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Branching-selection particle system with semiexponential increments

Gałązkowy system cząstek z selekcją, z semieksponencjalnymi przyrostami

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Introduction

In 1937, Fisher [9] and simultaneously Kolmogorov, Pietrowski, Piskunov [12] in their study on the evolution of biological populations, introduced the following heat equation:

$$\frac{\partial u}{\partial t} = \Delta u + u(1-u) \tag{0.1}$$

where $u = u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$. Currently, (0.1) is known as Fisher – Kolmogorov, Pietrowski, Piskunov equation (FKPP) and has found extensive applications in modeling population growth and wave propagation in various fields, including physics and chemistry.

The FKPP equation is known for its specific stable solutions of the form

$$u(x,t) = v(x \pm ct)$$

for some speed $c > 0$ and shape v – an increasing function such that

$$\lim_{z\to-\infty} v(z) = 0, \qquad \lim_{z\to+\infty} v(z) = 1.$$

Such solutions are called travelling wave solutions and are of significant interest in studying this equation. Important connected aspect of the FKPP equation is the long-term behaviour of solutions, including the shape of the limiting solution and the asymptotic position of the FKPP front. For some initial conditions branching Brownian motion can help in the exploration of these problems.

Branching Brownian motion (BBM) is a continuous time spatial branching process defined as follows. At time $t = 0$, a single particle starts at origin and moves as a standard Brownian motion. At random time with distribution $Exp(1)$ it divides into two particles. Those new particles moves as independent Brownian motions, each having a random lifetime with distribution $Exp(1)$. They split in the same fashion as their parent.

Both FKPP and BBM can be seen as modelling the evolution of a population. While the FKPP equation is deterministic and considers the population as a whole, the BBM model is stochastic and explicitly incorporates individual variation. Despite their differences, these two models are directly related to each other by McKean's duality.

As McKean has shown in [13] if one calls R_t the position of the rightmostpart particle in a BBM then the distribution function of R_t, that is

$$u(x,t) = \mathbb{P}(R_t > x)$$

is a solution of FKPP equation with a initial condition $u_0(x) = \mathbb{1}_{\{x<0\}}$. What's more, if $m(t)$ denotes the median of u, i.e. $u(m(t),t) = 1/2$, then

$$u(x + m_t, t) \to v(x), \qquad t \to \infty$$

and v is the wave solution of FKPP at speed $2^{1/2}$.

In this paper, we consider a model that can be interpreted as a discrete version of branching Brownian motion with selection, called N–branching random walk (N–BRW). It is a system of N particles positioned on the real line, which evolves through iterations following a specific set of rules. In each iteration, every particle undergoes a splitting process, resulting in two offspring particles. These particles then perform random jumps, guided by a prescribed displacement distribution supported on $\mathbb{R}$. The system retains only N rightmost particles, discarding the remaining ones.

This model was first introduced in 1997 by E. Brunet and B. Derrida in [4]. The inspiration for this model stemmed from Derrida's earlier investigations into directed polymers. As N–BRW can be seen as a discrete version of branching Brownian motion with selection, it can be assumed that some of the connections with FKPP equation remain. In the mentioned paper Brunet and Derrida demonstrated by simulations that, as N grows to infinity, there arises a connection between the velocity of rightmost particle of N–BRW and a discrete version of the FKPP equation with a cutoff value of $1/N$, that is

$$\frac{\partial u}{\partial t} = \Delta u + u(1-u)\mathbb{1}\{u \geq 1/N\}. \tag{0.2}$$

Interestingly, depending on the distribution of jumps, the behaviour of N–BRWs for large N differs significantly. The properties of this system are now well understood in the case where the displacement distribution admits exponential moments and in the case of displacements with regularly varying tails.

In the case of displacements admitting exponential tails in 1997 Brunet and Derrida in [4] gave non-rigorous arguments on the speed of convergence and the spatial distribution that later obtained rigorous proofs. Bérard and Gouéré in [6] proved that the asymptotic speed converges to a finite limiting speed as $N \to \infty$, with a slow rate $(\log N)^{-2}$. So this limiting speed is the same as the speed of the rightmost particle in a classical branching random walk without selection with exponentially decaying tails. In [8], [2] can be found arguments on the spatial distribution that the fraction of particles to the right of a given position at a given time evolve according to discrete analogue of FKPP equation. There are also results on the shape of genealogy in this case that can be found in [5].

In the case of displacements with polynomial tails, in 2014 Bérard and Maillard in [1] proved that the speed of the particle cloud grows as $a_N/\log_2 N$ in N, and the propagation is linear or superlinear (but at most polynomial) in time. In the classical branching random walk without selection in a heavy-tailed setting the propagation is exponentially fast in time, so these behaviours differ. In 2022, Penington, Roberts and Talyigás in [14] showed that in this case the majority of the population is located close to the leftmost particle and found the typical large time shape of the genealogy.

In this paper, we consider the case of semiexponential displacements and we are mainly interested in the behaviour of the maximal position of the system.

The main objective of this paper was to investigate the behaviour of the rightmost position's speed in the N–BRW as N becomes large. However, we encountered more significant challenges than initially anticipated, and as a result, we were only able to obtain partial asymptotic results. To establish our proof, we followed the methods employed in the case where the displacement distribution admits exponential moments, as outlined in [6]. In their work, they extensively use the results from [11] on the branching random walk killed below a linear space–time barrier. The basis of their proof involved comparing the particle system with a family of N independent branching random walks killed below a linear space–time barrier.

Specifically, they required the asymptotic analysis of the probability that an infinite ray exists within the branching random walk, consistently positioned above the line with a slope of $\gamma - \epsilon$, where γ denotes the asymptotic speed of the rightmost position in the branching random walk. So our work to prove the result involved determining similar asymptotics in our specific case.

The paper is organised as follows. In Section 1 we formally state the problem and the main result of the paper (Theorem 1.1). In Section 2 we investigate previously mentioned asymptotics in the branching random walks killed below a linear space–time barrier. Section 3 provides a discussion of various elementary properties of the model we consider useful in the sequel. Sections 4 and 5 collectively present the proof of Theorem 1.1, as they contain the proofs of both the upper and lower bounds stated in the theorem.

1. Statement of the results

1.1. **The model with selection.** Let $c : \mathbb{R} \to [0, +\infty)$ be a locally bounded function and such that $c(x) \xrightarrow{x\to\infty} C$, where $C \in (0, +\infty)$. Then let the distribution p on $\mathbb{R}$ be of the form

$$p([x, +\infty)) = c(x)e^{-\lambda x^r} \qquad \text{for } x > 0,$$

for some $r \in (0, 1)$, $\lambda > 0$ and has finite expectation.

We consider a discrete-time particle system of N particles on $\mathbb{R}$ evolving through the repeated application of branching and selection steps defined as follows:

- Branching: each of the N particles is replaced by two new particles; the position of each new particle is that of the original particle shifted by a random walk step, according to the distribution p, where the steps for different particles are independent.
- Selection: only the N rightmost particles are kept among the $2N$ obtained at the branching step, to form the new generation of N particles.

We call this system a N–particle branching random walk. Every repetition of the branching-selection mechanism we call a step.

The formal definition of this system appears later, namely in Section 3. We give two formalisation of this system: in terms of Markov chain on the space of point measures and in terms of branching random walks, that are defined in section 2.

1.2. **Main result.** Let X_n^N be the positions of the population at the step n. Then $\max X_n^N$ is the position of the rightmost particle of such population.

Theorem 1.1. *The limit $v_N = \lim_{n\to\infty} \max X_n^N/n$ exists almost surely and in L^1 and there exist such $\alpha_*, \alpha^* \in \mathbb{R}$, $0 < \alpha_* \leq \alpha^*$ that*

$$\alpha_* \leq \liminf_{N\to\infty} \frac{v_N}{(\log N)^{1/r-1}} \leq \limsup_{N\to\infty} \frac{v_N}{(\log N)^{1/r-1}} \leq \alpha^*. \tag{1.1}$$

The proof of Theorem 1.1 is split into three parts: the proof of the existence of v_N (Section 3, Proposition 3.13), the proof of the lower bound in (1.1) (Section 4) and the proof of the upper bound in (1.1) (Section 5).

It should be emphasised that the existence of v_N follows from general and known arguments and we adapt the proof formulated in [6]. The main contribution of this paper in the investigation of the behaviour of N–BRW with semiexponential increments is finding the asymptotics of v_N, so exactly the proof of the existence of α_* and α^* such that (1.1).

2. Killed branching random walk

In this section we consider a different model. It is called branching random walk with binary spitting and we denote it as BRW. It is somehow similar to the model defined in subsection 1.1, but starts with only one particle and doesn't consist on the selection step. As mentioned in the introduction, we want to investigate branching random walks killed below a barrier and some of the results given in this section will be useful in the proof of Theorem 1.1.

At the beginning of this section we formally introduce the BRW. Defining it requires recalling some definitions and notations concerning trees.

2.1. **The model without selection.** By the binary tree we mean the set of finite words

$$\mathfrak{T} = \{\varnothing\} \cup \bigcup_{n=1}^{\infty} \{0, 1\}^n.$$

$\varnothing$ is called a *root*, for $u, v \in \mathfrak{T}$, uv denotes the concatenation of u and v. If $v = u0$ or $v = u1$ then u is called a *parent* of v. More generally, if $v = uw$ for some $u, w \in \mathfrak{T}$, then we say that u is an *ancestor* of v and we write $u \leq v$. In the sequel, for fixed vertex u, $v \leq u$ means all the ancestors of u including u and $v < u$ means all the ancestors of u excluding u.

Given $m \geq 1$, we say that a sequence $u_0, \ldots, u_m \in \mathfrak{T}$ is a descending path if, for all $i \in [\![1, m]\!]$, u_{i-1} is the parent of u_i, where $[\![k, l]\!] := \{k, k+1, \ldots, l\}$ for $k < l$.

We also let $|u|$ be the length of the word u and we call it the *depth* of u. And by $\mathfrak{T}(n)$ we mean all the vertices of $\mathfrak{T}$ of depth n.

Definition 2.1. BRW is a pair $(\mathfrak{T}, \Phi)$, where $\mathfrak{T}$ is a binary tree, and $\Phi : \mathfrak{T} \to \mathbb{R}$ is a random map, such that $\Phi(\varnothing) = 0$ and

$$\Phi(v) = \sum_{\substack{v \leq u, \\ v \neq \varnothing}} X_v,$$

where $\{X_v\}_{v \in \mathfrak{T} \setminus \{\varnothing\}}$ are i.i.d. random variables with common distribution p.

2.2. Definitions and statement of the main result.

Definition 2.2. Given $\alpha > 0$ and $n \in \mathbb{N}$, we say that a vertex $u \in \mathfrak{T}$ is (α, n)–**good** if there exists a finite descending path $u =: u_0, u_1, \ldots, u_n$, such that $\Phi(u_i) - \Phi(u_0) \geq \alpha n^{1/r-1} i$ for all $i \in [\![1, n]\!]$.

For this model, we are interested in the probability that in BRW there exists a descending path above the barrier, so the probability of the existence of such (α, n)–good particles. So for $\alpha > 0$ and $n \in \mathbb{N}$ we consider an event that in BRW the root is (α, n)–good. Let

$$A_{\alpha,n} = \{\exists u \in \mathfrak{T} : |u| = n, \Phi(v) \geq \alpha n^{1/r-1} |v| \text{ for all } v \leq u\}.$$

Let the probability of this event be

$$\rho(\alpha, n) = \mathbb{P}[A_{\alpha,n}],$$

and let us also define

$$r(\alpha, n) = -\log \rho(\alpha, n).$$

The main result of this section is stated below.

Theorem 2.3. *There exists a sequence $\{n_k\}$ such that for every $\alpha > 0$, $r(\alpha, n_k)/n_k$ converges. Then*

$$r(\alpha) := \lim_{k \to \infty} r(\alpha, n_k)/n_k$$

is continuous and strictly increasing. What's more

$$\lambda \gamma_0^r \leq r(\alpha) \leq (\lambda \alpha^r) \wedge \left(\lambda \left(\frac{\lambda \alpha}{\log 2} \right)^{r/(1-r)} \right)$$

where $\gamma_0 \in (0, (\alpha / \log 2)^{1/(1-r)})$ is such that $\lambda \gamma_0^r = \gamma_0^{r-1} \alpha - \log(2)$.

The proof of this theorem is split into a series of propositions and lemmas. We start with finding the bounds on $r(\alpha, n)/n$. Then we prove the existence of such sequence $\{n_k\}$ that $r(\alpha, n_k)/n_k$ converges for all α rational. By some technical lemmas we are able to extend the definition of $r(\alpha)$ to all $\alpha \in \mathbb{R}$. At the end we check that such function is actually continuous and strictly increasing.

2.3. Bounds on $r(\alpha, n)$.

The first step to prove Theorem 2.3 is finding the bounds on $r(\alpha, n)$. We will give two upper bounds. Depending α and r the first or the second gives a better estimate.

Although we start with a simple remark on the distribution p, that will be useful in all the paper.

Remark 2.4. Since c is locally bounded and it converges, it is also bounded. Thus in the sequel M stands for the bound of c, so $M > 0$ is such that $c(x) \leq M$ for all $x \in \mathbb{R}$.

Upper bound on $r(\alpha, n)$. The idea behind the first proof is that since all the jumps are non negative, then it is enough for the path to meet the conditions of $A_{\alpha,n}$ to perform an sufficiently big jump at first generation.

Proposition 2.5. *For any* $\alpha > 0$ *we have*

$$\limsup_{n\to\infty} \frac{r(\alpha, n)}{n} \leq \lambda\alpha^r$$

Proof. Take $R \geq 0$ and let $\gamma := \alpha + Rn^{1-1/r}$. Assume that BRW is such that in the first generation exists one particle u_1 that performs jump not less than $\gamma n^{1/r}$. It happens with probability

$$1 - \left(1 - \mathbb{P}(X \geq \gamma n^{1/r})\right)^2 = 2\mathbb{P}(X \geq \gamma n^{1/r})(1 + o(1)).$$

If we also assume that in this BRW exists a descending path $u_2 \leq \cdots \leq u_n$ such that $X_{u_k} \geq -R$ for all $k \in [\![2, n]\!]$, then such BRW meets the condition of $A_{\alpha,n}$. It follows from the fact that for all $k \in [\![1, n]\!]$ we have

$$\Phi(u_k) \geq \gamma n^{1/r} - kR \geq \alpha n^{1/r-1}k = \alpha n^{1/r-1}|u_k|.$$

To find the probability that there exists such a descending path that all the jumps are $\geq -R$ let us consider a Galton–Watson process $\{Z_n^{(R)}\}_{n\geq 0}$ with the offspring distribution

$$\begin{gathered}\mathbb{P}(\xi = 0) = p((-\infty, -R))^2, \quad \mathbb{P}(\xi = 2) = p([-R, +\infty))^2, \\ \mathbb{P}(\xi = 1) = 2p((-\infty, -R))p([-R, +\infty)).\end{gathered} \tag{2.1}$$

Then the probability of the existence of the mentioned path is non-smaller than the probability of the survival of $\{Z_n^{(R)}\}$. Since $\mathbb{E}\xi = 2p([-R, +\infty))$, then for R large enough the process is supercritical. By Theorem 4.3.12. in [7] (p. 232) $\mathbb{P}[(\forall n)\ Z_n^{(R)} \geq 1] = q > 0$.

