Report on the thesis
COMBINATORICS OF ASYMPTOTIC REPRESENTATION THEORY OF THE SYMMETRIC GROUPS
by Maciej Dołęga

Prof. dr hab. Paweł Hitczenko
Department of Mathematics, Drexel University

1. Topics

The topics covered in the proposed thesis lie at the heart of algebraic combinatorics. More specifically, they concern certain combinatorial aspects of representation theory of the symmetric group. This is a timely and well studied topic that has drawn a lot of attention among the researchers in the past and it continues to be an area of a very active research today.

The topics covered in the dissertation concern very fresh developments. The author presents a number of results concerning the properties of the coefficients of the so-called Kerov polynomials. Kerov polynomials are polynomials that arise as expressions for the (normalized) characters of the irreducible representations of the symmetric group in terms of the free cumulants. Their existence was proved in 2000 by Kerov for the 'classical' case and in 2009 by Lassalle for more general case of the so–called Jack characters (the former situation corresponds to a special case of $\alpha = 1$ of the parameter associated with Jack polynomials; the details are explained in Chapter 2 of the dissertation).

As frequently happens in various aspects of combinatorics, the coefficients of expressing one quantity in terms of other quantities have integer values, often non–negative (or alternating in signs). Once this is known, it is then of interest to provide combinatorial interpretation of these coefficients (this is usually accomplished by proving that the coefficients count certain object of combinatorial interest). One, very classical, example is provided by the Stirling numbers of the first and the second kind which appear when expressing the falling factorials in terms of powers (and vice versa).

In the case of Kerov polynomials, only partial results are known and the picture is far from complete. The contributors to the development of this theory include such mathematicians as Biane, Féray, Goulden, Lassalle, Rattan, Stanley, Śniady, and Kerov himself. The dissertation under review makes further contributions to our understanding of the structure of Kerov polynomials.

2. Results

Original results are presented in Chapters 3, 4 and 5 of the thesis.

Results presented in Chapter 3 concern the form of the homogeneous part, $K_{k,d}$, of degree d of the Kerov polynomial in the free cumulants $(R_i)_{i\geq 2}$ (treating R_i as having degree i). Specifically, if

$$K_k = K_k(R_2, \ldots, R_{k+1})$$

is the Kerov polynomial and (R_i) are the free cumulants, we decompose it as

$$K_k = \sum_d K_{k,d},$$

where

$$K_{k,d} := \sum_{\mu \in P(d)} \beta_{k,\mu} \prod_i R_i^{m_i(\mu)},$$

and where $P(d)$ is the set of all integer partitions of the number d and for $\mu \in P(d)$, $m_i(\mu)$ is the multiplicity of part i (i.e. $\sum_{i\geq 1} i m_i(\mu) = d$ for each such μ); this notation differs from what is used in the thesis. The question is then: how does the $K_{k,d}$ look like? Kerov himself proved that $K_{k,d} \equiv 0$ unless $d = k+1-2g$ for a non–negative integer g and conjectured that if $g = 0$ then $K_{k,k+1} = R_{k+1}$. This was proved by Biane in 2003 who, in turn, conjectured that if $g - 1$ then

$$K_{k,k-1} = \frac{1}{4}\binom{k+1}{3} \sum_{\mu \in P(k-1)} \binom{\sum_i m_i(\mu)}{m_2(\mu), \ldots, m_{k-1}(\mu)} \prod_{i \geq 2} ((i-1)R_i)^{m_i(\mu)}.$$

This was proved by Śniady in 2006. For small values of g, it is possible to obtain the explicit expressions for $K_{k,k+1-2g}$ from the work of Goulden and Rattan, but according to Lassalle this is impractical for $g \geq 3$. Lassalle conjectured the form of $K_{k,k+1-2g}$ for *any* $g \geq 1$ (the exact formulation is given as the formula (3.2) of Theorem 3.1.1 of the thesis. He also conjectured the form of $K_{k,k+1-2g}$ written in terms of different expressions (still involving the free cumulants R_i), given as formula (3.3) in the same theorem.

The main result of Chapter 3 of the thesis is Theorem 3.1.1 which substantiates both of these conjectures. This is definitely a valuable result and a big step forward. Its proof requires a number of new ideas and goes well beyond what has been done earlier by Śniady or Lassalle. The results (obtained jointly with Śniady, the doctoral supervisor of Mr. Dołęga) were published in the *Journal of Combinatorial Theory*, arguably the world's top journal in combinatorics, having a very high standard and international reach.

