Report on the PhD Thesis
"Hardy spaces associated with certain semigroups of linear operators"
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The thesis under review is composed of four articles:

(A1) "Local atomic decomposition for multidimensional Hardy spaces", joint work with P. Plewa and M. Preisner, appeared in Rev. Mat. Complutense;
(A2) "Riesz transform characterisation for multidimensional Hardy spaces", joint work with M. Preisner;
(A3) "The atomic Hardy space for a general Bessel operator", single-authored;
(A4) "Sharp multiplier theorem for multidimensional Bessel operators", joint work with M. Preisner, appeared in J. Fourier Anal. Appl.

Each paper discusses a different problem, but they have as a common theme the analysis of Hardy spaces associated with semigroups.

As is well known (see, e.g., the seminal work of Fefferman and Stein [5]), the classical Hardy space $H^1(\mathbb{R}^n)$ can be characterised in many equivalent ways. This includes the *atomic decomposition*

$$H^1(\mathbb{R}^n) = \left\{ \sum_j \lambda_j a_j \,:\, a_j \text{ atom},\ \lambda_j \in \mathbb{C},\ \sum_j |\lambda_j| < \infty \right\}, \tag{1}$$

where atoms are functions $a : \mathbb{R}^n \to \mathbb{C}$ such that

$$\operatorname{supp} a \subseteq B, \qquad \int a \, d\mu = 0, \qquad \|a\|_\infty \leq \mu(B)^{-1} \tag{2}$$

for some ball $B \subseteq \mathbb{R}^n$; here μ is the Lebesgue measure. There is also a *Riesz transform characterisation*

$$H^1(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) \,:\, \partial_{x_j} (-\Delta)^{1/2} f \in L^1(\mathbb{R}^n) \text{ for } j = 1, \dots, n \right\}, \tag{3}$$

as well as a (radial) *maximal function characterisation* in terms of the semigroup generated by the Laplacian or one of its fractional powers:

$$H^1(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) \,:\, \sup_{t>0} |\exp(-t(-\Delta)^\gamma) f| \in L^1(\mathbb{R}^n) \right\}, \tag{4}$$

where $\gamma \in (0, 1]$.

A local variant $h^1(\mathbb{R}^n)$ of the above Hardy space has also been studied (see, e.g., the work of Goldberg [6]). Again, multiple equivalent characterisations hold for $h^1(\mathbb{R}^n)$. For example, $h^1(\mathbb{R}^n)$ admits an atomic decomposition similar to (1), where however the atoms are "local atoms at scale r_0" for some fixed $r_0 > 0$ (namely, either functions a satisfying (2) for some ball B of radius at most r_0, or functions of the form $|B|^{-1}\mathbf{1}_B$ for a ball B of radius r_0). Similarly, $h^1(\mathbb{R}^n)$ has a Riesz transform characterisation analogous to (3), where the Riesz transforms $\partial_{x_j}(-\Delta)^{1/2}$ are replaced by the "local Riesz transforms" $\partial_{x_j}(c_0 - \Delta)^{1/2}$ for some $c_0 > 0$. Furthermore, $h^1(\mathbb{R}^n)$ has a maximal function characterisation analogous to (4), where the supremum in the definition of the maximal function is restricted to $t \in (0, t_0)$ for some fixed $t_0 > 0$.

An important area of research in the last few decades has been the analysis of local and global Hardy spaces in settings other than the Euclidean, including the problem whether the equivalence of characterisations similar to the above hold true in more general settings. One of the settings that have been studied in depth is that of doubling metric measure spaces (or spaces of homogeneous type; see, e.g., the work of Coifman and Weiss [1]). In this setting, local and global Hardy spaces can be introduced by means of atomic decompositions as in the Euclidean case; moreover (possibly under additional technical assumptions), maximal function characterisations similar to the above can be proved in terms of semigroups satisfying Poisson-type bounds (see, e.g., the work of Uchiyama [9]).

Articles (A1) to (A3) in the thesis are specifically devoted to the study of Hardy spaces

$$H^1(L) = \left\{ f \in L^1(X) \,:\, \sup_{t>0} |\exp(-tL) f| \in L^1(X) \right\} \tag{5}$$

associated to a semigroup of operators $\exp(-tL)$ acting on $L^1(X)$ for some measure space (X, μ), and to prove equivalent characterisations of $H^1(L)$ in terms of suitable atomic decompositions or Riesz transforms. In all the cases considered there, X is actually an open subset of $\mathbb{R}^n$ (more specifically, a product of intervals), and the underlying measure μ is either the Lebesgue measure or a weighted variant thereof. Nevertheless, the aforementioned classical theory does not directly apply to the semigroups considered in (A1)-(A3), and some interesting phenomena emerge.