So for every $n \in \mathbb{N}$,

$$\begin{aligned}\rho(\alpha, n) &\geq 2\mathbb{P}(X \geq \gamma n^{1/r})(1 + o(1)) \cdot q \\ &= 2c(\alpha n^{1/r}) \exp\{-\lambda\gamma^r n\}(1 + o(1)) \cdot q,\end{aligned}$$

thus by the definition of γ,

$$-\frac{\log \rho(\alpha, n)}{n} \leq -\frac{1}{n} \log\left(2c(\alpha n^{1/r})(1 + o(1))q\right) + \lambda(\alpha + Rn^{1-1/r})^r. \tag{2.2}$$

Since $r < 1$ and the term $\log\left(2c(\alpha n^{1/r})q(1 + o(1))\right)$ is bounded, by letting $n \to \infty$ in (2.2) we conclude the lemma. $\square$

To prove the next upper bound we will use the following lemma on the supercritical Galton–Watson process. It is adapted from [6], but we prove it for completeness.

Lemma 2.6 (Lemma 3. from [6])**.** *Let* $(M_n)_{n\geq 0}$ *denote the population size of a supercritical Galton–Watson process with square-integrable offspring distribution started with* $M_0 = 1$. *Let* $m := \mathbb{E}M_1 > 1$, *then there exists* $r > 0$ *such that, for all* $n \geq 0$,

$$\mathbb{P}(M_n \geq m^n) \geq r.$$

Proof. Let us denote as m, σ^2 the mean and the variance of the offspring distribution, respectively. By the assumption on the process $m > 1$ and $\sigma^2 < \infty$. Let $W_n = M_n/m^n$, then by the second moment method

$$\mathbb{P}\left(\frac{M_n}{m^n} \geq 1\right) \geq \frac{(\mathbb{E}W_n)^2}{\mathbb{E}W_n^2}.$$

By the definition of W_n we have $\mathbb{E}W_n = 1$, so it is enough to find the second moment of W_n. By Example 4.4.9. in [7] (p. 283) we have that $\mathbb{E}W_n^2 = 1+\sigma^2\frac{1-m^{-n}}{m(m-1)}$. So for all $n \geq 1$ we have

$$\mathbb{P}(M_n \geq m^n) \geq \frac{1}{1+\sigma^2\frac{1-m^{-n}}{m(m-1)}} > 0.$$

□

Proposition 2.7. *For any $\alpha > 0$ we have*

$$\limsup_{n\to\infty} \frac{r(\alpha,n)}{n} \leq \lambda\left(\frac{\lambda\alpha}{\log 2}\right)^{r/(1-r)}.$$

Proof. We saw in the previous proof that one particle that performs a jump big enough can enable its descendants of several future generations to meet the condition of $A_{\alpha,n}$. The idea of this proof is similar to the previous one. Only this time we take into account also the reproduction of these offspring and the possibility of them performing another big jump. So we look how big jump is needed at first generation to trigger a chain reaction, i.e. for some of the offspring already meeting the condition of $A_{\alpha,n}$ to perform a next similar big jump with big probability.

Similarly as in the previous proof we fix $R \geq 0$ and consider the Galton–Watson process $\{Z_n^{(R)}\}_{n\geq 1}$ with offspring distribution given by (2.1). Let denote $m_R := \mathbb{E}\xi = 2\mathbb{P}(X \geq -R)$.

Let

$$\beta_0 = (1+\delta)\left(\frac{\lambda(\alpha+Rn^{1-1/r})}{\log m_R}\right)^{1/(1-r)}$$

for some $\delta > 0$. Assume that a BRW is such that in the first step one particle, called u_1, performs jump not less than $\beta_0 n^{1/r}$. This happens with probability $2\mathbb{P}(X \geq \beta_0 n^{1/r})(1+o(1))$.

Let $k_1 = n\frac{\beta_0}{\alpha+Rn^{1-1/r}}$. Assume also that in the BRW there exists a descending path $u_1 \leq \cdots \leq u_{k_1}$, $|u_i| = i$ for $i \leq k_1$ and $X_{u_i} \geq -R$. Then for $i \in [\![1,k_1]\!]$ we have

$$\Phi(u_i) \geq \beta_0 n^{1/r} - Ri \geq \alpha n^{1/r-1} i = \alpha n^{1/r-1}|u_i|.$$

So such path meets the condition of $A_{\alpha,n}$. By Lemma 2.6 we know that for R large enough there are $\geq m_R^{k_1}$ such paths with probability $\geq q$ for some $q > 0$. Now, we show that with big probability between that many descendants there appear a big jump.

So assume that $Z_{k_1}^{(R)} \geq m_R^{k_1}$. For n sufficiently large one of these $m_R^{k_1}$ particles performs a jump not less that $\beta_0 n^{1/r}$ with probability $\geq \frac{1}{2}$. It follows from the definition of β_0, since

$$\frac{(\log m_R)\beta_0}{\alpha+Rn^{1-1/r}} - \lambda\beta_0^r = ((1+\delta)-(1+\delta)^r)\left(\frac{\lambda(\alpha+Rn^{1-1/r})}{\log m_R}\right)^{1/(1-r)} > 0$$

and using the inequality $1-x \leq e^{-x}$ we have for n large enough

$$\begin{aligned}
\mathbb{P}(\max_{|v|=k_1} X_v \geq \beta_0 n^{1/r}) &\geq 1-\left(1-\mathbb{P}(X \geq \beta_0 n^{1/r})\right)^{m_R^{k_1}} \\
&\geq 1-\exp\{-m_R^{k_1}\mathbb{P}(X \geq \beta_0 n^{1/r})\} \\
&= 1-\exp\{-m_R^{k_1} c(\beta_0 n^{1/r})\exp\{-\lambda\beta_0^r n\}\} \\
&\geq 1-\exp\left\{-M\exp\left\{n\Big((\log m_R)\frac{\beta_0}{\alpha+Rn^{1-1/r}} - \lambda\beta_0^r\Big)\right\}\right\} \\
&\geq \frac{1}{2}.
\end{aligned}$$

Let the particle that performed second big jump be called u_{k_1}. Let now $k_2 = 2n\frac{\beta_0}{\alpha+Rn^{1-1/r}}$. Then again with probability greater than $\mathbb{P}(Z^{(R)}_{k_1} \geq m_R^{k_1})$ there exist more than $m_R^{k_1}$ particles v, descendants of u_{k_1}, $|v| = k_2$ such that for any $k_1 \leq i \leq k_2$ and any offspring $u_i \leq v$, $|u_i| = i$ we have that

$$\Phi(u_i) \geq 2\beta_0 n^{1/r} - Ri \geq \alpha n^{1/r-1} i.$$

And one of them performs a jump not less than $\beta_0 n^{1/r}$ with probability $\geq \frac{1}{2}$. This jump enables $n\frac{\beta_0}{\alpha+Rn^{1-1/r}}$ more generations to meet the condition of $A_{\alpha,n}$, so it prolongs our path. After $K := \lceil \frac{\alpha+Rn^{1-1/r}}{\beta_0} \rceil$ such steps we obtain the whole descending path needed in $A_{\alpha,n}$.

So

$$\begin{aligned}\rho(\alpha, n) &\geq 2\mathbb{P}(X \geq \beta_0 n^{1/r})(1 + o(1))(1/2)^K \mathbb{P}(Z^{(R)}_{k_1} \geq m_R^{k_1})^K \\ &\geq 2c(\beta_0 n^{1/r})(1 + o(1))(1/2)^K q^K \exp\{-\lambda\beta_0^r n\}\end{aligned}$$

thus

$$-\frac{\log \rho(\alpha, n)}{n} \leq -\frac{1}{n} \log\{2c(\beta_0 n^{1/r})(1 + o(1))(1/2)^K q^K\} + \lambda\beta_0^r.$$

Since the term $\log\{2c(\beta_0 n^{1/r})(1 + o(1))(1/2)^K q^K\}$ is bounded and $m_R \xrightarrow{R\to\infty} 2$, by letting $n \to \infty$ and taking arbitrarily large R we conclude the lemma. $\square$

Lower bound. Now we give a lower bound which is more complicated than the previous ones, but its main idea is similar – we again investigate the big jumps performed by the walk.

We again start with a remark on the distribution p that will be useful in the proof of the bound.

Remark 2.8. By the assumptions p has finite expectation. Although it doesn't have to have finite any other moments. If we assume that p is the distribution on $[0, +\infty)$, then it has all moments finite and it follows from the computations.

Let X be a variable with distribution p, then for every k we have

$$\begin{aligned}\mathbb{E}[X^k] &= k\int_0^\infty x^{k-1}\mathbb{P}(X > x)dx \leq kM\int_0^\infty x^{k-1}e^{-\lambda x^r}dx \\ &= \frac{kM}{r\lambda^{k/r}}\int_0^\infty t^{\frac{k}{r}-1}e^{-t}dt < \infty.\end{aligned}$$

Theorem 2.9. *For any $\alpha > 0$ we have*

$$\lambda\gamma_0^r \leq \liminf_{n\to\infty} \frac{r(\alpha, n)}{n},$$

where $\gamma_0 \in (0, (\lambda\alpha/\log 2)^{1/(1-r)})$ is such that $\lambda\gamma_0^r = \gamma_0^{r-1}\alpha - \log(2)$.

Remark 2.10. It may not be obvious at first, but $\gamma_0(\alpha) \in (0, (\lambda \log 2/\alpha)^{1/(r-1)})$ given as $\lambda\gamma_0^r = \gamma_0^{r-1}\alpha - \log 2$ actually grows with α. It follows from the Implicit function theorem, which applied to $f(\alpha, \gamma_0) = \gamma_0^{r-1}\alpha - \lambda\gamma_0^r - \log 2$ gives that $\gamma_0'(\alpha) = \frac{\gamma_0}{\gamma_0 r\lambda-(r-1)\alpha} > 0$ for $\gamma_0 > 0$ and $\alpha > 0$. What's more $\gamma_0(\alpha) \xrightarrow{\alpha\to+\infty} +\infty$.

Proof. First, let us notice that w.l.o.g. we can assume that p is the distribution on $[0, +\infty)$, just replacing X – variable with distribution p, by $X^+ = X\mathbb{1}_{\{X\geq 0\}}$, since the probability of the existence of a descending path for such changed distribution is definitely non smaller.

Let $H_k = \{\exists u \in \mathfrak{T} : |u| = k,\ X_u \geq \gamma n^{1/r}\}$ and $H_0 = (\bigcup_{k=1}^n H_k)^c$. Then we have

$$\rho(\alpha, n) = \sum_{k=1}^n \mathbb{P}(A_{\alpha,n} \cap H_k \cap (\bigcup_{j=1}^{k-1} H_j)^c) + \mathbb{P}(A \cap H_0). \tag{2.3}$$

Denote as S_m the sum of m independent variables of distribution p. Let also $E = \gamma n^{1/r}$. Then we have

$$\begin{aligned}
&\mathbb{P}(A_{\alpha,n} \cap H_k \cap (\bigcup_{j=1}^{k-1} H_j)^c) \\
&\quad \le \mathbb{P}(\exists |u| = k : \ (\forall v \le u) \sum_{\substack{\varnothing < w \le v \\ w \ne \varnothing}} X_w \ge \alpha n^{1/r-1}|v|, \quad (\forall v < u)\, X_v < E, \quad X_u \ge E) \\
&\quad \le \mathbb{P}(\exists |u| = k : \sum_{\substack{v<u \\ v \ne \varnothing}} X_v \ge \alpha n^{1/r-1}(k-1), \quad (\forall v < u)\, X_v < E, \quad X_u \ge E) \\
&\quad \le \mathbb{E}\Bigg[\sum_{|u|=k} \mathbb{1}\Bigg\{\sum_{\substack{v<u \\ v \ne \varnothing}} X_v \ge \alpha n^{1/r-1}(k-1), \quad (\forall v < u)\, X_v < E, \quad X_u \ge E\Bigg\}\Bigg] \\
&\quad = 2^k \cdot \mathbb{P}\left(S_{k-1} \ge \alpha n^{1/r-1}(k-1), \quad (\forall i \in [\![1, k-1]\!])\, X_i < E, \ \ X_k \ge E\right) \\
&\quad \le 2^k \cdot \mathbb{P}\left(\sum_{i=1}^{k-1} X_i^{(E)} \ge \alpha n^{1/r-1}(k-1)\right) \cdot \mathbb{P}(X_k \ge \gamma n^{1/r}),
\end{aligned} \tag{2.4}$$

where $X_i^{(E)} = X_i \mathbb{1}_{\{X_i < E\}}$. By Chebyshev's inequality and the independence of $X_i^{(E)}$ we have that

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{k-1} X_i^{(E)} \ge D\right) &\le \exp\{-sD\}\, \mathbb{E}\left[\exp\left\{s \sum_{i=1}^{k-1} X_i^{(E)}\right\}\right] \\
&= \exp\{-sD\}\, \mathbb{E}\left[\exp\{sX^{(E)}\}\right]^{k-1},
\end{aligned} \tag{2.5}$$

so by (2.4) and (2.5) we conclude that

$$\begin{aligned}
&\mathbb{P}(A \cap H_k \cap (\bigcup_{j=1}^{k-1} H_j)^c) \le \\
&\quad \le 2^k \cdot \exp\{-s\alpha n^{1/r-1}(k-1)\}\, \mathbb{E}\left[\exp\{sX^{(E)}\}\right]^{k-1} \mathbb{P}(X_k \ge \gamma n^{1/r}).
\end{aligned} \tag{2.6}$$

