REPORT ON "ON MODEL THEORY OF FIELDS WITH OPERATORS"

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This PhD thesis is concerned with the geometry and model theory of *fields with operators*. By that, one means a field (or more generally, a commutative ring) endowed with a fixed number and type of additive endomorphisms, satisfying additional relations. Classical examples include differential and difference fields (i.e., fields endowed with a derivation or an endomorphism, respectively), which are amply and actively studied, both algebraically and model theoretically. These examples, along with additional ones share a number of common features, and it is clearly profitable to provide a general framework that generalises them.

One appealing such framework was suggested by Moosa and Scanlon in a series of papers, where they produced a geometric theory of fields with operators, and applied it (in the "free" case) to study the basic model theoretic properties of such fields. Their approach can be described as follows: the operators and the relations among them are described by a suitably defined "monoid" M, and the action of these operators on a field K is given by an action $M \times X \to X$ of the monoid on $X = \mathrm{spec}(K)$. In the case of usual ("discrete") monoids, one obtains as usual (upon choosing generators) an action by endomorphisms, satisfying some relations. The "free" case corresponds to a free monoid, where one can then work with a (pointed) map $M_0 \times X \to X$ for $M_0 \subseteq M$ corresponding to the generators.

The generalised setup of the thesis is obtained, in these terms, by replacing the functor $X \mapsto M_0 \times X$ be a more general functor $X \mapsto F(X)$ along with a (functorial, faithfully-flat) map $F(X) \to X$, and a "base-point" map $X \to F(X)$ over X. It seems a bit surprising (to me) that one can obtain new interesting examples in this way, and indeed it is shown in §2.1 that in characteristic 0 there are no new examples. But in positive characteristic there are, and in fact the formalism includes the "derivation of the Frobenius map" previously studied by Kowalski. The point is that in positive characteristic the additive group structure does not determine the scalar multiplication, which can be twisted by the Frobenius. The situation is explained very nicely in Remark 2.8.

The thesis is structured very simply: after recalling some preliminaries, there is a section (§2) dedicated to algebra and geometry in this setup, and then a second section (§3) concerned with the model theory. The algebraic part mostly deals with the free case. The first subsection, which appears to be the most novel, contains a classification result for algebra schemes (i.e., functors as above). The next subsection defines the action of such operators, and (in the local case) studies extension properties for such operators, which are needed later for the study of existentially closed models and model companions. This part is mostly a direct generalisation of the special case given by actions. The next subsection studies the right adjoint of F, i.e., prolongations. Their application here and in the model theoretic treatment is standard, but the construction of the prolongation does not follow directly from Weil restriction in this case, so is done directly. Finally, there is a subsection about the general (non-free) case, which is approached differently than in the monoid description above (or in Moosa–Scanlon): instead of having a general monoid, relations are given by a universal first order formula (of a particular shape). This appears to correspond to the case of finitely presented monoids.

The section on model theory studies some basic model theoretic properties: definability of existential-closedness and existence of model companions, quantifier elimination, algebraic and definable closure, stability and description of forking, restriction to PAC structures (most of the results are in the "strongly local" case, i.e., when Frobenius is 0 on the kernel of the base point; this is not surprising, since these conditions are required even in the special case of actions). While most of the methods here are standard, the author is able to isolate and formalise the necessary conditions very nicely. As a result, he obtains very clean axiomatisation, and simplified proofs (e.g., Theorem 3.13 and Theorem 3.18). The results are formulated in terms of existential closedness in a suitable class of fields, and can therefore be applied in several variants: PAC (corresponding to regular extensions), separably closed and large (the definition of largeness in this context is also new).

To summarise, I find the thesis to be very interesting, containing new and interesting ideas, and very elegant in its presentation. The author managed to clarify and unify scattered ideas in a clear manner, deal with new cases and solve some open questions. I believe the dissertation substantially promotes the state of knowledge in the subject, and the author clearly admits a good understanding of the ideas involved. I therefore **strongly recommend that the thesis is accepted, and Jakub Gogolok is awarded a PhD.** ***I further recommend awarding the degree with distinction.***

Detailed comments. The following are some comments and minor corrections I would like suggest. There were some typos in addition to the ones I mention, that neglected to note. None of these issues contradict my positive assessment, but some of them would hopefully be addressed.

- **p6, 3para:** The setup of [2] is only equivalent to [34] in the free case
- **§1.5:** Is k a field here? Previously it was a general ring, and in this case, one probably needs some flatness (and for Fact 1.3 one definitely needs a field)
- **p17:** "Can not" on line 10 should be "cannot". Line -3 "definite" should be "define" (I omit most other typos)

Prop. 2.6: It might be helpful to point out explicitly, somewhere around here, that though both sides are "vector space schemes", the isomorphism need not preserve this structure (i.e., the group structure does not determine the vector space structure)

Def. 2.7: Since an isomorphism with $\mathbb{G}_a^e$ is fixed (as per the paragraph before it), it will be clearer to say that the data is just the k-algebra structure on $\mathbb{G}_a^e$, and denote the first projection by π (one could ask, in this case, how much of the sequel depends on this identification, but the it is rather clear that not much).

Rem. 2.8.3: I'm not sure what is meant by the last sentence, the ring scheme is just the ring of functions on the resulting product.

Def. 2.37: This is rather imprecise. For general $m > n$, there are several maps $\mathcal{B}^{(m)} \to \mathcal{B}^{(n)}$, depending on the choice of coordinates. Is it meant that one chooses the first n, or all of them?

Def. 2.39: The composition here appears to be reversed (unless I'm misunderstanding the previous definition; in any case, it is better to elaborate a bit)

p40,l-5: It took me some time to understand how $\bar{X}$ is viewed as an element of $\mathcal{B}(K[\bar{X}])$, there are many options, and it is better to say X_i in the i-th factor, or something similar. Also, this is one place where it was not completely clear that the construction is independent of the identification (though this is clear a posteriori, once the universal property is deduced).

Def. 2.63.1: Generated by A over what? What is the quantification on A here? Why can't we take $A = K$?

p52,para 2: Seems like there should be something else in place of "Proposition 2.29".

p53,para 2: In what theory should being of type $\mathfrak{K}$ be definable?

Ex. 3.2.3: A instance of the previous question, why are we allowed to change the theory to SCF?

p53, 1st line after Def 3.3: "see" is missing.

Notation 3.8: I don't see why $\mathfrak{K}_{fin}$ is definable.

p56,l1: Should be $f(a) \neq 0$.

p58: Why are partial differential fields in characteristic 0 covered by 3.14? It seems $\mathcal{B}$ is nice in this case.

Rem. 3.19: I don't agree that E looks strange and unnatural ☺. In the simplest case when $B = k^S$ for some pointed finite set $0 \in S$, The prolongation of V is simply V^S, so the prolongation W^S of $W \subseteq V^S$ naturally sits inside $(V^S)^S = V^{S\times S}$. The equaliser condition then states that we should be looking only at those $f \in V^{S\times S}$ satisfying $f(0,s) = f(s,0)$ for all $s \in S$. This is an obvious necessary condition.

Lemma 3.23: I assume that λ_0 is the partial inverse of Frobenius.

Lemma 3.30 (proof): It seems that dcl is the more interesting case, since that is where there is a difference in the non-local case.

Rem. 3.34: While this is true, the approach here also provides the description of independence, which is of course interesting on its own.

Def. 3.40: A condition is missing on the second line.

Rem. 3.48: What is $DCF_{p,m}$? I thought that m is the number of derivations, but then what is the case $m = 0$?

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