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Review of the PhD thesis of Tomasz Rzepecki — Bounded Invariant Equivalence Relations

To the University of Wrocław

Background

Notions of strong type play an important role in the study of first order theories. A *strong type* (over $\emptyset$) is a class of an automorphism-invariant equivalence relation on $\mathfrak{C}^\alpha$ which is bounded (i.e., the quotient has small cardinality) and refines equality of types. The phrase "strong type" by itself often refers to a *Shelah strong type,* which is simply a type over the algebraic closure of $\emptyset$ (in T^{eq}). In other words, two sequences have the same Shelah strong type if they are equivalent with respect to every definable equivalence relation with finitely many classes. Refining this is the notion of *KP strong type* ($\equiv^\alpha_{KP}$), in which two α-tuples are equivalent if they are equivalent with respect to every bounded type-definable equivalence relation. Finally, the *Lascar strong type* ($\equiv^\alpha_L$) is the finest notion of strong type (on α-tuples).

For a long time it was unknown whether non-G-compact theories exist, i.e., theories in which $\equiv_L \neq \equiv_{KP}$. The first example was found in [CLPZ01]. But once this was found, the natural question to ask is: how many Lascar strong types are there in a given KP-strong type C? This question was answered by Newelski [New03], showing that if there is more than one Lascar strong type in C, then there are at least $2^{\aleph_0}$.

This is also related to the study of the connected components G^{00} (the smallest type definable group of bounded index) and G^{000} (the smallest invariant group of bounded index) of a definable or type definable group G. For a long time it was not known of a type definable group G with $G^{00} \neq G^{000}$. The first such example was found in [CP12].

As opposed to Shelah and KP strong types, the space of Lascar strong types does not come equipped with a Hausdorff topology. It is therefore unclear to what category this quotient belongs. In [KPS12], the authors suggest viewing it through the framework of descriptive set theory (this idea was already mentioned in [CLPZ01]). When the theory is countable, they formally interpret equality of Lascar-strong types as a Borel equivalence relation over a compact Polish space, and then consider the position of this relation in the Borel reducibility hierarchy. They go on to compute specific examples, most of which relate to connected components of groups, i.e., G^{00} and G^{000}.

Given two Polish spaces X and X' and two Borel equivalence relations E and E' respectively on X and X', we say that E is *Borel reducible* to E', denoted by $E \leq_B E'$, if there is a Borel map f from X to X' such that $x \; E \; y \iff f(x) \; E' \; f(y)$ for all $x, y \in X$. Two relations are *Borel bi-reducible* if each is Borel reducible to the other. The quasi-order of Borel reducibility is a well-studied object in descriptive set theory, and enjoys a number of remarkable properties. One of them is given by the Harrington-Kechris-Louveau dichotomy, which asserts that a Borel equivalence relation is either smooth (Borel reducible to equality on 2^ω) or at least as complicated as $\mathbb{E}_0$ (eventual equality on 2^ω).

One of the main conjectures in [KPS12] was that if the Lascar strong type relation is not the same as the KP-strong type relation, then it is non-smooth. This was resolved in [KMS13] and some generalizations of that approach to a class of F_σ strong types were given in [KM14, KR16]. However, a big open question remained: does a similar result (i.e., closedness = smoothness) hold for general Borel strong types?

In a striking tour de force, the work of Krupinski, Pillay and Rzepecki [KPR15] completely solved this question, providing a full solution and even added new information: for any Borel strong type, when it is restricted to a type definable E-saturated set on which the automorphism group acts transitively (like the set of realizations of a complete type), there are precisely three options: E has finitely many classes and is (relatively) definable, E has continuum many classes and it is closed, or E is non-smooth. The method of the solution were completely different than the previous approach, and used mainly techniques from topological dynamics, applied in the model theoretic settings.

More recently, Krupinski and Rzepecki [KR18] improved the results of [KPR15], by topologically representing the quotient space of strong types in a countable theory as a quotient of a compact Polish group (and in the case when the theory is NIP, in terms of Borel cardinality). This allowed them to recover the main results of [KPR15] in a more streamlined way using properties of Polish groups. This also allowed them to deal with the analogous context of bounded quotients of type definable groups (which was not done in [KPR15]).

In [Rze18], Rzepecki is interested in the question of when the closedness of the classes of a strong type E imply the closedness of E, which is an easy exercise when the domain of E is a complete type. For this he defines the notion of a weakly orbital equivalence relation (for a G-invariant equivalence relation), which generalizes the case where G acts transitively on the domain of E, and the case of orbital equivalence relations (i.e., orbit equivalence relations for some subgroup H of G). This allows him to get the equivalence of "closedness" and "smoothness" for in this more general setting.

Review of the thesis

The thesis under review is the ultimate summation of all of the above (literally it contains all the main results from [KPR15, KR18, Rze18]). It is very well-written, and very cohesive: in many if not most theses there are different chapters, each about different results with some (perhaps very loose) common thread. Not so much in this one: it deals with a single subject — the analysis of strong types via descriptive set theory and topological dynamics. The thesis develops this subject from the very beginning up to the current state of the art (including many examples). The basic idea of the thesis is to find the correct abstraction to all the results in the papers mentioned above, so that all the model theoretic results will be easy consequences. I believe that Rzepecki should consider publishing that abstraction (which did not appear in the published papers), so that it will be familiar and available for non-logicians, and it may find completely different applications. In any case it will be an excellent reference for anyone interested in the subject.