At the end of the proof we show that for $s = \lambda E^{r-1}/(1+2\epsilon)$ for $\epsilon > 0$

$$\mathbb{E}\left[\exp\{sX^{(E)}\}\right] \le 1 + s \cdot \mathbb{E}[X] + o(s). \tag{2.7}$$

Using the inequality $1 + x \le e^x$ on the RHS of (2.7) it implies that

$$\mathbb{E}\left[\exp\{sX^{(E)}\}\right]^{k-1} \le \exp\{s(k-1)\mathbb{E}[X] + ko(s)\}, \tag{2.8}$$

so by (2.6)

$$\begin{aligned}
&\mathbb{P}(A_{\alpha,n} \cap H_k \cap (\bigcup_{j=1}^{k-1} H_j)^c) \le \\
&\quad \le 2^k c(\gamma n^{1/r}) \exp\{-s\alpha n^{1/r-1}(k-1) + s(k-1)\mathbb{E}[X] + ko(s) - \lambda\gamma^r n\}.
\end{aligned} \tag{2.9}$$

Similarly

$$\begin{aligned}
\mathbb{P}(A_{\alpha,n}\cap H_0) =&\mathbb{P}\Bigg(\exists |u|=n:\ (\forall v\le u)\ \sum_{\substack{w\le v\\ w\neq\varnothing}} X_w \ge \alpha n^{1/r-1}|v|,\quad (\forall v\le u)\ X_v<E\Bigg)\\
\le&\mathbb{P}\Bigg(\exists |u|=n:\ \sum_{\substack{v\le u,\\ v\neq\varnothing}} X_v \ge \alpha n^{1/r},\quad (\forall v\le u)\ X_v<E\Bigg)\\
\le&\mathbb{E}\left[\sum_{|u|=n}\mathbb{1}\left\{\sum_{\substack{v\le u,\\ v\neq\varnothing}} X_v \ge \alpha n^{1/r},\quad (\forall v\le u)\ X_v<E\right\}\right]\\
\le&2^n\cdot\mathbb{P}\left(S_n\ge \alpha n^{1/r},\quad (\forall i\in[\![1,n]\!])\ X_i<E\right)\\
\le&2^n\cdot\mathbb{P}\left(\sum_{i=1}^n X_i^{(E)}\ge \alpha n^{1/r}\right)\\
\le&2^n\exp\{-s\alpha n^{1/r}\}\mathbb{E}\left[\exp\{sX^{(E)}\}\right]^n\\
\le&2^n\exp\{-s\alpha n^{1/r}+sn\mathbb{E}[X]+no(s)\}.
\end{aligned}\tag{2.10}$$

So by (2.3), (2.9) and (2.10) we get that

$$\begin{aligned}
\rho(\alpha,n)\le& M\exp\{-\lambda\gamma^r n\}\sum_{k=1}^n 2^k\exp\{-s\alpha n^{1/r-1}(k-1)+s(k-1)\mathbb{E}[X]+ko(s)\}\\
&+2^n\exp\{-s\alpha n^{1/r}+sn\mathbb{E}[X]+no(s)\}\\
\le& M\exp\{-\lambda\gamma^r n\}\frac{2\exp\{o(s)\}}{1-2\exp\{-s\alpha n^{1/r-1}+s\mathbb{E}[X]+o(s)\}}\\
&+2^n\exp\{-s\alpha n^{1/r}+sn\mathbb{E}[X]+o(sn)\}\\
\le& MH\exp\{-\lambda\gamma^r n\}+2^n\exp\{-s\alpha n^{1/r}+sn\mathbb{E}[X]+o(sn)\}
\end{aligned}\tag{2.11}$$

where $H=2\exp\{o(s)\}/(1-2\exp\{-s\alpha n^{1/r-1}+s\mathbb{E}[X]+o(s)\})$ and supposing that

$$2\exp\{-s\alpha n^{1/r-1}+s\mathbb{E}[X]+o(s)\}<1,\tag{2.12}$$

since we used a formula for the sum of a geometric series. The definition of s, $s=\lambda(\gamma n^{1/r})^{r-1}/(1+2\epsilon)$ yields that the assumption (2.12) is equivalent to the assumption

$$\gamma<\left((\log 2+s\mathbb{E}[X]+o(s))\cdot\frac{1+2\epsilon}{\lambda\alpha}\right)^{1/(r-1)}.$$

So by (2.11) we have

$$\begin{aligned}
\rho(\alpha,n)\le& MH\exp\{-\lambda\gamma^r n\}+\\
&+2^n\exp\{\lambda\gamma^{r-1}(-\alpha n+n^{2-1/r}\mathbb{E}[X])/(1+2\epsilon)+o(n^{2-1/r})\}
\end{aligned}$$

Let $b_n=\lambda\gamma^{r-1}n^{2-1/r}\mathbb{E}[X]/(1+2\epsilon)+o(n^{2-1/r})$, then we have

$$\begin{aligned}
&\log\sqrt[n]{\rho(\alpha,n)}\le\\
&\qquad\le\log\sqrt[n]{M\exp\{-\lambda\gamma^r n\}+\exp\{-\lambda\alpha\gamma^{r-1}n/(1+2\epsilon)+n\log 2+b_n\}}.
\end{aligned}$$

Since $b_n\ll n$, then

$$\limsup_n\log\sqrt[n]{\rho(\alpha,n)}\le\max\{-\lambda\gamma^r,\log 2-\lambda\alpha\gamma^{r-1}/(1+2\epsilon)\}.\tag{2.13}$$

And by the fact that $s \xrightarrow{n\to\infty} 0$, then (2.13) holds for all for every $\gamma < \left(\log 2 \cdot \frac{1+2\epsilon}{\lambda\alpha})\right)^{1/(r-1)}$ and we can minimize it by γ. Let $f(\gamma) = -\lambda\gamma^r$, $g(\gamma) = \log 2 - \lambda\alpha\gamma^{r-1}/(1+2\epsilon)$. f is strictly decreasing and g is strictly increasing, they are both negative for considered γ. Then minimum of $\max\{f(\gamma), g(\gamma)\}$ is reached in γ_0 such that $f(\gamma_0) = g(\gamma_0)$. Since ϵ can be taken arbitrarily small, the conclusion follows.

Proof of (2.7). In this proof we follow [10] (more precisely the proof of (18) in the proof of Theorem 2).

Using the estimate $e^x \le 1 + x + \frac{1}{2}x^2 + \cdots + x^k/k! + (1/(k+1)!)x^{k+1}e^x$, we have

$$\begin{aligned}(2.14)\quad \mathbb{E}\left[\exp\left\{sX^{(E)}\right\}\right] \\ &\le 1 + s\mathbb{E}[X^{(E)}] + \frac{1}{2}s^2\mathbb{E}[(X^{(E)})^2] + \cdots + \frac{s^k}{k!}\mathbb{E}[(X^{(E)})^k] \\ &\quad + \frac{s^{k+1}}{(k+1)!}\mathbb{E}[(X^{(E)})^{k+1}\exp\left\{sX^{(E)}\right\}].\end{aligned}$$

By Remark 2.8 we know that $\limsup_E \mathbb{E}[(X^{(E)})^k] < \infty$ for every k. So it suffices to show that the term $\mathbb{E}[(X^{(E)})^{k+1}\exp\left\{sX^{(E)}\right\}]$ remains bounded for $n \to \infty$.

Fix $\epsilon > 0$. Due to Cauchy-Schwarz inequality,

$$\begin{aligned}(2.15)\quad \mathbb{E}\left[(X^{(E)})^{k+1}\exp\left\{sX^{(E)}\right\}\right] \\ &\le \mathbb{E}\left[(X^{(E)})^{(k+1)(1+\epsilon)/\epsilon}\right]^{\epsilon/(1+\epsilon)}\mathbb{E}\left[\exp\left\{(1+\epsilon)sX^{(E)}\right\}\right]^{1/(1+\epsilon)}.\end{aligned}$$

Again, by Remark 2.8 the first term on the RHS of (2.15) remains bounded for $n \to \infty$, so we only investigate the second term.

Now we use the fact that for a non-negative random variable X and $t, K > 0$ we have

$$\mathbb{E}\left[\exp\left\{tX\mathbb{1}_{\{X\le K\}}\right\}\right] = \int_0^K te^{ts}\mathbb{P}[X > s]ds + 1 - (e^{tK} - 1)\mathbb{P}(X > K).$$

We get that for n large enough

$$\begin{aligned}&\mathbb{E}\big[\exp\{(1+\epsilon)sX^{(E)}\}\big] \\ &\quad\le 1 + \int_0^E (1+\epsilon)s\exp\{(1+\epsilon)sx\}\mathbb{P}(X > x)dx \\ &\quad\le 1 + \int_0^E M(1+\epsilon)s\exp\{(1+\epsilon)sx - \lambda x^r\}dx.\end{aligned}$$

Taking $s = \lambda E^{r-1}/(1+2\epsilon)$ we have

$$\begin{aligned}\mathbb{E}\left[\exp\{(1+\epsilon)sX^E\}\right] &\le 1 + M\int_0^E \lambda E^{r-1}\exp\left\{\frac{1+\epsilon}{1+2\epsilon}\lambda E^{r-1}x - \lambda x^r\right\}dx \\ &= 1 + M\int_0^1 \lambda E^r\exp\left\{\lambda E^r\left(\frac{1+\epsilon}{1+2\epsilon}y - y^r\right)\right\}dy.\end{aligned}$$

Since $\frac{1+\epsilon}{1+2\epsilon} \le 1 - \delta$ for some $\delta > 0$ and for $y \in [0,1]$ we have $y - y^r \le 0$, so

$$\begin{aligned}\mathbb{E}\left[\exp\{(1+\epsilon)sX^{(E)}\}\right] &\le 1 + M\int_0^1 \lambda E^r\exp\{\lambda E^r(-\delta y)\}dy \\ &= 1 - \frac{M}{\delta}\left[\exp\{-\lambda E^r\delta\} - 1\right] \\ &= 1 - \frac{M}{\delta}\left[\exp\{-\lambda\gamma^r n\delta\} - 1\right].\end{aligned}$$

So the second term in (2.15) also remains bounded for $n \to \infty$. $\square$

2.4. **Definition of the rate function.**

Proposition 2.11. *There exist a sequence n_k such that $r(\alpha, [\beta n_k])/[\beta n_k]$ is convergent for any rational $\alpha, \beta > 0$.*

Proof. By the lemmas 2.5, 2.7 and 2.9 we know that $\frac{r(\alpha,[\beta n])}{[\beta n]}$ is bounded for every α and β. So, for every α and β, and for every sequence $\{n_k\}$ there exists such a subsequence $\{n_{k_i}\}$ that $r(\alpha, [\beta n_{k_i}])/[\beta n_{k_i}]$ converges.

Now, let $\{(\alpha_k, \beta_k) : k \in \mathbb{N}\}$ be the sequence of all the pairs of positive rational numbers. We construct a sequence $\{n_k\}$ in a following way:

(1) In first step, as $\{n_k^1\}$ we take the sequence such that $r(\alpha_1, [\beta_1 n_k^1])/[\beta_1 n_k^1]$ converges and take $n_1 := n_1^1$;
(2) In mth step, as $\{n_k^m\}$ we take a subsequance of $\{n_k^{m-1}\}$ such that $r(\alpha_m, [\beta_m n_k^m])/[\beta_m n_k^m]$ converges. Take $n_m = n_m^m$.

From the construction $\{n_k\}$ follows that it is convergent for every rational $\alpha, \beta > 0$. $\square$

In the sequel for a rational $\alpha > 0$ we will write

$$r(\alpha) = \lim_{k\to\infty} r(\alpha, n_k)/n_k, \tag{2.16}$$

where $\{n_k\}$ is the sequence obtained by Lemma 2.11.

We aim for the extension of the definition of $r(\alpha)$ for all $\alpha \in \mathbb{R}$. For this purpose we will prove the continuity of $r(\alpha)$ for $\alpha \in \mathbb{Q}_+$.

Firstly we prove a technical lemma on the convergence in (2.16).

Lemma 2.12. *For a rational $\alpha > 0$,*

$$\frac{r(\alpha, n_k \pm 1)}{n_k} \to r(\alpha).$$

Proof. Since $A_{\alpha,n+1} \subseteq A_{\alpha,n}$ for every $n \in \mathbb{N}$, then $\rho(\alpha, n+1) \le \rho(\alpha, n)$, so

$$\frac{r(\alpha, n+1)}{n} \ge \frac{r(\alpha, n)}{n}.$$

So $\liminf_k \frac{r(\alpha,n_k+1)}{n_k} \ge r(\alpha)$. Similarly we get that $\limsup_k \frac{r(\alpha,n_k-1)}{n_k} \le r(\alpha)$.

To get the second inequality, at the end of the proof we show that for m large enough

$$c(\varepsilon(m)m^{1/r})e^{-\lambda\varepsilon(m)^r m} \cdot \rho(\alpha, m) \le \rho(\alpha, m+1), \tag{2.17}$$

where $\varepsilon(m) = \alpha\left(\left(1 + \frac{1}{m}\right)^{1/r} - 1\right)$.

Then, taking $m = n_k$ we get that

$$-\frac{\log c(\varepsilon(n_k)n_k^{1/r})}{n_k} + \lambda\varepsilon(n_k)^r + \frac{r(\alpha, n_k)}{n_k} \ge \frac{r(\alpha, n_k+1)}{n_k}. \tag{2.18}$$

Since c is bounded and $\varepsilon(m) \xrightarrow{m\to\infty} 0$, by taking $k \to \infty$ in (2.18) we get that $\limsup_k \frac{r(\alpha,n_k+1)}{n_k} \le r(\alpha)$. Similarly, taking $m = n_k - 1$ we get that

$$-\frac{\log c(\varepsilon(n_k-1)(n_k-1)^{1/r})}{n_k} + \lambda\varepsilon(n_k-1)^r \cdot \frac{n_k-1}{n_k} + \frac{r(\alpha, n_k-1)}{n_k} \ge \frac{r(\alpha, n_k)}{n_k}.$$

So $\liminf_k \frac{r(\alpha,n_k-1)}{n_k} \ge r(\alpha)$ and it will finish the proof.