The results presented in Chapter 4 concern similar, but more general subject, namely the study of Kerov character polynomials associated with the family of Jack polynomials $(J_\mu^{(\alpha)})$ indexed by partitions μ and having an additional parameter α (assumed to be non–negative, although this does not seem to be mentioned anywhere in the thesis). Setting $\alpha = 1$ gives the situation considered in Chapter 3. Since there is an additional parameter α, various statements take a bit different form and there are additional issues (like the nature of dependence on α, for example). The main result of this chapter, given as Theorem 4.1.2, states that the coefficients of Jack characters when these are expressed in terms of the analogs of the free cumulants, depend polynomially on α. This results partially confirms another conjecture of Lassalle (which states, in addition, that these polynomials in α have non–negative coefficients). Even though the author was unable to prove the full strength of Lassalle's conjecture, he obtained some bounds on the degrees of these polynomials. This, together with other results proved in that chapter, allowed him to derive a number of consequences. Some of those that I consider most interesting are:

- an extension of the celebrated Kerov–Vershik result about the limiting shape of a randomly (with respect to a suitable measure) selected Young

diagram and fluctuations around that shape (the extension consists in the fact that with each value of the parameter α corresponds a probability measure on Young tableaux of size n; the Kerov–Vershik results corresponds to the value $\alpha = 1$, in which case the measure is the classical Plancherel measure). These results are gathered in Sections 4.5–4.7 (Theorems 4.5, 4.61, and 4.7.1, in particular).
- a weaker version of a conjecture of Goulden and Jackson. This conjecture asserts that certain quantities (see Section 4.3.6 for a more detailed description) which are known to be rational functions are, in fact, polynomials with non–negative coefficients. The author proves that these quantities are, indeed, the polynomials. The non–negativity of their coefficients remains open.
- two conjectures of Matsumoto. The conjectures are about a form of certain expressions (see the end of Section 4.3.7 for more detailed description).
- some new results concerning the form of the coefficients of Kerov polynomials of Jack characters. They provide supporting evidence for a conjecture of Lassalle (see Theorem 4.2.2 and Remark 4.4.3)
- new proofs of known results, including a result by Lapointe and Vinet about the polynomiality of the coefficients of the Jack polynomials expansion in the monomial symmetric basis (quoted as Theorem 4.1.1).

The first four items make, in my opinion, important new contributions and answer (some partially) open questions posed by other researchers. Some of the results were presented during the latest *Formal Power Series and Algebraic Combinatorics* conference (the most prominent conference in combinatorics), an extended abstract (co–authored with V. Féray) will appear in the proceedings of this conference, and I have no doubts that further results presented in this chapter will be published in respected journal(s).

Chapter 5 centers around Conjecture 5.1.1, which specifies a form of the coefficients in Stanley's character formula for Jack characters, given in the second bullet in this conjecture. The content of this conjecture is the form of the coefficients, therein called wt_M. The claim is that the wt_M as functions of $\gamma := (1-\alpha)/\sqrt{\alpha}$ are polynomials with non–negative rational coefficients of a degree bounded by a specified function of M (which is a map from a bipartite graph to a Young diagram λ); the details are described in Sections 2.7 and 2.8.

This conjecture has been put forth in the thesis. The author then formulates a more specific version of this conjecture (given as Conjecture 5.1.2). The specificity consists in giving a candidate expression for the wt_M. This is the 'measure of non–orientability' of M, mon_M. The conjecture (with $\mathrm{wt}_m = \mathrm{mon}_M$) is then verified in some instances. This is the place, where I get a bit confused, since, if I understand it correctly, the author also knows that the conjecture (still with $\mathrm{wt}_m = \mathrm{mon}_M$) is *not* true in general, so there would seem to be a little point in trying to verify it for *some types* of partitions. Nonetheless, the results are written (in a paper co–authored by Féray and Śniady) and available on `arxiv` (and presumably submitted for publication).

3. Comments

3.1. **General comments.** I have two general points to make. The first is that *all* of the results presented in the dissertation are (in the process of being) published in manuscripts that are co–authored with other researchers. In addition, both of those researchers are senior as compared with Mr. Dołęga. At the same time the role and contributions Mr. Dołęga has made to those manuscripts is not clarified in the materials I received. It is therefore impossible to evaluate contributions made by the candidate himself. I will proceed with my recommendations assuming that Mr. Dołęga's contributions to each of the manuscripts is, roughly, inversely proportional to the number of the authors and with the understanding of the special role that the scientific adviser (promotor) plays in the preparation of a doctoral dissertation. I would, however, suggest that the issue is clarified before the defense of this dissertation is taken up by the Scientific Council of the Department of Mathematics and Computer Science of the University of Wrocław.

