Review of the thesis “Dynamics and computability in Geometric Group Theory” by Karol Duda

Jarek Kędra

University of Aberdeen
kedra@abdn.ac.uk

September 12, 2023

It is my pleasure to review the dissertation by Karol Duda. The thesis consists of two parts: the first is concerned with computable aspects of amenability, while the second is about group actions on simply connected small cancellation complexes. The results are non-trivial and the proofs demonstrate technical strength of the author. The dissertation satisfies all statutory requirements to award the doctoral degree.

Let me comment of the first part of the dissertation and discuss a sample of results. The concept of amenability was initially motivated by the question of Ruziewicz as to whether the Lebesgue probability measure is a unique *finitely additive* measure on a sphere invariant under rotations. This led to the discovery of paradoxical decompositions by Banach and Tarski and it is now a well established area of mathematics.

Recall that a group is non-amenable if and only if it admits a paradoxical decomposition. The concept has been generalised to pseudo-groups of transformations of spaces and the thesis focuses on the computability aspects of it. Namely, assuming that a group action on a countable set is non-amenable can one find a computable paradoxical decomposition? Here, a computable subset of the set $\mathbf{N}$ of natural numbers is a set whose characteristic function can be computed by a Turing machine.

The first result (Theorem I.2) states that if (G, X) is a pseudo-group of computable transformations which does not satisfy Følner condition (a criterion of amenability) then X has an effective paradoxical G-decomposition.

An effective paradoxical decomposition of X is a tuple $(X_1, X_2, \gamma_1, \gamma_2)$ such that $X = X_1 \sqcup X_2$ is a partition into computable sets and $\gamma_i \in G$ are computable with $\gamma_i \colon X_i \to X = \gamma_i(X_i)$. The above result has applications in obtaining computable paradoxical decomposition for pseudo-group of so called wobbly transformations of a countable discrete metric space as well as for genuine group actions.

A crucial step of the proof relies of a computable version of Hall's Harem Theorem which is also proven in the dissertation (Theorem I.1).

In 1920's, von Neumann asked whether amenability of a group is equivalent to containing a non-abelian free group as a subgroup. This is not true as shown by Olshanskii. However, a special case of this question has an affirmative answer as proven by Whyte (for certain discrete metric spaces) and generalised by Schneider (for some coarse spaces).

The non-amenability in these cases implies existence of certain regular forests. The computable version of these results (Theorem 4.6.2) follow from a computable version of Hall's Harem Theorem (Theorem 4.1.3) which is the main result of Chapter 4. It's proof is a careful construction of a matching.

All the results are very strong and their proofs are difficult. However, their wider context or their applicability is not clear to me (see the questions below). This could be better explained.

The results of this part are contained in two papers by the student, one of which is co-authored with his supervisor.

Question: How far are (some of) the above computability results from being converted into algorithms and computer programs? As the results are presented in the dissertation, they rely on the choice of an unspecified bijection from the countable group to the set of natural numbers. Are there examples, where they can be applied in a more practical context?

The second part of the dissertation is concerned with small cancellation groups. Its first chapter offers an introduction to small cancellation theory. It provides all necessary definitions and a selection of know techniques used in the subsequent sections.

The main result of this part (Theorem II.1) states that torsion subgroups of a group satisfying the C(6) or C(4)-T(4) or C(3)-T(6) small cancellation condition are finite cyclic. It is a consequence of a more general result (Theorem II.2) about group actions on simply connected complexes satisfying suitable small cancellation conditions.

Another major result (Corollary II.5) is a version of the Tits Alternative for groups acting almost freely on simply connected C(3)-T(6)–complexes. It states that such a group is either virtually cyclic or virtually $\mathbf{Z}^2$ or contains a non-abelian free group.

The proof of Theorem II.2 is explicit, careful and very technical. Its summary is clearly explained in a three paragraph text and its main part is broken into a number of lemmas and propositions. The nature of the results requires considering cases depending on the relevant small cancellation conditions. In contrast with the long proof of Theorem II.2, the (beautiful) short proof of Theorem II.3 presents a construction of an *obvious* CAT(0)-metric on a C(3)-T(6)–complex. This could be included as an exercise in the Bridson-Haefliger book. The proofs throughout this part use the state of the art tools and results of geometric group theory.

The dissertation is very well written and shows mathematical maturity of the author. It also shows an expertise in two relatively distinct areas of mathematics which is rare. However, lack of examples and more concrete applications in both parts is a drawback. Especially in the second part, providing a whole family of non-trivial examples of groups and complexes satisfying various cancellation conditions would considerably improve the presentation. Also, are there more concrete consequences of Theorem II.1 or Theorem II.6?

In my opinion, the dissertations satisfies all statutory requirements and I recommend to award Karol Duda the degree of Doctor of Mathematical Sciences. I also recommend to award the degree with distinction.