To prove (2.17) consider an event B such that in BRW in the first generation there exists a particle v_1 that performs a jump $X_{v_1} \ge \varepsilon(m)m^{1/r}$, and then its descendants perform an independent BRW that belongs to $A_{\alpha,m}$. It formally means that in $\mathfrak{T}$ exists a descending path

$v_1 \le v_2 \le \cdots \le v_{m+1}$, such that $\Phi(v_1) \ge \varepsilon(m)m^{1/r}$ and $\Phi(v_i) - \Phi(v_1) \ge \alpha m^{1/r-1}(i-1)$ for $i \ge 2$. By the independence of the jumps the probability of BRW belonging to B is equal

$$p([\varepsilon(m)m^{1/r}, +\infty)) \cdot \rho(\alpha, m) = c(\varepsilon(m)m^{1/r})e^{-\lambda\varepsilon(m)^r m} \cdot \rho(\alpha, m).$$

Now we claim that $B \subseteq A_{\alpha,m+1}$. So we need to check that $\Phi(v_i) \ge \alpha(m+1)^{1/r-1}i$ for each $i \in [\![1, m+1]\!]$, so check if for each $i \in [\![1, m+1]\!]$

$$\varepsilon(m)m^{1/r} + \alpha m^{1/r-1}(i-1) \ge \alpha(m+1)^{1/r-1}i. \tag{2.19}$$

It holds for $i = m+1$ since $\varepsilon(m) = \alpha\left((1+\frac{1}{m})^{1/r} - 1\right)$, so

$$\varepsilon(m)m^{1/r} + \alpha m^{1/r} = \alpha(m+1)^{1/r}.$$

If (2.19) holds for i, then it holds for $i-1$ since $\alpha m^{1/r} \le \alpha(m+1)^{1/r}$. So (2.19) holds for all $i \in [\![1, m+1]\!]$. □

Lemma 2.13. *For any rational $\alpha, \varepsilon > 0$,*

$$r(\alpha+\varepsilon) \ge r(\alpha) \ge r(\alpha+\varepsilon) - \lambda\varepsilon^r. \tag{2.20}$$

Proof. Since $A_{\alpha+\varepsilon,n} \subseteq A_{\alpha,n}$ for every $n \in \mathbb{N}$, then we have

$$\frac{r(\alpha+\varepsilon, n_k)}{n_k} \ge \frac{r(\alpha, n_k)}{n_k}$$

so, by taking $k \to \infty$ we get the LHS inequality in (2.20).

To prove the RHS inequality we show that

$$c(\delta(m)m^{1/r})e^{-\lambda\delta(m)^r m} \cdot \rho(\alpha, m-1) \le \rho(\alpha+\varepsilon, m) \tag{2.21}$$

for $\delta(m) = \varepsilon - \alpha\left((1-\frac{1}{m})^{\frac{1}{r}} - 1\right)$ and for m large enough.

To prove (2.21) we proceed similarly as in the proof of (2.19). Let us consider an event C such that in BRW in the first generation there exists a particle v_1 that performs a jump $X_{v_1} \ge \delta(m)m^{1/r}$ and then its descendants perform an independent BRW that belongs to $A_{\alpha,m-1}$. It formally means that in $\mathfrak{T}$ exists a descending path $v_1 \le v_2 \le \cdots \le v_m$, such that $\Phi(v_1) \ge \delta(m)m^{1/r}$ and $\Phi(v_i) - \Phi(v_1) \ge \alpha(m-1)^{1/r-1}(i-1)$ for $i \in [\![2, m]\!]$. By the independence of jumps the probability of BRW belonging to C is equal

$$p([\delta(m)m^{1/r}, +\infty)) \cdot \rho(\alpha, m-1) = c(\delta(m)m^{1/r})e^{-\lambda\delta(m)^r m} \cdot \rho(\alpha, m-1).$$

Now we claim that $C \subseteq A_{\alpha+\varepsilon,m}$. So we need to check that $\Phi(v_i) \ge (\alpha+\varepsilon)m^{1/r-1}i$ for each $i \in [\![1, \ldots, m]\!]$, so if for each $i \in [\![1, \ldots, m]\!]$

$$\delta(m)m^{1/r} + \alpha(m-1)^{1/r-1}(i-1) \ge (\alpha+\varepsilon)m^{1/r-1}i. \tag{2.22}$$

For $i = 1$ the condition holds if $\delta(m)m^{1/r} \ge (\alpha+\varepsilon)m^{1/r-1}$, so if

$$\left(\varepsilon - \alpha\left((1-\frac{1}{m})^{\frac{1}{r}} - 1\right)\right) \cdot m \ge \alpha + \varepsilon. \tag{2.23}$$

For m large the LHS of the inequality behaves like $m\varepsilon - \alpha/r$, so (2.23) holds for m large enough.

To check the condition (2.22) for $i \ge 2$ it is enough to check it for $i = m$, since $\alpha(m-1)^{1/r-1} \le (\alpha+\varepsilon)m^{1/r-1}$. For $i = m$ (2.22) holds by the choice of δ

$$\delta(m)m^{1/r} + \alpha(m-1)^{1/r} = (\alpha+\varepsilon)m^{1/r}.$$

So (2.22) holds for all $i \in [\![1, m]\!]$, which gives (2.21).

By (2.21) for $m = n_k$ it follows that for k large enough

$$-\frac{\log c(\delta(n_k)n_k^{1/r})}{n_k} + \lambda\delta(n_k)^r + \frac{r(\alpha, n_k-1)}{n_k} \ge \frac{r(\alpha+\varepsilon, n_k)}{n_k},$$

where $\delta = \varepsilon - \alpha\left((1-\frac{1}{n_k})^{\frac{1}{r}} - 1\right)$. Since c is bounded and $\delta(m) \xrightarrow{m\to\infty} \varepsilon$, then by Lemma 2.12 by taking $k\to\infty$ we obtain the RHS inequality in (2.20).

□

Proposition 2.14. *The sequence $r(\alpha, n_k)/n_k$ is convergent for any $\alpha > 0$. What's more $r(\alpha) = \lim_k r(\alpha, n_k)/n_k$ is continuous and non-decreasing.*

Proof. We choose an $\alpha > 0$ and fix $\epsilon > 0$. Then we find such $\alpha_1, \alpha_2 \in \mathbb{Q}$ that $\alpha_1 \le \alpha \le \alpha_2$ and $\alpha_2 - \alpha_1 < \epsilon$. Then

$$\rho(\alpha_1, n_k) \le \rho(\alpha, n_k) \le \rho(\alpha_2, n_k),$$

so

$$r(\alpha_1) \ge \limsup_{k\to\infty} \frac{r(\alpha, n_k)}{n_k} \ge \liminf_{k\to\infty} \frac{r(\alpha, n_k)}{n_k} \ge r(\alpha_2).$$

By Lemma 2.13

$$0 \le r(\alpha_2) - r(\alpha_1) \le \lambda(\alpha_2 - \alpha_1)^r \le \lambda\epsilon^r. \tag{2.24}$$

Since ϵ can be chosen arbitrarily, then we get that

$$\limsup_{k\to\infty} \frac{r(\alpha, n_k)}{n_k} = \liminf_{k\to\infty} \frac{r(\alpha, n_k)}{n_k}.$$

By (2.24) the function r is continuous and non-decreasing. □

2.5. **Analysis of the rate function.** We want to prove also that $r(\alpha)$ is strictly increasing. To do that we prove one more technical lemma.

Lemma 2.15. *For any $\alpha, \delta > 0$ and rational $\varepsilon \in (0,1)$ such that $\alpha \ge \delta\varepsilon^{1/r-1}$,*

$$r(\alpha) + \varepsilon r(\delta) \ge (1+\varepsilon) r(\beta),$$

where

$$\beta = \frac{\alpha + \delta\varepsilon^{1/r}}{(1+\varepsilon)^{1/r}}.$$

Proof. We will prove that for every $\alpha, \delta > 0$ and $\varepsilon \in (0,1)$

$$\rho(\alpha, m)\cdot\rho(\delta, [\varepsilon m]) \le \rho(\beta, [(1+\varepsilon)m]) \tag{2.25}$$

for large enough $m \in \mathbb{N}$. It will give us that

$$\frac{r(\alpha, n_k)}{n_k} + \varepsilon\frac{r(\delta, [\varepsilon n_k])}{\varepsilon n_k} \ge (1+\varepsilon)\cdot\frac{r(\beta, [(1+\varepsilon)n_k])}{(1+\varepsilon)n_k}.$$

So by Lemma 2.12, that

$$r(\alpha) + \varepsilon r(\delta) \ge (1+\varepsilon) r(\beta)$$

which will prove the lemma.

To prove (2.25) we choose α, δ and ε and consider an event D that in BRW there exists a descending path such that in first m generations it belongs to $A_{\alpha,m}$ and in generations between m and $[(1+\varepsilon)m]$ it belongs to $A_{\delta,[\varepsilon m]}$.

It formally means that in $\mathfrak{T}$ exists a descending path $\{v_i\}_{i=1}^{[(1+\varepsilon)m]}$, $v_1 \le \cdots \le v_{[(1+\varepsilon)m]}$, $|v_i| = i$, such that $\Phi(v_i) \ge \alpha m^{1/r-1} i$ for $i \in [\![1, m]\!]$ and $\Phi(v_i) \ge \alpha m^{1/r} + \delta[\varepsilon m]^{1/r-1}(i-m)$ for $i \in [\![m+1, [(1+\varepsilon)m]]\!]$.

We claim that $D \subseteq A_{\beta,[(1+\varepsilon)m]}$. We need to check that for every $i \in [\![1, [(1+\varepsilon)m]]\!]$

$$\Phi(v_i) \ge \beta[(1+\varepsilon)m]^{1/r-1} i. \tag{2.26}$$

It holds for $i = 1$ since $\beta \le \frac{\alpha+\delta\varepsilon^{1/r}}{(1+\varepsilon)^{1/r}}$ and $\alpha \ge \delta\varepsilon^{1/r-1}$, so

$$\beta[(1+\varepsilon)m]^{1/r-1} \le \frac{\alpha+\delta\varepsilon^{1/r}}{1+\varepsilon} m^{1/r-1} \le \alpha m^{1/r-1} \le \Phi(v_1).$$

Since (2.26) holds for $i = 1$ and $\beta[(1+\varepsilon)m]^{1/r-1} \le \alpha m^{1/r-1}$, then it holds for every $i \in [\![1, m]\!]$.

Then we know that by the definition of δ and the fact that $\alpha \ge \delta\varepsilon^{1/r-1}$

$$\alpha m^{1/r} + \delta(\varepsilon m)^{1/r} \ge (\alpha + \delta\varepsilon^{1/r})m^{1/r} \ge \beta[(1+\varepsilon)m]^{1/r}$$

and

$$\beta((1+\varepsilon)m)^{1/r-1} \ge \frac{\alpha+\delta\varepsilon^{1/r}}{1+\varepsilon} m^{1/r-1} \ge \delta(\varepsilon m)^{1/r-1} \ge \delta[\varepsilon m]^{1/r-1}$$

So for m large enough

$$\Phi(v_{[(1+\varepsilon)m]}) \ge \alpha m^{1/r} + \delta[\varepsilon m]^{1/r} \ge \beta[(1+\varepsilon)m]^{1/r}$$

so (2.26) holds for $i = [(1+\varepsilon)m]$. And since for m large enough

$$\beta[(1+\varepsilon)m]^{1/r-1} \ge \delta[\varepsilon m]^{1/r-1},$$

then (2.26) holds for every $i \in [\![m+1, [(1+\varepsilon)m]]\!]$. It ends the proof. □

Proposition 2.16. *Function $r(\alpha)$ is strictly increasing.*

Proof. Let us start by noticing that by Lemma 2.15 we have

$$r(\alpha) - r\left(\frac{\alpha+\delta\varepsilon^{1/r}}{(1+\varepsilon)^{1/r}}\right) \ge \varepsilon \cdot \left(r\left(\frac{\alpha+\delta\varepsilon^{1/r}}{(1+\varepsilon)^{1/r}}\right) - r(\delta)\right), \tag{2.27}$$

for all $\delta \le \alpha\varepsilon^{1-1/r}$. By the assumption on δ we have that $\beta := \frac{\alpha+\delta\varepsilon^{1/r}}{(1+\varepsilon)^{1/r}} \le \alpha$.

Fix $\epsilon > 0$ and take such small $\varepsilon > 0$ that

$$\alpha - \frac{\alpha}{(1+\varepsilon)^{1/r}} < \epsilon.$$

Now, let us choose such small δ smaller than $\alpha\varepsilon^{1-1/r}$ that

$$r\left(\frac{\alpha}{(1+\varepsilon)^{1/r}}\right) - r(\delta) > \epsilon.$$

We are able to find such $\delta > 0$, since $r(0) = 0$, r is non-decreasing and, by lemmas 2.5 and 2.9, for all $\eta > 0$, $0 < r(\eta) < \lambda\eta^r$, so it is enough for δ to be smaller than $\left(r\left(\frac{\alpha}{(1+\varepsilon)^{1/r}}\right)/\lambda\right)^{1/r}$ and $\alpha\varepsilon^{1-1/r}$.

For such chosen ε and δ, by (2.27) and by the fact that r is non-decreasing we have that

$$r(\alpha) - r(\beta) \ge \varepsilon \cdot \left(r\left(\frac{\alpha}{(1+\varepsilon)^{1/r}}\right) - r(\delta)\right) > \varepsilon\epsilon > 0,$$

for $\beta < \alpha$ such that $\alpha - \beta < \epsilon$. The conclusion follows by the fact that ϵ is chosen arbitrarily. □

3. Elementary properties of the branching-selection system

In this section we give a formal definition of the model defined in the subsection 1.1 and we examine its basic properties. The main result of this section is the proof of the existence of $v_N = \lim_{n\to\infty} \max X_n^N/n$. All the definitions and proofs in this subsection are adapted from [6].

3.1. **Notations and definitions.** It is convenient to represent finite populations of particles as processes on the space of finite point measures on $\mathbb{R}$. That's why we start with the introduction of the notion of such measures.

Definition 3.1. A finite point measure on $\mathbb{R}$ is a measure μ, for which there exist such $M(\mu) \in \mathbb{N}$ and a sequence $\{x_i\}_{i=1}^{M(\mu)}$, $x_1 \geq \cdots \geq x_{M(\mu)}$ that

$$\mu = \sum_{i=1}^{M(\mu)} \delta_{x_i}.$$

Then $M(\mu)$ is called **the total mass of** μ and $\{x_i\}$ we call the support of μ.

