Title: Limit Theorems for BM-independent random variables and related topics
Author: L. Oussi

Scientific content

The present thesis is concerned with limit theorems of Poisson type in the framework of BM-independence, a notion introduced by the advisor of the candidate, J. Wysoczanski some time ago and consists of three chapters:

(1) A detailed introduction (40p)
(2) BM-law of small numbers (30p)
(3) Models for Poisson elements on BM-Fock space (30p)

the latter two chapters are drawn from two joint papers of the candidate and the advisor. The reviewer had the opportunity to discuss the contents of the thesis with the author and supervisor during a one week visit at the end of June.

Overview

Chapter 1.

The first half of the introductory chapter (Section 1-3) presents the basic notions from operator algebras and noncommutative probability. In particular, Wysoczanski's axioms for BM independence are presented, generalizing Muraki's notion of monotone independence from linear ordered sets to partially ordered sets.

Two final sections present limit theorems and Fock spaces, respectively, in the framework of Boolean and monotone independence. These are the concepts to be generalized in the remaining two chapters.

Chapter 2.

This chapter establishes the law of small numbers or Poisson limit theorem for BM-independent arrays. Three partial orders are considered, induced from the cone $\mathbb{R}_n^+$, the cone of positive definite matrices and the Lorentz cone.

Establishing a limit theorem in noncommutative probability consists in understanding the combinatorics of the limit moments (i.e., eliminate asymptotically inessential terms), and, if possible, to solve the moment problem and to derive an analytic expression for the distribution if possible. In this chapter the combinatorial problem is solved in all three models and involves the volume characteristic numbers already observed previously by Wysoczanski. An analytic expression for the limit seems out of reach.

In the case of the Lorentz cone the proof of Lemma 2.4.11 contains a gap on page 67. In the middle the author claims that the condition $\mu \not\sim \rho \preceq \xi$ implies $p_\mu \leq \pi_\rho \leq p_\xi$ where for a Minkowski vector $\xi = (t; x)$ one defines $p_\xi = \frac{1}{\sqrt{2}}(t - \|x\|)$. This claim is wrong as we found out during a discussion, a counterexample is the following:

$$\mu = (2; \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \qquad \rho = (2; \begin{pmatrix} 2 \\ 0 \end{pmatrix}) \qquad \xi = (4; \begin{pmatrix} 0 \\ 0 \end{pmatrix}).$$

The gap was fixed by J. Wysoczanski who found a simpler alternative proof which will be included in the submitted paper.

Chapter 3.

In this chapter the distribution of certain operators on Fock space is computed. These operators are natural analogues of models for Poisson distribution from classical, free and monotone probability.

§3.2 The construction of the discrete BM-Fock space is taken from ref. [38] (Wysoczanski 2007). Here the combinatorics of the corresponding creation and annihilation operators are considered in detail and a Poisson limit theorem is calculated. The proof is rather technical.

We would like to indicate a more elementary proof of Proposition 3.3.1 and Lemma 3.3.3.

First observe that in order to evaluate the moments with respect to the vector state ϕ it suffices to consider the cyclic invariant subspace U_ξ generated by the vacuum vector Ω. Since both A_ξ^+ and A_ξ^- are nilpotent and A_ξ^o is idempotent, it was already noted in the proof of Lemma 3.2.2 that this subspace has dimension 2 with basis consisting of the vectors Ω and $g_\xi = A^+\Omega$. Therefore the moments of $S(\lambda)$ are the same as the moments of the matrix representation of the restriction of $S(\lambda)$ to U_ξ which is easily seen to be

$$M_\lambda = \begin{bmatrix} 0 & 1 \\ 1 & \lambda \end{bmatrix}.$$

The characteristic polynomial is $\chi(z) = |z - M_\lambda| = z^2 - \lambda z - 1$ and the Cauchy transform therefore is

$$G(z) = (z - M_\lambda)^{-1}_{1,1}$$

which is exactly formula (3.16). From this the recursion (3.11) for the moments can be deduced using Newton's relations.

The set D_p defined after Remark 3.4.4 is well known under the name of *Motzkin paths*, see the classic paper of Flajolet *Combinatorial aspects of continued fractions*, Discr.Math. 32 (1980).

Overall assessment

The results of the thesis are interesting and the proofs are quite involved technically. However the presentation is very clear and a pleasure to read, despite many orthographic mistakes due to the linguistic background of the author. Mathematically the author shows full masterships of all the concepts involved.

Remarks on style and notation. In some places notation is not very clear.

(1) The notation $C(\mu)$ in formula (2.7) hides the fact that the quantity in question depends on the partition π as well and not only on the indices μ.
(2) In section 3.2.1 creation, annihilation and preservation operators $A^+_{g_\xi}$, $A^-_{g_\xi}$, $A^o_{g_\xi}$ are defined. Starting on p. 77 however they are denoted by A^+_ξ, A^-_ξ A^o_ξ without stating explicitly that a fixed unit vector g_ξ is assumed. This is confusing, in particular if one reads ref. [38] in parallel where A_ξ is a more general operator operating on Fock space.
(3) on p.93, the set $\{0, g_\eta, w \otimes g_\eta\}$ where $w = \ldots$ should be reformulated more clearly.

Future developments

What makes this thesis the most interesting are the questions it raises. Many generalizations seem possible. As we noted in our discussions, the model fits well into the framework of the recent paper arXiv:1901.09158 *An Operad of Noncommutative Independences Defined by Trees* by Jekel and Liu. Among many problems to consider the following ones seem the most interesting.

(1) Elucidate the role of BM-independence in Jekel/Liu's theory.
(2) Compute explicitly the cumulants according to the formulas of Jekel/Liu.
(3) Compute explicitly the convolutions according to the formulas of Jekel/Liu.
(4) Prove Poisson limit theorems and Fock space models for other models from Jekel/Liu's framework.

On the other hand it would be interesting to extend the results to other partial orders. The proofs in Chapter 2 involve some Følner-type estimates familiar from the theory of amenable groups. We conjecture that the methods carry over to amenable ordered groups and it will be interesting to see what happens for nonamenable ordered groups as index sets.

Conclusion

The thesis presents a solid work and my overall assessment is very good, with some minor reserves due to the stylistic issues addressed above.

Graz, July 31, 2019 F. Lehner