leqslant\mathbb{P}\left[2\max_{j\geqslant 1}\sum_{k\in I^c_{n,\varepsilon}}Y^k_{j-S_{k-1}}>\bar\varepsilon a_n\right]&+\mathbb{P}\left[\max_{j\geqslant 1}\sum_{k\in I_{n,\varepsilon}}(Y^k_{j-S_{k-1}}+Y^k_{j-S_{k-1}+1})>(x-\bar\varepsilon)a_n\right].\end{aligned}\tag{5.5.13}$$

Note that because of (5.5.8) we expect that for large n the set $I_{n,\varepsilon}$ should be distributed rather uniformly on $\{1,\dots n\}$, so that the large blocks are far from each other. Indeed, since $nP[\xi>\varepsilon a_n]\to\varepsilon^{-\beta}$, for any sequence b_n such that $b_n=o(n)$,

$$\mathrm{P}\left[(\exists k,l\in I_{n,\varepsilon})\ k\neq l,|k-l|\leqslant b_n\right]\leqslant n\mathrm{P}[\xi>\varepsilon a_n]\cdot b_n\mathrm{P}[\xi>\varepsilon a_n]\to 0\qquad\text{as } n\to\infty.$$

That is, with high probability, large blocks are at distance at least b_n from each other. On the other hand, we know that the extinction occurs very often in our process, which should mean that as the process evolves, no two bloodlines of immigrants from large blocks coexist at one time. Let

$$D_{k,n}=\left\{\mathbb{Y}^k_{\sqrt{n}}=0\right\}$$

be an event that the progeny of immigrants from k'th block does not survive more than $\sqrt{n}$ blocks. Then, by Lemma 5.3.1,

$$\mathbb{P}\left[\bigcup_{k\leqslant n}D^c_{k,n}\right]\leqslant n\mathbb{P}[\tau_1>\sqrt{n}]\leqslant ne^{-c\sqrt{n}}\mathbb{E}e^{c\tau}\to 0$$

as $n\to 0$. Therefore the probability of the set

$$D_n=\bigcap_{k\leqslant n}D_{k,n}$$

converges to 1 as $n\to\infty$ and so does the probability of

$$A_n=\{(\forall k,l\in I_{n,\varepsilon})\,k\neq l\implies|k-l|>2\sqrt{n}\}.$$

Moreover, on the set $A_n\cap D_n$, the progeny of immigrants from each large block dies out before the next large block occurs. That is, $\max_{j\geqslant 1}\sum_{k\in I_{n,\varepsilon}}(Y^k_{j-S_{k-1}}+Y^k_{j-S_{k-1}+1})$ is really a maximum of independent maxima of Y^k such that $k \in I_{n,\varepsilon}$. Therefore,

$$\mathbb{P}\left[\max_{j\geqslant 1}\sum_{k\in I_{n,\varepsilon}}(Y^k_{j-S_k}+Y^k_{j-S_k+1}) > xa_n,\, A_n\cap D_n\right] = \mathbb{P}\left[\max_{k\in I_{n,\varepsilon}} M^k_{\xi_k}\,\mathbb{1}_{\bar{D}_{k,n}} > xa_n,\, A_n\right].$$

By Proposition 5.5.4, this quantity converges to

$$1-\exp\left(-x^{-\beta}\mathbb{E}M_\infty^\beta\,\mathbb{1}_{M_\infty<x/\varepsilon} - \varepsilon^{-\beta}\mathbb{P}[M_\infty \geqslant x/\varepsilon]\right)$$

as $n\to\infty$. Going back to (5.5.13), we have

$$1-\exp\left(-x^{-\beta}\mathbb{E}M_\infty^\beta\,\mathbb{1}_{M_\infty<x/\varepsilon} - \varepsilon^{-\beta}\mathbb{P}[M_\infty \geqslant x/\varepsilon]\right) \leqslant \liminf_{n\to\infty}\mathbb{P}\left[\max_{j<S_n}(Z_j+Z_{j+1}) > xa_n\right]. \tag{5.5.14}$$

On the other hand, by Proposition 5.5.3,

$$\limsup_{n\to\infty}\mathbb{P}\left[2\max_{j\geqslant 1}\sum_{k\in I^c_{n,\varepsilon}} Y^k_{j-S_{k-1}} > \bar{\varepsilon}a_n\right] \leqslant C_5(\bar{\varepsilon}/2)^{-\beta-\delta}\varepsilon^\delta,$$

which means that

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{P}\left[\max_{j<S_n}(Z_j+Z_{j+1}) > xa_n\right] &\leqslant C_5(\bar{\varepsilon}/2)^{-\beta-\delta}\varepsilon^\delta \\ &+ 1-\exp\left(-(x-\bar{\varepsilon})^{-\beta}\mathbb{E}M_\infty^\beta\,\mathbb{1}_{M_\infty<(x-\bar{\varepsilon})/\varepsilon} - \varepsilon^{-\beta}\mathbb{P}[M_\infty \geqslant (x-\bar{\varepsilon})/\varepsilon]\right).\end{aligned} \tag{5.5.15}$$

Observe that, since $\mathbb{E}M_\infty^{\beta+\delta} < \infty$ (see Remark 5.5.2), we have

$$\varepsilon^{-\beta}\mathbb{P}[M_\infty \geqslant x/\varepsilon] \leqslant \varepsilon^\delta x^{-\beta-\delta}\mathbb{E}M_\infty^{\beta+\delta} \to 0 \quad \text{as } \varepsilon\to 0,$$

while by the monotone convergence theorem,

$$\mathbb{E}M_\infty^\beta\,\mathbb{1}_{M_\infty<x/\varepsilon} \to \mathbb{E}M_\infty^\beta \quad \text{as } \varepsilon\to 0.$$

Therefore passing with ε to 0 in (5.5.14) gives

$$1-\exp\left(-x^{-\beta}\mathbb{E}M_\infty^\beta\right) \leqslant \liminf_{n\to\infty}\mathbb{P}\left[\max_{j<S_n}(Z_j+Z_{j+1}) > xa_n\right],$$

and similarly in (5.5.15),

$$\limsup_{n\to\infty}\mathbb{P}\left[\max_{j<S_n}(Z_j+Z_{j+1}) > xa_n\right] \leqslant 1-\exp\left(-(x-\bar{\varepsilon})^{-\beta}\mathbb{E}M_\infty^\beta\right),$$

which ends the proof since $\bar{\varepsilon}>0$ is arbitrary. □

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