We denote by $\max \mu$ and $\min \mu$ respectively the maximum and minimum of the support of μ, and the diameter $d(\mu) := \max \mu - \min \mu$.

As $\mathcal{C}$ we denote the set of all finite point measure on $\mathbb{R}$. For $N \geq 1$, we denote the set of finite point measures on $\mathbb{R}$ with total mass equal to N as $\mathcal{C}_N$.

It is easy to notice that the model that we consider, so the branching-selection system is actually a Markov chain, since the positions of the system in nth step, conditioning on the step number $n-1$ are independent on the previous steps. This explains the following definition.

Definition 3.2. Let $\{X_n^N\}_{n\geq 0}$ be a Markov chain on $\mathcal{C}_N$ that starts at a deterministic value $X_0^N \in \mathcal{C}_N$ and whose transition probabilities are given by the branching-selection mechanism with N particles defined in the subsection 1.1.

We assume that this Markov chain is defined on a probability space denoted by $(\Omega, \mathcal{F}, \mathbb{P})$.

We will also need a notion of stochastic ordering.

Definition 3.3. For two real random variables X and Y we write $X \preceq Y$ and say that X is stochastically smaller than Y, if

$$\mathbb{P}[X \geq x] \leq \mathbb{P}[Y \geq x]$$

for all $x \in \mathbb{R}$.

Similarly, we define the ordering in the set of finite point measures.

Definition 3.4. Given $\mu, \nu \in \mathcal{C}$ we say that $\mu \preceq \nu$, if

$$\mu([x, +\infty)) \leq \nu([x, +\infty)) \text{ for all } x \in \mathbb{R}.$$

We will consider such ordering only on finite point measures. It turns out there is an easier formulation of such ordering in this case.

Remark 3.5. For $\mu = \sum_{i=1}^{M(\mu)} \delta_{x_i}$, $\nu = \sum_{i=1}^{M(\nu)} \delta_{y_i}$, where $x_1 \geq \cdots \geq x_{M(\mu)}$, $y_1 \geq \cdots \geq y_{M(\nu)}$, $\mu \preceq \nu$ is equivalent to $M(\mu) \leq M(\nu)$ and $x_i \leq y_i$ for all $i \in [\![1, M(\mu)]\!]$.

3.2. **Elementary Properties of the Model.** First in this subsection we prove that assuming that their existence the limits $\min X_n^N/n$ and $\max X_n^N/n$ for $n \to \infty$ are equal. Then we prove the existence of the limit of $\max X_n^N/n$.

At first let us notice that the expectation of $\max X_n^N$ is finite.

Remark 3.6. Let m be the maximum of $2Nn$ i.i.d. variables of distribution p. The expectation of m is finite, since the maximum can be bounded by the sum, so

$$\mathbb{E}[m] \leq 2Nn\mathbb{E}|X| < \infty,$$

where X has distribution p. So $\mathbb{E} \max X_n^N$ is finite as it is non greater than $\mathbb{E}m$.

3.2.1. *Estimates on the diameter.*

Proposition 3.7. *Let $u_N := \lceil \frac{\log N}{\log 2} \rceil + 1$. Let $m_N^{(1)}$ and $m_N^{(2)}$ denote respectively the minimum and the maximum of the $2Nu_N$ steps performed by the system between times $n - u_N$ and n. Then for all $N \geq 1$, any initial population $X_0^N \in \mathcal{C}_N$, and all $n \geq u_N$,*

$$d(X_n^N) \leq u_N \cdot (m_N^{(2)} - m_N^{(1)}).$$

Proof. Fix N, X_0^N and $n \geq u_N$. Let $y := \max X_{n-u_N}^N$ and let u be the particle that in the $n - u_N$th step reaches the maximal position y. We want to study the evolution of the branching-selection system between times $n - u_N$ and n.

Assume first, that for all $k \in [\![n - u_N + 1, n]\!]$, the condition $\min X_k^N < y + (k - (n - u_N))m_N^{(1)}$ holds. Let us examine all the descendants of u. Since all the steps performed between times $n - u_N$ and n are at least $m_N^{(1)}$, the position of a descendent of u in kth generation is greater than $y + (k - (n - u_N))m_N^{(1)}$, so greater that $\min X_k^N$. Which means that all the descendants of u from generation n are in the system X_n^N. But there is $2^{u_N} > N$ of them, which is a contradiction.

So, there exists $k \in [\![n - u_N + 1, n]\!]$, such that

$$\min X_k^N \geq y + (k - (n - u_N))m_N^{(1)}. \tag{3.1}$$

Again, by the fact that all the steps performed between times $n - u_N$ and n are at least $m_N^{(1)}$, if the condition (3.1) holds in the step kth, it also holds in the steps $k + 1, \dots, n$. So it has to hold in nth step, so we conclude that $\min X_n^N \geq y + u_N m_N^{(1)}$.

Now, by the definition of $m_N^{(2)}$ and y, we have that $\max X_n^N \leq y + u_N m_N^{(2)}$, so

$$d(X_n^N) = \max(X_n^N) - \min(X_n^N) \leq u_N \cdot (m_N^{(2)} - m_N^{(1)}).$$

□

Corollary 3.8. *For all $N \geq 1$ and any initial population $X_0^N \in \mathcal{C}_N$*

$$\lim_{n\to\infty} n^{-1} d(X_n^N) = 0,$$

with probability 1 and in $L^1(\mathbb{P})$.

Proof. Using the notation of Proposition 3.7, let $F_N := m_N^{(2)} - m_N^{(1)}$. The expectation of F_N is finite, which follows from Remark 3.6, as the expectation of F_N is non greater than the expectation of $m_N^{(2)}$.

By Proposition 3.7 we get that

$$\mathbb{E}[n^{-1} d(X_n^N)] \leq n^{-1} u_N \mathbb{E}[F_N],$$

for all $n \geq u_N$. So it yields the convergence of $\mathbb{E}[n^{-1} d(X_n^N)]$ to 0 in $L^1(\mathbb{P})$ for $n \to \infty$.

Again, by Proposition 3.7 we have that for any $x > 0$

$$\sum_{n \geq u_N} \mathbb{P}(n^{-1} d(X_n^N) \geq x) \leq \sum_{n \geq u_N} \mathbb{P}(n^{-1} u_N F_N \geq x) \leq u_N \mathbb{E}(F_N)/x.$$

So applying the Borel-Cantelli lemma yields that

$$\mathbb{P}(\limsup_{n\to\infty} n^{-1} d(X_n^N) \geq x) = 0$$

for every $x > 0$. Since x can be chosen arbitrarily, we conclude the convergence of $\mathbb{E}[n^{-1} d(X_n^N)]$ to 0 with probability 1. □

3.2.2. *Monotonicity properties.* Now we prove a technical lemma. Its corollary states that for two branching-selection systems ordered in first step we are able to find such a coupling that keeps their order. This property will be useful in the proof that $\{v_N\}_N$ is monotonic.

Lemma 3.9. *For $N_1, N_2 \in \mathbb{N}$ such that $1 \le N_1 \le N_2$, and $\mu_1 \in \mathcal{C}_{N_1}$, $\mu_2 \in \mathcal{C}_{N_2}$ such that $\mu_1 \preceq \mu_2$, there exists a pair of random variables (Z^1, Z^2) taking values in $\mathcal{C}_{N_1} \times \mathcal{C}_{N_2}$ such that:*

- *the distribution of Z^i for $i \in \{1,2\}$ is that of the population of particles obtained by performing one branching-selection step (with N_i particles) starting from the population μ_i,*
- *with probability one, $Z^1 \preceq Z^2$.*

Proof. For $k \in \{1,2\}$, we write $\mu_k = \sum_{i=1}^{N_k} \delta_{x_i(k)}$, with $x_1(k) \ge \cdots \ge x_{N_k}(k)$. From the assumption that $\mu_1 \preceq \mu_2$, by Remark 3.5, we deduce that $x_i(1) \le x_i(2)$ for all $i \in [\![1, N_1]\!]$.

Now consider an i.i.d. family $\{\varepsilon_{i,j}\}_{i\in[\![1,N_2]\!],j\in\{1,2\}}$ with common distribution p. Those are the jumps performed by the particles of our systems in the first step. So let $T_k := \sum_{i=1}^{N_k} \sum_{j\in\{1,2\}} \delta_{x_i(k)+\varepsilon_{i,j}}$, and define Z^k as being formed by the N_k rightmost particles in T_k. Since $x_i(1) + \varepsilon_{i,j} \le x_i(2) + \varepsilon_{i,j}$, for all $1 \le i \le N_1$ and $j \in \{1,2\}$, we deduce that $T_1 \preceq T_2$, whence $Z^1 \preceq Z^2$. The conclusion follows. □

Remark 3.10. Let us notice that the statement of Lemma 3.9 holds for any number of branching-selection systems. The proof of Lemma 3.9 is naturally modified to the proof of the existence of such coupling for k systems, since it just requires considering greater family of epsilons, i.e. $\{\varepsilon_{i,j}\}_{i\in[\![1,N]\!],j\in[\![1,k]\!]}$ where $N = \max_{l\le k} N_l$ and constructing more variables $\{T_l\}_{l=1}^k$.

Corollary 3.11. *For $N_1, N_2 \in \mathbb{N}$ such that $1 \le N_1 \le N_2$, and $\mu_1 \in \mathcal{C}_{N_1}$, $\mu_2 \in \mathcal{C}_2$ such that $\mu_1 \preceq \mu_2$, there exists a coupling $(Z_n^1, Z_n^2)_{n\ge0}$ between two versions of the branching-selection particle system, with N_1 and N_2 particles respectively, such that $Z_0^1 := \mu_1$, $Z_0^2 := \mu_2$, and $Z_n^1 \preceq Z_n^2$ for all $n \ge 0$.*

To prove the existence of the limit of $\max X_n^N/n$ we will use the Kingman's subadditive ergodic theorem. Below we present one of its versions. The proof can be found in [7] (Theorem 6.4.1., p. 343).

Theorem 3.12 (Kingman's subadditive ergodic theorem)**.** *Suppose a family of random variables $X_{m,n}$ satisfy:*

(i) $X_{0,n+m} \le X_{0,n} + X_{n,m}$.
(ii) $\{X_{nk,k}\}_{n\ge1}$ *are i.i.d. for every* k.
(iii) The distribution of $\{X_{m,k}\}_{k\ge1}$ *doesn't depend on* m.
(iv) $\mathbb{E}X_{0,1}^+ < \infty$ *and for each* n, $\mathbb{E}X_{0,n} \ge \gamma_0 n$, *where* $\gamma_0 > -\infty$.

Then $X = \lim_{n\to\infty} X_{0,n}/n$ exists a.s. and in L^1 and $X = \inf_m \mathbb{E}X_{0,m}/m$ a.s.

Proposition 3.13. *There exists v_N such that, with probability 1, and in $L^1(\mathbb{P})$,*

$$\lim_{n\to\infty} \min X_n^N/n = \lim_{n\to\infty} \max X_n^N/n = v_N \tag{3.2}$$

and $v_N = \inf_n \mathbb{E}[\max X_n^N/n]$.

Proof. Firstly, let us note that, in view of Corollary 3.8, if either of the two limits in (3.2) exists, then the other must exist too and have the same value. So it is enough to prove the statement with (3.2) replaced by

$$\lim_{n\to\infty} \max X_n^N/n = v_N. \tag{3.3}$$

Let us also note that it is enough to prove the result for such systems that $X_0^N = N\delta_0$. It follows from the observation that the limit in the thesis is translation invariant, meaning that shifting all the particles by a translation on $\mathbb{R}$ doesn't change the limit. Having a system

$\{X_n^N\}_{n\geq 0}$ we take two systems: $\{Y_n^N\}_{n\geq 0}$ such that $Y_0^N = N\delta_{\min X_0^N}$ and $\{Z_n^N\}_{n\geq 0}$ such that $Z_0^N = N\delta_{\max X_0^N}$. Then by Remark 3.10 we get that

$$Y_n^N \preceq X_n^N \preceq Z_n^N$$

for all n, and so by Remark 3.5

$$\max Y_n^N \leq \max X_n^N \leq \max Z_n^N.$$

So having (3.3) for $\{Y_n^N\}$ and $\{Z_n^N\}$, we conclude it for $\{X_n^N\}$.

So we perform the proof of (3.3) for such system $\{X_n^N\}$ that $X_0^N = N\delta_0$.

Consider a family $\{\varepsilon_{l,i,j}\}_{l\geq 0, i\in\{1,\dots,N\}, j\in\{1,2\}}$ of i.i.d. random variables with common distribution p. For $l \geq 0$, we denote by $\{W_{l,k}^N\}_{k\geq 0}$ a N–BRW such that:

- $W_{l,0}^N := N\delta_0$;
- For $k \geq 0$, we write $W_{l,k}^N = \sum_{i=1}^N \delta_{x_i}$, where $x_1 \geq \dots \geq x_N$, and we define $T_{l,k} := \sum_{i=1}^N \delta_{x_i+\varepsilon_{l+k,i,1}} + \delta_{x_i+\varepsilon_{l+k,i,2}}$. Then $W_{l,k+1}^N$ is obtained from $T_{l,k}$ by keeping only the N rightmost particles.

Observe that for every l, $\{W_{l,n}^N\}_{n\geq 0}$ has the same distribution as $\{X_n^N\}_{n\geq 0}$.

Below on Figure 1 we present 6 generations of simulated W_1^N, W_2^N and W_3^N. We shift W_2^N and W_3^N in time to outline the relation between the jumps of those processes.

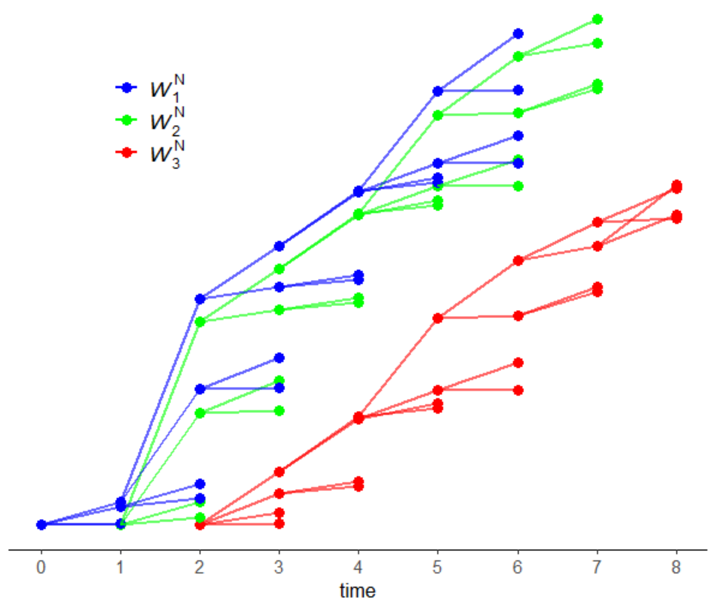

FIGURE 1. Simulated 6 generations of W_1^N, W_2^N and W_3^N where p is $Weibull(1)$

Now, we are going to use the Kingman's subadditive ergodic theorem. So below we check the assumptions in the same order as stated in Theorem 3.12.

