Review of G. Świderski's doctoral dissertation entitled "Spectral properties of unbounded Jacobi matrices"

by Jacob S. Christiansen, Lund University

August 3, 2016

The present dissertation consists of 3 articles that we shall refer to as Paper I, II, III and an introduction (incl. preliminaries) providing the necessary background.

The dissertation deals with Jacobi matrices and orthogonal polynomials. As explained in the introduction, these are classical objects which are closely related. A Jacobi matrix is a tridiagonal symmetric matrix of the form

$$A = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}, \tag{1}$$

where $a_n > 0$ and $b_n \in \mathbb{R}$ for all $n \geq 0$. The sequences $a = (a_n : n \geq 0)$ and $b = (b_n : n \geq 0)$ are also called the Jacobi parameters. Note that A naturally defines a symmetric operator, J, on the Hilbert space $\ell^2(\{0, 1, 2, \ldots\})$.

Interestingly, the three-term recurrence relation

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x), \quad n \geq 0, \tag{2}$$

with initial conditions $p_{-1}(x) \equiv 0$ and $p_0(x) \equiv 1$ generates a sequence of orthonormal polynomials. That is, there exists a probability measure μ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} p_n(x) p_m(x) d\mu(x) = \delta_{n,m}. \tag{3}$$

Conversely, if (3) holds and $p_n(x)$ has positive leading coefficient, then we also get (2) for some choice of Jacobi parameters.

The measure μ in (3) need not be unique. But when the Jacobi parameters are bounded (i.e., J is a bounded selfadjoint operator), this is always the case and supp(μ) is compact. In fact, there is a one-one correspondence between bounded Jacobi parameters and (non-trivial) compactly supported probability measures. The associated spectral theory which describes how qualitative features of the Jacobi parameters are reflected in the measure μ – and vice versa – is well-developed. It covers, e.g., the cases of constant, periodic, or almost periodic Jacobi parameters and compact perturbations thereof.

As is also mentioned in the introduction, the measure μ of orthogonality is unique if and only if the symmetric operator J is (essentially) selfadjoint. In the non-selfadjoint case, J has deficiency indices $(1,1)$ and different selfadjoint extensions lead to different measures of orthogonality. This can only happen if $a_n \to \infty$ sufficiently fast. For the well-known Carleman criterion states that $\sum 1/a_n = \infty$ implies uniqueness of μ. There is a rich and beautiful theory for the situation where μ is non-unique. It peaks with the so-called Nevanlinna parametrization of all measures of orthogonality.

The present dissertation does *not* deal with bounded Jacobi parameters nor parameters that lead to non-uniqueness of μ. It deals with the case in between (e.g., $a_n = n^\alpha$ for some $\alpha \in (0,1]$ and $b_n = 0$) and greatly advances the theory in that situation. Before describing the results of the dissertation in more detail, let me just mention one very interesting consequence. The new ideas of Grzegorz enables him to solve a 25 year old problem of Chihara that has been presented at open problems sessions at several conferences.

The main results are 1) Theorem A, 2) Theorems B–C, and 3) Theorems D–E. While 1) is contained in Paper I which has already been published in Constructive Approximation, a high impact leading journal in approximation theory, 2) and 3) are contained in Paper II and III, respectively. As explained below, Papers II and III are closely linked to one another and both are currently under review for publication.

When the symmetric operator J defined from (1) is unbounded, but selfadjoint, it is most natural to ask about its spectrum. Under which conditions is $\sigma(J)$ all of $\mathbb{R}$ (or $[0,\infty)$ if J is known to be positive)? When are there 'gaps' in the spectrum? What can we say about the spectral type? Are there conditions which ensure that $\sigma(J)$ is, e.g., purely absolutely continuous? Etc, etc.

Already Paper I gives a partial answer to some of these questions. Theorem A provides conditions under which $\sigma(J) = \mathbb{R}$ and $\sigma_p(J) = \emptyset$ (i.e., eigenvalues are absent). Results of this type is already in the literature, but not in the same generality. The advantage of Grzegorz's new result is the extra amount of freedom that comes with the sequence $\alpha = (\alpha_n : n \geq 0)$. This enables him to cover many more cases than previously and on top of that it leads to a solution of Chihara's problem (by utilizing a clever result relating the spectrum of certain restrictions of J^2 to the spectrum of J, see Prop. 3). The main idea of the proof of Theorem A is to consider the commutator $BA - AB$, where B is defined from the sequence α by

$$B = \begin{pmatrix} 0 & \alpha_0 & 0 & 0 & \cdots \\ -\alpha_0 & 0 & \alpha_1 & 0 & \cdots \\ 0 & -\alpha_1 & 0 & \alpha_2 & \cdots \\ 0 & 0 & -\alpha_2 & 0 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}. \tag{4}$$

Papers II and III are closely related and may be considered as a whole. While Paper II provides the basic theoretical results about generalized eigenfunctions, Paper III gives the main application, namely a formula for the ac density of the measure μ of orthogonality. A generalized eigenvector is a sequence $u = (u_n : n \geq 0) \not\equiv \underline{0}$ which satisfies

$$a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1} = \lambda u_n, \quad n \geq 1 \tag{5}$$

for some real number λ, also called the spectral parameter. Note that u need not be an ordinary eigenvector for the matrix A nor belong to ℓ^2. Theorems B and C specify conditions on A so that

$$a_n(u_{n-1}^2 + u_n^2), \quad n \geq 1 \tag{6}$$

is bounded above and below (by some constant times $u_0^2 + u_1^2$) for *any* generalized eigenvector u. The two results differ at the level of spectral parameters. In Theorem B (the regular case), the bounds are uniform for λ in any compact interval $I \subset \mathbb{R}$. This in turn implies that μ is purely absolutely continuous on $\mathbb{R}$ and has no singular part. In Theorem C (the critical case), this is only so for compact intervals that do not intersect a certain interval $[\lambda_-, \lambda_+]$. Both results are established by use of transfer matrices. It is mentioned in the accompanying corollaries that some combination of the entries of the eigenvectors, called S_n, converges uniformly on compact intervals to some limiting function of definite sign. Compared with previous results, Theorems B and C have again some extra degree of freedom which allows for more applications. They are both formulated for an arbitrary integer $N \geq 1$, not only $N = 1, 2$.

Theorems D and E complement Theorems B and C, respectively. Under the same assumptions (plus the Carleman condition), Grzegorz shows that the N-shifted Turán determinants have a limit which in turn is intimately related to the ac density of the measure μ of orthogonality. A result of this type was previously only known for bounded Jacobi matrices. It is of interest as it provides a method to recover the measure μ directly from the orthogonal polynomials. The main idea of the proof is to truncate A and extend it N-periodically. Then a careful analysis allows for passing to the limit. Note that part of the legwork was already done in Paper II by showing that S_n has a limit (which is now being identified). Paper III also contains results on assymptotics of Christoffel functions and sheds new light on a conjecture of Ignjatović.

My overall assessment is very positive and I'm delighted to nominate the work of Grzegorz for a doctoral dissertation award. The dissertation is well-organized and well-written, it has a clear scope and contains several new/interesting results.

Jacob Christiansen