A particularly striking feature of the results obtained in these three papers is the mixture between "local" and "global" aspects of the Hardy space theory (which does not appear in the aforementioned classical results on $\mathbb{R}^n$ or on spaces of homogeneous type, but is already present, e.g., in previous work of Dziubański–Zienkiewicz [4] on Schrödinger operators). Namely, while the supremum in the maximal function in (5) is taken over all $t > 0$ (thus suggesting that the space under consideration is a "global" space), the obtained atomic decompositions are "local" in nature. More precisely, to each studied semigroup $\exp(-tL)$ the authors associate an "admissible covering" $\mathcal{Q}$ of X made of cuboids, and the atoms in the obtained decomposition of $H^1(L)$ are either functions of the form $\mu(Q)^{-1}\mathbf{1}_Q$ for some $Q \in \mathcal{Q}$, or functions $a : X \to \mathbb{C}$ satisfying (2) for some cuboid B contained in a $Q \in \mathcal{Q}$ (or a slight enlargement thereof). It should be pointed out that the cuboids in an admissible covering $\mathcal{Q}$ may have different diameters; this is another important difference compared to the aforementioned classical results, as here the "scale" of the atoms can change from point to point, thus taking into account the specific "geometric" properties of the semigroup $\exp(-tL)$ under consideration.

A common strategy underlying these three papers appears to be the comparision of the semigroup $\exp(-tL)$ under consideration with another "model" semigroup $\exp(-tL_0)$ to which the aforementioned classical theory of Hardy spaces applies (specifically, in (A1) and (A2), L_0 is the Euclidean Laplacian or a fractional power thereof, while in (A3) L_0 is a classical Bessel operator). In order to obtain the atomic decomposition of local type for $H^1(L)$ associated to an admissible covering $\mathcal{Q}$, the following three assumptions on $\exp(-tL)$ are essentially considered:

(a) The semigroup $\exp(-tL)$ has a nonnegative integral kernel P_t satisfying upper Gaussian- or Poisson-type bounds (of the same type of those satisfied by the kernel H_t of the model semigroup $\exp(-tL_0)$).

(b) P_t satisfies an additional "far-diagonal integrability condition" of the form

$$\sup_{y \in Q^*} \int_{X \setminus Q^{**}} \sup_{t>0} P_t(x,y)\, d\mu(x) \lesssim 1 \qquad \forall Q \in \mathcal{Q}$$

(here Q^* denotes a suitable enlargement of the cuboid Q).

(c) Finally, the "remainder" $P_t - H_t$ satisfies a near-diagonal integrability condition for small times:

$$\sup_{y \in Q^*} \int_{Q^{**}} \sup_{0<t<d_Q^{2\gamma}} |P_t(x,y) - H_t(x,y)|\, d\mu(x) \lesssim 1 \qquad \forall Q \in \mathcal{Q}$$

(here d_Q is the diameter of Q).

It should be pointed out that the additional "far-diagonal integrability" stated in (b) above for $\exp(-tL)$ is not satisfied by the model semigroup $\exp(-tL_0)$, and its validity for $\exp(-tL)$ explains why the atomic decomposition of $H^1(L)$ has a local character. Indeed, under the above assumptions, the proof of the atomic decomposition of $H^1(L)$ goes, roughly speaking, as follows: by using assumptions (a) and (b), one can localise in time and space the maximal function defining $H^1(L)$; at that point, the assumption (c) on the remainder can be used to replace the semigroup $\exp(-tL)$ in the maximal function with the model semigroup $\exp(-tL_0)$, and then use the already known atomic decomposition for $H^1(L_0)$ to derive the one for $H^1(L)$.

Additional assumptions of the same kind as (a)-(c) above, but involving derivatives of the integral kernels, are considered in the paper (A2) to deal with the more delicate Riesz transform characterisation; again, the strategy is to reduce the problem for L to already known characterisations for L_0, by means of suitable estimates for the remainder $P_t - H_t$.

The strategy presented in the three papers (A1)-(A3) proves to be fairly effective, in that the authors show that the required assumptions are satisfied in a number of different examples (including Bessel and Laguerre operators, as well as certain Schrödinger operators on $\mathbb{R}^n$). In light of this flexibility, a natural question would be whether the "model operator" L_0 must be one of those considered in those papers (Euclidean Laplacian or classical Bessel operator) or could be a more general operator to which the classical theory of (local and global) Hardy spaces applies. Indeed, the assumption (c) on the remainder $P_t - H_t$ stated above appears to be a quite "rigid" one if the model semigroup H_t is fixed, and there might be opportunities to exploit a similar strategy to prove more general and "robust" results. Alternatively, one might ask whether assumption (c) could be replaced by a more "intrinsic" one that only involves P_t and the covering $\mathcal{Q}$, and avoids the comparison with another semigroup. In any case, all these questions appear to indicate that not only do the presented results constitute some interesting contributions to the theory of Hardy spaces associated with operators, but they are also likely to stimulate further research in the area.