Firstly, let us notice that for all $n, m \geq 0$

$$\max W_{0,n+m}^N \leq \max W_{0,n}^N + \max W_{n,m}^N. \tag{3.4}$$

If we have two systems that perform the same branching steps, and at some point one is greater than another, then it will be greater in every next step by the same argument as in the proof of Lemma 3.9. At (3.4) we compare maximums of two systems that perform the same branching steps (due to the construction of $\{W_{l,k}^N\}$, the epsilons used in construction of

$W^N_{0,n+m}$ and $W^N_{0,n}$, $W^N_{n,m}$ are exactly the same), but at moment n one of them is shifted to $N\delta_{\max W^N_{0,n}}$. Since $W^N_{0,n} \preceq N\delta_{\max W^N_{0,n}}$, (3.4) holds.

Secondly, let us notice that $\{W^N_{nd,d}\}_{n\geq 0}$ is an i.i.d. family. It follows from the fact that the random measure $W^N_{nd,d}$ is obtained by performing d branching-selection steps that depend only on a collection $\mathcal{E}_d := \{\varepsilon_{d(n+1),i,j} : i \in [\![1,N]\!], j \in \{1,2\}\}$, where $\{\mathcal{E}_d : d \geq 0\}$ are i.i.d. So we deduce that the sequence $\{\max W^N_{nd,d}\}_{n\geq 0}$ is i.i.d.

Thirdly, the distribution of $\{\max W^N_{l,k}\}_{k\geq 0}$ doesn't depend on l, since the distribution of $\{W^N_{l,k}\}_{k\geq 0}$ doesn't.

Finally, let us note that $\max W^N_{0,1} = \max_{i\in[\![2N]\!],j\in\{1,2\}} \varepsilon_{0,i,j}$, so

$$\mathbb{E}|\max W^N_{0,1}| \leq 2N\mathbb{E}|\varepsilon| < +\infty$$

and

$$\mathbb{E}\max W^N_{0,1} \geq \mathbb{E}[\varepsilon],$$

where ε has distribution p. Inductively, we prove that $\mathbb{E}[\max W^N_{0,n}] \geq n\cdot\mathbb{E}[\varepsilon]$, since $\max W^N_{0,n+1} \geq \max W^N_{0,n} + \max_{j\in\{1,2\}} \varepsilon_{n,1,j}$.

Theorem 3.12 gives that $\lim_{n\to\infty} W^N_{0,n}/n = v_N = \inf_n \mathbb{E}[\max W^N_{0,n}/n]$ in $L^1(\mathbb{P})$ and with probability 1. Since $\{W^N_{0,n}\}_n$ has the same distribution as $\{X^N_n\}_n$, the conclusion follows. □

Proposition 3.14. *The sequence $(v_N)_{N\geq 1}$ is non-decreasing.*

Proof. Take $N_1, N_2 \in \mathbb{N}$, such that $N_1 \leq N_2$. By Corollary 3.11 since $N_1\delta_0 \preceq N_2\delta_0$ we get the coupling of two branching-selection systems with N_1 and N_2 particles $\{X^{N_1}_n, Y^{N_2}_n\}_{n\geq 0}$ such that $X^{N_1}_0 = N_1\delta_0$, $Y^{N_2}_0 = N_2\delta_0$, and $X^{N_1}_0 \preceq Y^{N_2}_0$ for all $n \geq 0$. Then, by Proposition 3.13 we get that

$$v_{N_1} = \lim_{n\to+\infty} \max X^{N_1}_n/n \leq \lim_{n\to+\infty} \max Y^{N_2}_n/n = v_{N_2}.$$

□

3.3. **Coupling with a family of N branching random walks.** We now give one more formal definition of the branching-selection system. It is equivalent to the one given in the definition 3.2, but uses the BRWs defined in section 2. It will be useful to combine the concept of branching-selection system with the killed branching random walks.

Let $\{BRW_i\}_{i\in[\![1,N]\!]}$ denote N independent copies of a branching random walk BRW. Each BRW_i consists of a binary tree $\mathfrak{T}_i$ and a map Φ_i. For $i \in [\![1,N]\!]$ and $n \geq 0$ we define the disjoint union

$$\mathcal{T}^N_n := \mathfrak{T}_1(n) \sqcup \cdots \sqcup \mathfrak{T}_N(n).$$

For every n, we fix an a priori total order on $\mathcal{T}^N_n$ depending only on the tree structure.

We now define by induction a sequence $\{G^N_n\}_{n\geq 0}$ such that, for each $n \geq 0$, G^N_n is a random subset of $\mathcal{T}^N_n$ containing exactly N elements:

- $G^N_0 := \mathcal{T}^N_0$;
- Given $n \geq 0$ and G^N_n, let $H^N_n \subseteq \mathcal{T}^N_{n+1}$ contains the children of G^N_n. Then, let $G^N_{n+1} \subseteq H^N_n$ contains the N vertices that are associated with the largest values of the underlying random walks Φ_is (breaking the ties by using the priori order on $\mathcal{T}^N_n$).

Now let $\mathfrak{X}^N_n$ denote the distribution of the values of corresponding Φ_is on the vertices from G^N_n. The sequence $\{\mathfrak{X}^N_n\}_{n\geq 0}$ has the same distribution as $\{X^N_n\}_{n\geq 0}$ started from $X^N_0 := N\delta_0$. Thus, we can take for our reference probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the one on which $BRW_1, \ldots, BRW_N$ are defined, and let X^N_n has the distribution $\mathfrak{X}^N_n$. This way we obtain a coupling between $\{X^N_n\}_{n\geq 0}$ (such that $X^N_0 = N\delta_0$) and the branching random walks $BRW_1, \ldots, BRW_N$.

4. Upper bound on v_N

Theorem 4.1. *There exists such* $\alpha^* > 0$ *that*

$$\limsup_{N\to\infty} \frac{v_N}{(\log N)^{1/r-1}} \leq \alpha^*.$$

What's more, $\alpha^* \geq \log 2/\lambda^{1/r}$.

In the proof of this theorem a following combinatorial lemma is useful. It gives the connection between the theorem and the problem investigated in section 2 – the probability that the particle in BRW is (m, v)–good.

The lemma and its proof are adapted from [6].

Lemma 4.2. *Let* $v_1, v_2 \in \mathbb{R}$ *be such that* $v_1 < v_2$. *Let* $K > 0$. *Take* $n \in \mathbb{N}$, *such that* $n \geq 1$ *and* $m \in [\![1, n]\!]$. *Let now* $x_0, \ldots, x_n$ *be a sequence of real numbers such that* $x_0 = 0$ *and* $x_{i+1} - x_i \leq K$ *for all* $i \in [\![0, n-1]\!]$. *Let*

$$I := \{i \in [\![1, n-m]\!] :\ x_i - x_i \geq v_1(j-i) \text{ for all } j \in [\![i, i+m]\!]\}.$$

If $x_n \geq v_2 n$, *then* $\#I \geq \frac{v_2-v_1}{K-v_1}\frac{n}{m} - K/(K-v_1)$.

Proof. Let $0 =: x_0, \ldots, x_n$ be as in the statement of the lemma. We define inductively a sequence $\{\tau_i\}_i$.

- $\tau_0 := 0$.
- Given $\tau_i \leq n$ we define

  $$\tau_{i+1} := \inf\{j \in [\![\tau_i + 1, n]\!] : x_j < x_{\tau_i} + v_1(j - \tau_i) \text{ or } j = \tau_i + m\},$$

  where we assume that $\inf \varnothing = n+1$.

Let $G := \max\{i : \tau_i < n\}$, then $\tau_G \geq n - m$.

Now we "colour" the integers $0, \ldots, n-1$. If $x_{\tau_{i+1}} \geq x_{\tau_i} + v_1(\tau_{i+1} - \tau_i)$ and $\tau_{i+1} \leq n$, then $\tau_i, \ldots, \tau_{i+1} - 1$ are coloured red. Note that then τ_i belongs to I. The remaining integers in $0, \ldots, n-1$ are coloured blue.

Now, let V_{red} be the set of the integers coloured red in $[\![0, n-1]\!]$, and V_{blue} of the blue terms. Since for every k we have $x_{k+1} - x_k \leq K$, and for every blue τ_k we have $x_{\tau_{k+1}} - x_{\tau_k} < v_1(\tau_{k+1} - \tau_k)$ then

$$\begin{aligned}
x_n &= (x_n - x_{\tau_G}) + \sum_{k=0}^{G-1} (x_{\tau_{k+1}} - x_{\tau_k}) \\
&= \sum_{k=\tau_G}^{n-1} (x_{k+1} - x_k) + \sum_{\{k<G\colon \tau_k \in V_{red}\}} (x_{\tau_{k+1}} - x_{\tau_k}) + \sum_{\{k<G\colon \tau_k \in V_{blue}\}} (x_{\tau_{k+1}} - x_{\tau_k}) \\
&\leq K \cdot m + K \sum_{\{k<G\colon \tau_k \in V_{red}\}} (\tau_{k+1} - \tau_k) + v_1 \sum_{\{k<G\colon \tau_k \in V_{blue}\}} (\tau_{k+1} - \tau_k) \\
&\leq K \cdot m + K \cdot \#V_{red} + v_1 \cdot \#V_{blue}.
\end{aligned}$$

Since $\#V_{red} + \#V_{blue} = n$ and $v_2 n \leq x_n$, we have that

$$\#V_{red} \geq \frac{v_2 - v_1}{K - v_1} n - \frac{Km}{K - v_1}.$$

Thus, since at least $\#V_{red}/m$ terms belong to I, we have

$$\#I \geq \#V_{red}/m \geq \frac{v_2 - v_1}{K - v_1}\frac{n}{m} - \frac{K}{K - v_1}.$$

□

Proof of Theorem 4.1. Let $m = [\log N]$. Fix $\epsilon > 0$, then by Theorem 2.3 there exist such α^* that

$$\log(\rho(\alpha^*, m)) \leq -(1 + 2\epsilon)m + o(m),$$

so for sufficiently large N,

$$\rho(\alpha^*, m) \leq \frac{1}{N^{1+\epsilon}}. \tag{4.1}$$

Let now $n = [F \log{(N)}^2]$ for some $F > \delta^{-1}\left(\frac{1+\epsilon}{\lambda}\right)^{1/r}$, $v_1 = (\alpha^* + \delta)m^{1/r-1}$ for some $\delta > 0$, $v_2 = \alpha^* m^{1/r-1}$ and $\varphi = (\frac{1}{\lambda})^{\frac{1}{r}} (\log N)^{\frac{2}{r}(1+\zeta)}$ for some $\zeta > 0$. Then for N large enough $\varphi > v_2 n$ and

$$\begin{aligned} &\mathbb{E}[\frac{1}{n} \max X_n^N] \leq \\ &\leq v_2 + \mathbb{E}[\frac{1}{n} \max X_n^N \mathbb{1}_{\{\max X_n^N \in (v_2 n, \varphi)\}}] + \mathbb{E}[\frac{1}{n} \max X_n^N \mathbb{1}_{\{\max X_n^N > \varphi\}}] \\ &\leq v_2 + \frac{1}{n} \varphi \mathbb{P}(\max X_n^N > v_2 n) + \frac{1}{n} \mathbb{E}[\max X_n^N \mathbb{1}_{\{\max X_n^N > \varphi\}}]. \end{aligned} \tag{4.2}$$

So in the sequel we will show that $\frac{\varphi}{n}\mathbb{P}(\max X_n^N > v_2 n)$ and $\mathbb{E}[\frac{1}{n} \max X_n^N \mathbb{1}_{\{\max X_n^N > \varphi\}}]$ converge to 0 as $N \to \infty$.

Let us start by bounding $\mathbb{P}(\max X_n^N > v_2 n)$. For this purpose we use Lemma 4.2. Let $K = (\frac{1+\epsilon}{\lambda} \log{(N)})^{1/r}$. For such K and v_1, v_2, n, m stated previously we have that

$$\lim_{N\to\infty} \frac{v_2 - v_1}{K - v_1} \frac{n}{m} = \lim_{N\to\infty} \left(\frac{\delta[\log{(N)}]^{1/r-1}}{(\frac{1+\epsilon}{\lambda} \log{(N)})^{1/r}} \frac{F \log{(N)}^2}{\log{(N)}} \right) = \delta F \left(\frac{1+\epsilon}{\lambda} \right)^{-1/r}$$

and $\lim_{N\to\infty} \frac{K}{K - v_1} = 1$. By the assumption on F, $\delta F / (\frac{1+\epsilon}{\lambda})^{1/r} > 1$, so for N large enough we have

$$\frac{v_2 - v_1}{K - v_1} \frac{n}{m} - K/(K - v_1) > 0.$$

Now consider such $u \in G_n^N$ that $\Phi_i(u) = \max X_n^N$. Then u belongs to some BRW_i, so $u \in \mathfrak{T}_i$ and in $\mathfrak{T}_i$ exists a descending path $\varnothing = u_1 \leq u_2 \leq \cdots \leq u_n = u$ such that $u_l \in G_l^N$ for every $l \in [\![1, \ldots, n]\!]$. Using Lemma 4.2 to the sequence $\{\Phi_i(u_n)\}$ and v_1, v_2, K as above, we get that if $\Phi(u_n) = \max X_n^N \geq v_2 n$, then either one of random steps $\Phi_i(u_{l+1}) - \Phi_i(u_l)$ is greater of equal to K or there exists such $l \in [\![1, n - m]\!]$ that $\Phi_i(u_j) - \Phi_i(u_k) \geq v_1(j - k)$ for all $j \in [\![k, k + m]\!]$, so one of $u_0, \ldots, u_{n-m}$ is $(m, \alpha^* + \delta)$–good in BWR_i.