The second point concerns the Chapter 5 of the dissertation. Perhaps I misunderstood or missed something, but it seems to me that trying to prove a conjecture in *some* cases when it is known that it is false *in general* is of limited interest. Rather, seems to me, an effort should be made to try to re–formulate the conjecture, so that it has a chance of being true in general. (It is possible that the study of the measure of non–orientability is of interest in its own right. If so, however, this should have been made clearer in the thesis, and perhaps presented in different perspective, not in the context of Conjecture 5.1.1). So, I was much more impressed with the content of Chapters 3 and 4, than Chapter 5.

Having made those two points, let me state that even discounting Chapter 5, the material presented in Chapters 3 and 4 alone (and assuming that Mr. Dołęga's contributions are what they are expected to be), is certainly enough for a good doctoral thesis in mathematics. The results are significant, the proofs are highly non–trivial, and a number of conjectures posed in the literature are substantiated.

3.2. Specific comments: presentation.

- I have some issues with the presentation in Chapter 3. I felt that proving some results right after their statements, and postponing proofs of others till later was done a bit arbitrarily and inconsistently. I am not sure what would improve the presentation, but I found myself flipping the pages back and forth a bit too much.
- I would have found a section listing the notation for an easy reference *very* helpful.
- p. 105, last paragraph of Section 5.1.2: manuscript [CJŚ13] of is referred to by stating 'we will present', which is inappropriate, given that the candidate is *not* a co–author of that manuscript.
- p. 106, the next to the last sentence: I am not sure the statements like this ('missing notation' in particular) are suitable in a doctoral thesis.

3.3. Specific comments: minor omissions, typographical errors.

- p. v_9 and vii_8: I believe [Las08c] is meant here.
- p. 17^{11}: parentheses seem to be missing in the numerator and denominator in the last expressions.
- p. 33^{11}: looks like '$h_g(\mu) =$' is missing in front of '$h_g(\mu_1, \mu_2, \dots)$'.
- p. 36, item (a) in Lemma 3.2.3: I think it was meant here that $C(t)$ is the power–sum of the first *and* second kind (i.e. simultaneously, not 'respectively').'
- p. 62^8: I think 'at most' should be added at the end of this line.

- p. 64_{11}: I think that 'integrality' was meant to be used rather than 'integrity'.
- p. 68_8: looks like the expression after $min_{i=1,2,3}$ should be parenthesized.
- p. 75^5: If I understand the explanation for $g^{(2)}_{(k),(k);(1^k)} = k$, I should compare the coefficients of the suitable powers of n in the display one line above. When I try to do it by letting $i = k$ on the rhs I get $g^{(2)}_{(k),(k);(1^k)} = (n!)^2/(k-1)!$. Where do I go wrong?
- p. 85^2 (and p. 86, Theorem 4.5.4): Isn't '$= 0$' meant to be '$\to 0$' in those two instances?
- p. 87_7: $|h_n|$ is meant to be $|h|_n$, I believe.
- p. 88^{13}: there is an unnecessary '.' (period) at the end of the line.
- p. 89_6: the period at the end of the line should be two lines below.
- p. 98^{10}: the capital 'K' in this line should be the lowercase 'k'.
- p. 99^1: there is a ')' missing at the end of expression involving ω.
- p. 100_5: I am not sure what '*Pola*' means here.
- p. 113_{12}: $\frac{1}{}$ should be $\frac{1}{2}$.
- pp. 127–131: there are many inconsistencies like 'P. Biane' vs 'Philippe Biane', 'A. Czyżewska-Jankowska' vs 'Agnieszka Czyżewska–Jankowska', or 'V. Féray' vs 'Valentin Féray' throughout the references.
- there is a number of lingual errors, including misspellings like 'remainning' (which actually should have been 'remainder'), 'combinatorialy', 'finaly', 'recurrsion' (all of them could have been easily avoided), but since they do not obstruct the reading in any way I do not list them all.

4. Conclusion

The results presented in the thesis constitute a significant contribution to the understanding of the structure of Kerov polynomials. The author substantiates a number of conjectures put forth by M. Lassalle and S. Matsumoto concerning the structure of Kerov polynomials. He also obtains an extension of Kerov–Vershik result on the limiting shape of a random Young tableaux and the fluctuations around that shape. All of these results are valuable contributions to a dynamically developing part of modern algebraic combinatorics.

Some of the results were already published or accepted for publication in the prestigious journals with worldwide readership like *Journal of Combinatorial Theory*, other were submitted for publication.

In my opinion the quantity, the quality, and the relevance of the presented research to current trends in this area of algebraic combinatorics, is certainly enough to make it a good doctoral thesis. I have no doubt that after a satisfactory clarification of Mr.Dołęga's contributions the proposed dissertation will be found to easily comply with the requirements set forth by the Polish law. I therefore recommend that, the dissertation is accepted and that the PhD defense of Mr. Maciej Dołęga proceeds to its next phase.

Paweł Hitczenko Filadelfia, July 29, 2013