The remaining paper (A4) included in the thesis is somewhat different in character and focus. Indeed, the paper is devoted to the proof of a sharp spectral multiplier theorem for classical multidimensional Bessel operators

$$L_\alpha = -\sum_{j=1}^{N} \left(\partial_{x_j}^2 + \frac{\alpha_j}{x_j} \partial_{x_j} \right) \qquad \text{on } (0,\infty)^N,$$

where $\alpha \in (-1,\infty)^N$. The space $(0,\infty)^N$, with the Euclidean distance and the weighted Lebesgue measure $d\mu_\alpha(x) = \prod_j x_j^{\alpha_j}\, dx_j$, is a doubling metric measure space of homogeneous dimension

$$Q_\alpha = \sum_j \max\{1, 1+\alpha_j\}.$$

Moreover, the Bessel operator L_α is a positive self-adjoint operator on $L^2(\mu_\alpha)$, and the heat semigroup $\exp(-tL_\alpha)$ generated by L_α satisfies upper and lower Gaussian-type bounds. As a consequence, a number of classical results apply to this setting, including the fact that the Hardy space $H^1(L_\alpha)$ has an atomic decomposition à la Coifman–Weiss (see, e.g., previous work of Dziubański–Preisner [3]). Moreover, an L^p spectral multiplier theorem of Mihlin–Hörmander type holds for L_α (see, e.g., work by Hebisch [7] and Duong–Ouhabaz–Sikora [2]), yielding the weak type $(1,1)$ and L^p boundedness for $p \in (1,\infty)$ of operators of the form $F(L_\alpha)$ whenever the multiplier $F : \mathbb{R} \to \mathbb{C}$ satisfies a local scale-invariant smoothness condition of the form

$$\sup_{t>0} \|F(t\cdot)\eta\|_{L^q_s} < \infty \tag{6}$$

for $q = \infty$ and $s > Q_\alpha/2$; here $L^q_s(\mathbb{R})$ denotes the L^q Sobolev space of (fractional) order s, while $\eta \in C^\infty_c((0,\infty))$ is a nontrivial cutoff.

One of the aims of the paper (A4) is to sharpen the aforementioned multiplier theorem for the multidimensional Bessel operators L_α. Indeed, the authors succeed in showing that the smoothness condition (6) on the multiplier can be weakened by taking $q = 2$ instead of $q = \infty$. This improvement is obtained by following a general strategy described, e.g., in previous work by Duong–Ouhabaz–Sikora [2], namely, by checking that a certain "Plancherel-type estimate" for L_α holds true with an L^2 norm instead of an L^∞ norm of the multiplier.

Additionally, the authors show in (A4) that, under the same smoothness assumptions (6) on the multiplier (with $q = 2$ and $s > Q_\alpha/2$), the operator $F(L_\alpha)$ is also bounded on $H^1(L_\alpha)$. Actually, they prove their H^1 boundedness result in a more general and "abstract" setting, thus providing a natural Hardy-space counterpart to the general L^p spectral multiplier theorem of Duong–Ouhabaz–Sikora [2].

Finally, the authors investigate the sharpness of the condition $s > Q_\alpha/2$ in their multiplier theorem for Bessel operators. To this purpose, in the case $N = 1$, they study L^p and weak type boundedness properties of the imaginary powers $L_\alpha^{i\gamma}$ (for which explicit formulas can be derived by "subordination" to the heat kernel), and succeed in showing that the regularity threshold $Q_\alpha/2$ cannot be replaced by a smaller quantity.

This last result is particularly interesting in the case $\alpha > 0$, in which case $Q_\alpha = 1 + \alpha > 1 = N$. Indeed, N is the "local dimension" (topological dimension) of the underlying manifold, and (as L_α is an elliptic operator) a standard transplantation argument (see, e.g., the work of Kenig–Stanton–Tomas [8]) yields the lower bound $N/2$ to the optimal Mihlin–Hörmander threshold for L_α. However, the results of this paper show that, when $N = 1$ and $\alpha > 0$, the smoothness condition in the multiplier theorem cannot be pushed down to $s > N/2$, and that the global geometry (i.e., the volume growth at infinity taken into account by Q_α) plays a crucial role in this problem.

In conclusion, the thesis includes a number of very interesting results and contributions to the investigation of several research problems. The material is certainly worthy of publication, and indeed some of the articles have already appeared in international journals. The submitted work, including the overall introduction to the thesis, is well written and of good quality, and demonstrates the candidate's breadth and familiarity with a variety of techniques. For these reasons, I am happy to confirm my overall positive evaluation of the thesis.

References

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