So if

$$B_n := \#\{u \in G_0^N \cup \cdots \cup G_n^N : u \text{ is } (m, \alpha^* + \delta)\text{–good}\},$$

then by the union bound

$$\begin{aligned} &\mathbb{P}(\max X_n^N \geq v_2 n) \leq \\ &\leq \mathbb{P}(\max\{\Phi_i(vk) - \Phi_i(v) : |v| \leq n, i \in [\![1, N]\!], k \in \{1, 2\}\} \geq K) \\ &\qquad + \mathbb{P}(B_n \geq 1). \end{aligned} \tag{4.3}$$

By the fact that all of the variables $\{\Phi_i(vk) - \Phi_i(v) : |v| \leq n, i \in [\![1, N]\!], k \in \{1, 2\}\}$ are i.i.d. with common distribution p we have that

$$\begin{aligned} &\mathbb{P}(\max\{\Phi_i(vk) - \Phi_i(v) : |v| \leq n, i \in [\![1, N]\!], k \in \{1, 2\}\} \geq K) = \\ &= 1 - p((-\infty, K])^{2Nn} = 1 - (1 - c(K)e^{-\lambda K^r})^{2Nn} \\ &\leq 2Nnc(K)e^{-\lambda K^r} \leq 2MNne^{-(1+\epsilon)\log{(N)}} \\ &= 2Mn\frac{1}{N^\epsilon}. \end{aligned} \tag{4.4}$$

By the definition of B_n we have

$$B_n = \sum_{u\in\mathfrak{T}_1\cup\cdots\cup\mathfrak{T}_n} \mathbb{1}\{u \text{ is } (m,\alpha^*+\delta)\text{–good}\}\cdot \mathbb{1}\{u\in G_0^N\cup\cdots\cup G_n^N\}.$$

Fix now $u\in\mathfrak{T}_1\cup\cdots\cup\mathfrak{T}_n$ such that $|u|=l$. Then by the definition of the branching-selection system an event $\{u\in G_0^N\cup\cdots\cup G_n^N\}$ depends only on system steps performed at depth at most l. That is $\{u\in G_0^N\cup\cdots\cup G_n^N\}$ belongs to the σ-algebra generated by $\{\Phi_i(u_{k+1})-\Phi_i(u_k) : i\in[\![1,N]\!], |u_{k+1}|\le l\}$. The event $\{u \text{ is } (m,\alpha^*+\delta)\text{–good}\}$ on the other hand, depends only on the system steps performed at depth at least l, i.e. belongs to the σ-algebra generated by $\{\Phi_i(u_{k+1})-\Phi_i(u_k) : i\in[\![1,N]\!], |u_{k+1}|\ge l+1\}$. Thus, those two events are independent. So by (4.1) we have

$$\begin{aligned}\mathbb{E}B_n &= \sum_{u\in\mathfrak{T}_1\cup\cdots\cup\mathfrak{T}_n} \mathbb{P}(u \text{ is } (m,\alpha^*)\text{–good})\cdot\mathbb{E}[\mathbb{1}\{u\in G_0^N\cup\cdots\cup G_n^N\}]\\ &= \rho(\alpha^*,m)\cdot\mathbb{E}[\sum_{u\in\mathfrak{T}_1\cup\cdots\cup\mathfrak{T}_n}\mathbb{1}\{u\in G_0^N\cup\cdots\cup G_n^N\}] \\ &= \rho(\alpha^*,m)\cdot N(n+1)\le(n+1)\cdot N^{-\epsilon}.\end{aligned}\tag{4.5}$$

Then by Markov inequality ($\mathbb{P}[X>1]\le\mathbb{E}[X]$), (4.3), (4.4), (4.5) we have that

$$\mathbb{P}(\max X_n^N\ge v_2 n)\le 2Mn\cdot N^{-\epsilon}+(n+1)\cdot N^{-\epsilon}.\tag{4.6}$$

Thus, since $\varphi=(\frac{1}{\lambda})^{\frac{1}{r}}(\log N)^{\frac{2}{r}(1+\zeta)}$, we have that $\varphi\ll N^{-\epsilon}$, so

$$\frac{1}{n}\varphi\mathbb{P}(\max X_n^N\ge v_2 n)\xrightarrow{N\to\infty}0.\tag{4.7}$$

Now, we estimate $\mathbb{E}[\max X_n^N\mathbb{1}_{\{\max X_n^N>\varphi\}}]$. We divide this term into two parts by using a formula $\mathbb{E}[X\mathbb{1}_{\{X\ge a\}}]=a\mathbb{P}(X>a)+\int_a^\infty\mathbb{P}(X>x)dx$. So in our case

$$\mathbb{E}[\max X_n^N\mathbb{1}_{\{\max X_n^N>\varphi\}}]=\varphi\mathbb{P}(\max X_n^N>\varphi)+\int_\varphi^\infty\mathbb{P}(\max X_n^N>x)dx.\tag{4.8}$$

Then using the coupling with family of N branching random walks we have that

$$\begin{aligned}\mathbb{P}(\max X_n^N>x)&\le\mathbb{P}\Bigg((\exists i\in[\![1,N]\!])(\exists u\in\mathfrak{T}_i(n))\sum_{\substack{v\le u,\\ v\ne\varnothing}}X_v>x\Bigg)\\ &\le\mathbb{E}\Bigg[\sum_{i=1}^N\sum_{u\in\mathfrak{T}_i(n)}\mathbb{1}\{\sum_{\substack{v\le u,\\ v\ne\varnothing}}X_v>x\}\Bigg]=N2^n\mathbb{P}(S_n>x),\end{aligned}$$

where $S_m:=\sum_{i=1}^m Y_i$ and $\{Y_i\}$ are i.i.d. random variables of distribution p. Let also Y denote random variable of distribution p and $M_m=\max_{i\in[\![1,m]\!]}Y_i$. Then notice that for $\varepsilon>0$ we have

$$\mathbb{P}(S_n\ge n^{1/r}y)\le n\mathbb{P}(Y>(1-\varepsilon)n^{1/r}y)+\mathbb{P}(S_n>n^{1/r}y, M_n\le(1-\varepsilon)n^{1/r}y).$$

Using the computations performed previously in the proof of Theorem 2.9, precisely (2.5) for $D=n^{1/r}$, $E=(1-\varepsilon)n^{1/r}y$, (2.7) and (2.8) for $s=\lambda\left((1-\varepsilon)n^{1/r}y\right)^{r-1}/(1+\delta)$ for some $\delta>0$, we obtain that

$$\begin{aligned}&\mathbb{P}(S_n>n^{1/r}y, M_n\le(1-\varepsilon)n^{1/r}y)\le\\ &\quad\le\exp\{-\lambda(1-\varepsilon)^{r-1}ny^r/(1+\delta)+\lambda(1-\varepsilon)^{r-1}n^{2-1/r}y^{r-1}\mathbb{E}Y/(1+\delta)+no(s)\}\\ &\quad=\exp\{-\lambda Cny^r+\lambda Cn^{2-1/r}y^{r-1}\mathbb{E}Y+no(s)\},\end{aligned}$$

where $C = (1-\varepsilon)^{r-1}/(1+\delta)$. So we have

(4.9)
$$
\begin{aligned}
\mathbb{P}(\max X_n^N > \varphi) \leq N2^n \mathbb{P}(S_n > \varphi) \\
\leq & N2^n \left(nM \exp\{-\lambda(1-\varepsilon)^r \varphi^r\} + \exp\{-\lambda C\left(\varphi^r - n\varphi^{r-1}\mathbb{E}Y\right) + no(s)\}\right) \\
= & N2^n \big(nM \exp\{-(1-\varepsilon)^r n^{2(1+\zeta)}\} + \\
& + \exp\{-Cn^{2(1+\zeta)} + C\lambda^{1/r} n^{2(1+\zeta)(1-1/r)+1}\mathbb{E}Y + o(n^{3-1/r-r+\frac{2}{r}(1+\zeta)(r-1)})\}\big),
\end{aligned}
$$

where M is the bound on $c(x)$ as in the Remark 2.4. Since $n = [F \log N]^2$ and $2(1+\zeta) > 2(1+\zeta)(1-1/r)+1$, $2(1+\zeta) > 3-1/r-r+\frac{2}{r}(1+\zeta)(r-1)$, then the term in the last line in (4.9) goes to 0 as $N \to \infty$. Furthermore, $\varphi = \left(\frac{1}{\lambda}\right)^{1/r} n^{\frac{2}{r}(1+\zeta)}$, so $\exp\{n^{2(1+\zeta)}\} \gg \varphi N 2^n n$. So

$$
\varphi \mathbb{P}(\max X_n^N > \varphi) \xrightarrow{N\to\infty} 0. \tag{4.10}
$$

Now, we bound the second term of (4.8)

$$
\begin{aligned}
\int_\varphi^\infty \mathbb{P}(\max X_n^N > x)dx = & n^{1/r} \int_{\varphi n^{-1/r}}^\infty \mathbb{P}(\max X_n^N > n^{1/r} y)dy \\
\leq & N2^n n^{1/r} \Bigg(nM \int_{\varphi n^{-1/r}}^{+\infty} \exp\{-\lambda(1-\varepsilon)^r n y^r\}dy + \\
& + \int_{\varphi n^{-1/r}}^{+\infty} \exp\{-\lambda C n y^r + \lambda C n^{2-1/r} y^{r-1}\mathbb{E}Y + no(s)\}dy \Bigg)
\end{aligned}
$$

Since $r-1<0$, then for $y \in [\varphi n^{-1/r}, +\infty)$

$$
\exp\{\lambda C n^{2-1/r} y^{r-1}\mathbb{E}Y\} \leq \exp\{\lambda C n^{2-1/r} (\varphi n^{-1/r})^{r-1}\mathbb{E}Y\} = \exp\{\lambda C n \varphi^{r-1}\mathbb{E}Y\},
$$

similarly

$$
s = \lambda\left((1-\varepsilon)n^{1/r}y\right)^{r-1}/(1+\delta) \leq \lambda\left((1-\varepsilon)\varphi\right)^{r-1}/(1+\delta).
$$

So

$$
\begin{aligned}
\int_\varphi^\infty \mathbb{P}(\max X_n^N > x)dx \leq & \\
\leq & N2^n n^{1/r} \Bigg(nM \int_{\varphi n^{-1/r}}^{+\infty} \exp\{-\lambda(1-\varepsilon)^r n y^r\}dy + \\
& + \exp\{\lambda C n \varphi^{r-1}\mathbb{E}Y + o(n\varphi^{r-1})\} \int_{\varphi n^{-1/r}}^{+\infty} \exp\{-\lambda C n y^r\}dy \Bigg).
\end{aligned} \tag{4.11}
$$

Now let us note that for any $A>0$ we have

$$
\begin{aligned}
\int_{\varphi n^{-1/r}} \exp\{\lambda A n y^r\}dy = & \left| \begin{array}{c} \lambda A n y^r = z \\ r\lambda A n y^{r-1} dy = dz \\ dy = \frac{1}{r}\frac{1}{(An\lambda)^{1/r}} z^{1/r-1}dz \end{array} \right| \\
= & \frac{1}{r}\frac{1}{(An\lambda)^{1/r}} \int_{A\lambda\varphi^r}^\infty z^{1/r-1}e^{-z}dz
\end{aligned}
$$

and since for $a \geq 1$ and $b \geq 1$

$$\begin{aligned}\int_a^\infty t^{b-1}e^{-t}dt &= \int_0^\infty (u+a)^{b-1}e^{-(u+a)}du \\ &= a^{b-1}e^{-a}\int_0^\infty (1+\frac{u}{a})^{b-1}e^{-u}du \\ &\leq a^{b-1}e^{-a}\int_0^\infty (1+u)^{b-1}e^{-u}du \\ &= a^{b-1}e^{-a+1}\int_1^\infty u^{b-1}e^{-u}du \\ &\leq a^{b-1}e^{-a+1}\Gamma(b),\end{aligned}$$

we have

$$\begin{aligned}\int_{\varphi n^{-1/r}} \exp\{\lambda A n y^r\}dy &\leq \frac{1}{r}\frac{1}{(An\lambda)^{1/r}}(A\lambda\varphi^r)^{1/r-1}\exp\{-A\lambda\varphi^r+1\}\Gamma(1/r) \\ &= \frac{1}{rA\lambda n^{1/r}}\varphi^{1-r}\exp\{-A\lambda\varphi^r+1\}\Gamma(1/r).\end{aligned}\tag{4.12}$$

Combining (4.11) and (4.12) with substituted $A = (1-\varepsilon)^r$ and $A = C$ we have

$$\begin{aligned}&\int_\varphi^\infty \mathbb{P}(\max X_n^N > x)dx \leq \\ &\leq N2^n n^{1/r}\Bigg(nM\frac{1}{r(1-\varepsilon)^r\lambda n^{1/r}}\varphi^{1-r}\exp\{-(1-\varepsilon)^r\lambda\varphi^r+1\}\Gamma(1/r) \\ &\quad + \exp\{\lambda C n\varphi^{r-1}\mathbb{E}Y + o(n\varphi^{r-1})\}\frac{1}{rC\lambda n^{1/r}}\varphi^{1-r}\exp\{-C\lambda\varphi^r+1\}\Gamma(1/r)\Bigg) \\ &= N2^n\frac{1}{r\lambda}\varphi^{1-r}e\Gamma(1/r)\Bigg(nM\frac{1}{(1-\varepsilon)^r}\exp\{-(1-\varepsilon)^r\lambda\varphi^r\} \\ &\quad + \frac{1}{C}\exp\{-C\lambda(\varphi^r - n\varphi^{r-1}\mathbb{E}Y) + o(n\varphi^{r-1})\}\Bigg),\end{aligned}$$

so by the computations in (4.9) we get that

$$\int_\varphi^\infty \mathbb{P}(\max X_n^N > x)dx \xrightarrow{N\to\infty} 0.\tag{4.13}$$

Combining 4.8, (4.10) and (4.13) we conclude that

$$\frac{1}{n}\mathbb{E}[\max X_n^N \mathbb{1}_{\{\max X_n^N > \varphi\}}] \xrightarrow{N\to\infty} 0.\tag{4.14}$$

Now, by (4.2), (4.7), (4.14)

$$\begin{aligned}&\mathbb{E}[\frac{1}{n}\max X_n^N] - v_2 \leq \\ &\leq \frac{1}{n}\varphi\mathbb{P}(\max X_n^N > v_2 n) + \frac{1}{n}\mathbb{E}[\max X_n^N \mathbb{1}_{\{\max X_n^N > \varphi\}}] \xrightarrow{N\to\infty} 0.\end{aligned}$$

By Proposition 3.13 $v_N = \inf_n \mathbb{E}[\frac{1}{n}\max X_n^N]$, so

$$\limsup_{N\to\infty}\frac{v_N}{\log(N)^{1/r-1}} \leq \limsup_{N\to\infty}\frac{\mathbb{E}[\frac{1}{n}\max X_n^N]}{\log(N)^{1/r-1}} \leq \limsup_{N\to\infty}\frac{v_2}{\log(N)^{1/r-1}} = \alpha^*,$$

which completes the first part of the proof. The bound on the α^* can be obtained by Proposition 2.7 and (4.1). □

5. Lower bound on v_N

Theorem 5.1. *There exists such $\alpha_* > 0$ that*

$$\liminf_{N\to\infty} \frac{v_N}{(\log N)^{1/r-1}} \geq \alpha_*.$$

Proof of Theorem 5.1. Let $m = [\log(N)]$. Fix $\epsilon > 0$, then by Theorem 2.3 exists such $\alpha > 0$ that

$$\log(\rho(\alpha, m)) \geq -(1-2\epsilon)m + o(m),$$

so for sufficiently large N,

$$\rho(\alpha, m) \geq \frac{1}{N^{1-\epsilon}}. \tag{5.1}$$

To prove the theorem we will use Proposition 4 from [6] in following version adapted for our purpose .