I think that the results in this thesis are quite remarkable, utilizing deep understanding of all aspects of the subject: model theory, descriptive set theory and topological dynamics. I am sure that these results, especially Main Theorem C will have further applications in studying strong types, and may help in finding out more on the structure of strong types and in particular this may help to understand the partial order that Borel reducibility gives on strong types. For example, what are the possible Borel cardinalities that may appear for strong types and in particular for the Lascar strong type?

In conclusion, I strongly recommend the acceptance of the thesis. Based on the results attained and the breadth of the presentation, I propose to grade the thesis with "passed with high distinction" ("summa cum laude"). Moreover, I think that Rzepecki should be considered for an award for this thesis.

Comments on the text

Here are some small comments I found on specific details, addressed to the author.

Page 10. It is not clear why Main Theorem C doesn't just follow from Corollary 6.12 and Theorem 6.9.

Page 11, Main Theorem E: should add that in the relatively definable case, E has finitely many classes.

Remark 2.12, the definition of Q_s seems wrong (in the first sentence).

Proposition 2.20, in the end of page 18 the X's should probably be Y's.

Proposition 2.34, I think you should explain why the "In particular" case follows from the main body of the proposition.

Example 2.49 seems wrong as stated. For example it is not true that Polish spaces have only countably many clopen subsets. Moreover, G should probably have dimension continuum.

Fact 2.57 (8) doesn't seem accurate. $H(u\mathcal{M})$ should be the intersection of all τ-closures of all τ-neighborhoods of u, I think.

Definition 2.64. Shouldn't it be $\leq$? Also it is not clear immediately why this is equivalent to the definition from [Koh95].

Below Definition 2.64. A "topological isomorphism" is just a homeomorphism, no?

Fact 2.68. Perhaps you should explain the bit in parenthesis in the end.

Fact 2.71. Perhaps in the preliminaries you should also explain what is a G-equivariant quotient. On this note, you should probably define what is an orbit map.

Proof of Proposition 2.72 (3) implies (4). Perhaps you should give a reference to why the group of homeomorphisms is a Polish group. Also, and more importantly I didn't understand the proof here. I don't think that in general a subspace of a separable space has to be separable. Perhaps a better argument is to say that there is a bound on the set containing π_g for all $g \in G$ ($2^{\aleph_0}$), as it is contained in a Polish space. Thus the closure is bounded by $2^{2^{\aleph_0}}$ (this is true for any Hausdorff space, I think).

(4) implies (5). I don't understand the argument here for the same reason as above. However, it is ok, since the size of that set is at most continuum, this also bounds the number of ω-sequences (since $\left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0}$).

Proof of Fact 2.73. Perhaps give references to the two facts used: that the pointwise limit of a sequence of continuous functions is Borel and that there at most continuum many Borel measurable functions.

Fact 2.148. In the published version, this is [KM14, Corollary 3.37].

Corollary 3.8, line -1 in page 56, there is a typo in X_M/E^M. More importantly, couldn't you have used Corollary 3.3 instead in this proof?

End of page 57. You should probably explain when you identify $A/\equiv$ with $R\left[A\right]$ as sets, and when can you do it as topological spaces.

Proof of Lemma 4.2. Should also prove that D is a group.

Proof of Lemma 4.4. Can you elaborate on the the sentence in parenthesis "(which one may also be describe..." (there is a typo there).

In the diagram in the top of page 60, you are not claiming that these maps are continuous, just that they exist, so maybe stress this (that maps in diagrams need not be continuous).

Proposition 4.7. For the "by compactness, equivalently, Polish" remark you need some metrization theorem, no?

In Proposition 4.8, maybe stress that $\overline{u\mathcal{M}}$is the closure in EL.

Corollary 4.12. It is not clear that you can use Proposition 2.34 in the proof. I mean it seem to give something like $Core\left(H\left(u\mathcal{M}\right)D\right)$ and not what is written.

Definition 4.19, in the end: $\left(Aut\left(M\right), A_M\right)$ does not compile: $Aut\left(M\right)$ does not act on A_M.

Page 64, first paragraph, line 4. $S_\varphi\left(B\right)$ should be $S_\varphi\left(B_0\right)$.

Corollary 4.27. What do you mean by a "canonical parameter" for a type definable set? if you mean a hyper-imaginary, I am not sure this is helpful for the presentation.

Definition 4.28. I think it would be much easier to understand if you define $G^Y\left(M\right)$ as all automorphisms σ of M such that some (any) extensions are in G^Y (if this is correct).

Definition 5.1. The "In other words" part is not clear to me.

Theorem 5.52 (1). $E_{\hat{G}}$ should be $E|_{\hat{G}}$.

Page 90, towards the end, maybe remind why $H(u\mathcal{M})$ acts trivially on X/E.

Theorem 6.9, page 98 line 1, "of" should be "or".

Last line of page 124. "remark" should be "proposition".

After Definition 7.13, should probably remark that you are going to prove soon that orbital implies weakly orbital.

Proof of Proposition 7.22, write what is $\tilde{X}$ for the first half.

Sincerely,
Itay Kaplan

References

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