Proposition 5.2. *Let m be as previously stated. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Let*

$$B_{\varepsilon,n} := \{\min(X_k^N) < (\alpha-\varepsilon)m^{1/r-1}k \text{ for all } k \in [\![1,n]\!]\}.$$

Then for N large enough

$$v_N \geq (\alpha-\varepsilon)m^{1/r-1} - |(\alpha-\varepsilon)m^{1/r-1}|n\mathbb{P}(B_{\varepsilon,n}) - n\mathbb{E}(|\Theta_n|\mathbb{1}_{B_{\varepsilon,n}}),$$

where Θ_n is the minimum of $2nN$ i.i.d. random variables with distribution p.

Proof. We reuse the coupling constructed to use the Kingman's subbaditive ergodic theorem in the proof of Proposition 3.13. Then $\{X_n^N\}_{n\geq 0}$ is defined as $X_n^N := W_{0,n}^N$.

We now inductively define sequences $\{\Gamma_i\}_{i\geq 0}$, $\{L_i\}_{i\geq 0}$, $\{J_i\}_{i\geq 1}$.

- Let $\Gamma_0 := 0$, $J_0 := 0$.
- Given $i \geq 0$, Γ_i and J_i, let $L_{i+1} := \inf\{k \in [\![1,n]\!] : \min W_{\Gamma_i,k}^N \geq (\alpha-\varepsilon)m^{1/r-1}k\}$, where $\inf\emptyset := n$. Then let $\Gamma_{i+1} := \Gamma_i + L_{i+1}$, and let $J_{i+1} := J_i + \min W_{\Gamma_i,L_{i+1}}^N$.

By a similar argument as in the proof of (3.4) in the proof of Proposition 3.13 we have that

$$\min W_{0,\Gamma_i}^N \geq \sum_{k=0}^{i} \min W_{\Gamma_k,L_{k+1}}^N = J_i. \tag{5.2}$$

It follows from the fact that we compare minimums of two systems that perform the same steps, but at moments $\sum_{j=0}^k L_{j+1}$ for $k \in [\![0,i]\!]$ one of them is shifted to $N\delta_{\min W_{\Gamma_k,L_{k+1}}^N}$.

Let us observe that since $\{W_{l,k}^N\}_{l\geq 0}$ is an i.i.d. family, the variables $\{L_i\}_i$ are i.i.d. with common distribution $L := \inf\{k \in [\![1,n]\!] : \min X_k^N \geq (\alpha-\varepsilon)m^{1/r-1}k\}$ ($\inf\emptyset = n$). So variables $\{\Gamma_{i+1}-\Gamma_i\}_i$ are i.i.d. with common distribution L. Similarly $\{J_{i+1}-J_i\}$ are i.i.d. with distribution $\min X_L^N$.

By the law of large numbers and the fact that $v_N = \lim_n \min X_n^N/n$ (Proposition 3.13) we have that

$$\lim_{i\to\infty} \frac{\min X_{\Gamma_i}^N}{i} = \lim_{i\to\infty} \frac{\min X_{\Gamma_i}^N}{\Gamma_i} \cdot \frac{\Gamma_i}{i} = v_N \cdot \mathbb{E}L.$$

On the other hand, by (5.2) we have $\min X_{\Gamma_i}^N \geq J_i = \sum_{k=1}^i (J_k - J_{k-1})$. So since $\{J_{i+1}-J_i\}_i$ are i.i.d. with distribution $\min X_L^N$, by the law of large numbers we have that

$$\liminf_{i\to\infty} i^{-1}\min X_{\Gamma_i} \geq \mathbb{E}[\min X_L^N].$$

We conclude that $v_N \geq \mathbb{E}[\min X_L^N]/\mathbb{E}L$.

Now denote by Θ_n the minimum of all the jumps performed by the system between time 0 and n. By the definition of $B_{\varepsilon,n}$ we have that

$$\min X_L^N \geq (\alpha-\varepsilon)m^{1/r-1}L\mathbb{1}_{B^c_{\varepsilon,n}} + L\Theta_n\mathbb{1}_{B_{\varepsilon,n}}.$$

Since $1 \leq L \leq n$ we have that

$$\begin{aligned}\mathbb{E}[\min X_L^N] &\geq (\alpha-\varepsilon)m^{1/r-1}(\mathbb{E}L - \mathbb{E}(L\mathbb{1}_{B_{\varepsilon,n}})) + \mathbb{E}(L\Theta_n\mathbb{1}_{B_{\varepsilon,n}}) \\ &\geq (\alpha-\varepsilon)m^{1/r-1}(\mathbb{E}L - nP(B_{\varepsilon,n})) - n\mathbb{E}(\Theta_n\mathbb{1}_{B_{\varepsilon,n}}).\end{aligned}$$

It completes the proof since

$$\begin{aligned}v_N &\geq (\alpha-\varepsilon)m^{1/r-1}\left(1 - \frac{n}{\mathbb{E}L}P(B_{\varepsilon,n})\right) - \frac{n}{\mathbb{E}L}\mathbb{E}(\Theta_n\mathbb{1}_{B_{\varepsilon,n}}) \\ &\geq (\alpha-\varepsilon)m^{1/r-1} - |\alpha-\varepsilon|m^{1/r-1}nP(B_{\varepsilon,n}) - n\mathbb{E}(\Theta_n\mathbb{1}_{B_{\varepsilon,n}}).\end{aligned}$$

□

To prove Theorem 5.1 we want to choose such n and ε that

$$|(\alpha-\varepsilon)m^{1/r-1}|n\mathbb{P}(B_{\varepsilon,n}) + n\mathbb{E}(|\Theta_n|\mathbb{1}_{B_{\varepsilon,n}}) \xrightarrow{N\to\infty} 0.$$

To do that we use Lemma 2.6.

Let R be such that $R < \alpha$ and $p([R,+\infty)] \geq 2/3$. Consider a Galton–Watson tree with binomial offspring distribution with parameters 2 and $p([R,+\infty)]$. The average number of offspring is thus equal to $2p([R,+\infty)] \geq R) \geq 4/3 > 1$ with our assumptions. In the sequel, we use the notations r and ϕ to denote the numbers given by Lemma 2.6 for this offspring distribution.

Proposition 5.3. *Using the previous notation, let* $s_N := \lceil \frac{\log N}{\log \phi} \rceil + 1$ *and* $n := m + s_N$. *Let also* C_ε *be an event that in BRW there exist a least* ϕ^{s_N} *descending paths of the form* $\varnothing =: u_0, \ldots, u_n$ *such that* $\Phi(u_i) \geq (\alpha-\varepsilon)m^{1/r-1}i$ *for all* $i \in [\![1,n]\!]$. *Then*

$$\mathbb{P}(C_\varepsilon) \geq \rho(\alpha, m) \cdot r,$$

for

$$\varepsilon \geq \frac{s_N}{m^{1/r-1}(m+s_N)}(\alpha m^{1/r-1} - R). \tag{5.3}$$

In the sequel we use the notation s_N, n and ε as in the statement of this proposition.

Proof. The idea behind the proof is as follows. Assume that a BRW contains a descending path of length m starting at root that performs steps as in the event $A_{\alpha,m}$ and its last particle u_m has at least ϕ^{s_N} offspring at generation $n = m + s_N$. Then such BRW belongs to C_ε for some ε.

Now, let us note that by Lemma 2.6 applied to the Galton–Watson tree with binomial offspring distribution mentioned previously, there are at least ϕ^{s_N} offspring of u_m at generation n with the probability at least r. It means exactly that the probability of the existence of at least ϕ^{s_N} descending paths of the form $u_m, \ldots, u_n$ such that $\Phi(u_{k+1}) - \Phi(u_k) \geq R$ for all $k \in [\![m, n-1]\!]$ is $\geq r$. So by the independence of jumps the probability of the existence of at least ϕ^{s_N} paths of the form $\varnothing = u_0, \ldots, u_n$ such that $\Phi(u_i) \geq \alpha m^{1/r-1}i$ for $i \in [\![1,m]\!]$ and $\Phi(u_{i+1}) - \Phi(u_i) \geq R$ for $i \in [\![m, n-1]\!]$ is $\geq \rho(\alpha,m)\cdot r$. So to prove the proposition it is enough to check that such paths meet the assumptions of the event C_ε for ε as is the proposition.

So formally we need to check that if $\Phi(u_i) \geq \alpha m^{1/r} + R\cdot j$, then $\Phi(u_i) \geq (\alpha-\varepsilon)m^{1/r-1}(m+j)$ for every $j \in [\![1, s_N]\!]$. For ε as in the proposition we have

$$\varepsilon \geq \frac{j}{m^{1/r-1}(m+j)}(\alpha m^{1/r-1} - R),$$

for all $j \in [\![1, s_N]\!]$, so

$$\alpha m^{1/r} + R \cdot j \geq (\alpha - \varepsilon) m^{1/r-1}(m + j).$$

So the conclusion follows. □

Now, let D_ε be the event that for all $j \in [\![1, N]\!]$ BRW_j belongs to C_ε^c, so it doesn't contain more than ϕ^{s_N} distinct descending paths $\varnothing =: u_0, \dots, u_n$ such that $\Phi(u_i) \geq (\alpha - \varepsilon) m^{1/r-1} i$ for all $i \in [\![1, n]\!]$. Then, by Proposition 5.3 and the independence of $BRW_1, \dots, BRW_N$ we have that

$$\mathbb{P}(D_\varepsilon) \leq [1 - \rho(\alpha, m) r]^N.$$

By the inequality $1 - x \leq \exp\{-x\}$ and (5.1) we have that

$$\mathbb{P}(D_\varepsilon) \leq \exp\{-N \cdot N^{-(1-\epsilon)}\} = \exp\{-N^\epsilon\}.$$

We now claim that $B_{\varepsilon,n} \subseteq D_\varepsilon$. We show that by contradiction. First, let us note that D_ε doesn't depend on the selection procedure, but $B_{\varepsilon,n}$ does. Assume that $B_{\varepsilon,n} \cap D_\varepsilon^c$ occurs. Then there exists such $j \in [\![1, N]\!]$ that in BRW_j there are more than ϕ^{s_N} distinct descending paths $u_0 \dots, u_n$ such that $\Phi(u_i) \geq (\alpha - \varepsilon) m^{1/r-1} i$ for all $i \in [\![1, n]\!]$. Since $B_{\varepsilon,n}$ occurs, for every such path $\Phi(u_i) > \min X_i^N$ for all $i \in [\![1, n]\!]$. So for every i, $u_i \in G_i^N$, i.e. it survives through the selection procedure. As a consequence, in $G_{s_N}^N$ there are more than $\phi^{s_N} > N$ particles, which is a contradiction.

So we get that

$$\mathbb{P}(B_{\varepsilon,n}) \leq \mathbb{P}(D_\varepsilon) \leq \exp\{-N^{-\epsilon}\}. \tag{5.4}$$

To use Proposition 5.2 we bound Θ_n from above by the sum of the absolute values of $2nN$ corresponding i.i.d. variables and use Schwarz's inequality. We deduce that

$$\mathbb{E}(\Theta_n \mathbb{1}_{B_{\varepsilon,n}}) \leq 2nN \cdot \mathbb{E}\zeta \cdot \mathbb{P}(B_{\varepsilon,n})^{1/2}, \tag{5.5}$$

where ζ is a variable with distribution p, $\mathbb{E}\zeta < \infty$. Since $n = m + s_N \leq [\log N]\left(1 + \frac{1}{\log \phi}\right)$, by Proposition 5.2, (5.4) and (5.5) we have

$$\begin{aligned} v_N \geq& (\alpha - \varepsilon)[\log N]^{1/r-1} \\ &- \left|(\alpha - \varepsilon)[\log N]^{1/r}\right| \left(1 + \frac{1}{\log \phi}\right) \exp\{-N^{-\lambda}\} \\ &- 2\left(\left(1 + \frac{1}{\log \phi}\right)[\log N]\right)^2 N \cdot \mathbb{E}\zeta \cdot \exp\{-N^{-\lambda}/2\} \end{aligned} \tag{5.6}$$

for

$$\varepsilon \geq \frac{s_N}{m^{1/r-1}(m + s_N)}(\alpha m^{1/r-1} - R).$$

Since

$$\frac{s_N}{m^{1/r-1}(m + s_N)}(\alpha m^{1/r-1} - R) \xrightarrow{N \to \infty} \frac{\frac{1}{\log \phi}\alpha}{1 + \frac{1}{\log \phi}} = \frac{\alpha}{1 + \log \phi},$$

taking $N \to \infty$ in (5.6) we conclude that

$$\liminf_{N \to \infty} \frac{v_N}{(\log N)^{1/r-1}} \geq \alpha - \varepsilon$$

for $\varepsilon \geq \frac{\alpha}{1 + \log \phi}$, so

$$\liminf_{N \to \infty} \frac{v_N}{(\log N)^{1/r-1}} \geq \frac{\log \phi}{1 + \log \phi}\alpha.$$

Taking $\alpha_* := \frac{\log \phi}{1 + \log \phi}\alpha$ completes the proof.

□

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