PhD thesis report
On model theory of fields with operators
by Jakub Gogolok

Gogolok's PhD thesis makes significant contributions to the model theory of fields with operators – a subject within Mathematical Logic – . This branch of model theory somewhat focuses on determining the existence of the so-called model-companion of a given theory. The model companion yields a suitable setup to deploy the machinery of first order logic due to the fact that it is model complete, and hence first-order properties are preserved when passing to extensions that are models. In the context of fields with operators the existence of a model companion is usually equivalent to determining whether the existentially closed models form an elementary class; while the latter makes the companiability problem more approachable there are still several difficulties one needs to address.

The theory of fields with operators (from a model theoretic point of view) can be dated back to Robinson's work on differentially closed fields of characteristic zero DCF_0 in the late 60s; and then Wood's work in the characteristic $p > 0$ setup DCF_p in the early 70s. In characteristic zero, the theory DCF_0 satisfies tame properties: it is a complete ω-stable theory with elimination of quantifiers and imaginaries. On the other hand, in characteristic $p > 0$, the theory DCF_p is complete and stable but not superstable; it does not eliminates quantifiers nor imaginaries in the language of differential fields. It does, however, eliminates quantifiers after adding the p-th root function to the language. It is still unknown whether there is a natural language in which it eliminates imaginaries. In any case, the theory of differentially closed fields is the model companion of differential fields.

Later on, Macintyre followed Chatzidakis-Hrushovski studied the model theory of difference fields. They established the existence of a model companion ACFA. This theory, while not stable, it is simple (i.e., Shelah's forking is an abstract independence relation; in particular, it is symmetric). Furthermore, this theory has almost elimination of quantifiers.

In 2014, both contexts (differential and difference) were unified by Moosa-Scanlon using the framework of free operators. Namely, one is given a finite dimensional algebra $\mathcal{D}$ over a field k (with some additional properties) and operator(s) on a k-algebra R is a k-algebra homomorphism $e : R \to R \otimes_k \mathcal{D}$. By setting $\mathcal{D}$ to be the dual numbers one recovers the case of derivations; while setting $\mathcal{D} = k \times k$ one recovers automorphisms. Moosa and Scanlon then observed that, for any choice of $\mathcal{D}$, if $char(k) = 0$ then there is a model companion and it is simple and has almost quantifier elimination. Furthermore when $\mathcal{D}$ is a local ring, one gets stability and full quantifier elimination. A few years later Beyarslan-Hoffmann-Kowalski-Kamensky explored the case $char(k) = p > 0$ and noted that when $\mathcal{D}$ is local and the unique maximal ideal is contained in the kernel of the Frobenius, $Fr_{\mathcal{D}}$, then a model-companion exists, is stable and eliminates quantifiers after adding the p-th root function.

One shortcoming of the above results is that they do not include the case of *derivations of the Frobenius*; i.e., additive maps satisfying a Leibniz rule of the form $\delta(xy) = \delta(x)y^p + x^p\delta(y)$. One of the main objectives in Gogolok's thesis is to unify all above instances (including derivations of the Frobenius) to a single framework. Before the work of Gogolok, this idea seemed a bit far fetch (at least to me) since the classical context of Moosa-Scanlon does not allow twisting by Frobenius. It is indeed quite interesting that this novel framework (namely, coordinate k-algebra schemes) works uniformly.

In Chapter 1, a good amount of preliminaries are presented, which should aid the non-familiar reader acquire the necessary knowledge the read the rest of the thesis. Chapter 2 presents the general framework in which the rest of the dissertation is situated in. Namely, the notion of a coordinate k-algebra scheme $\mathcal{B}$ is presented – the most relevant point of the definition is that as a group scheme $\mathcal{B}$ is isomorphic to a product of the additive group scheme – and some examples are provided. In particular, it is noted (Proposition 2.20) that the context of Moosa-Scanlon is recovered whenever the group scheme isomorphism is also one of k-module schemes, in which case $\mathcal{B}$ is of the form $\mathcal{B}(k)_\otimes := -\otimes_k \mathcal{B}(k)$. Furthermore, it is also explained how derivations of the Frobenius can be recover as coordinate k-algebra schemes. This is followed by a classification of coordinate k-algebra schemes (Theorem 2.17) when k is perfect. Essentially they are all Frobenius twists of $\mathcal{B}(k)_\otimes$.

Part of Chapter 2 is devoted to adaptions of known results about extending free operators to the coordinate k-algebra scheme setup. This is done in a standard fashion by using results on étale and smooth ring extensions. Then, in 2.3.3, a construction of the prolongation spaces is performed. This is rather technical, partly because there is no Weil descent functor available to work with in the general setup of algebra schemes. Thus, the prolongation construction has to be done from the scratch. Then, in 2.4.2, an abstract notion of iterativity φ is introduced, as a well as notions of $\mathcal{B}_\varphi$-fields, $\mathcal{B}_\varphi$-varieties, and nice pairs $(\mathcal{B}, \varphi)$. A good amount of examples are discussed.

In Chapter 3, a class of extensions $\mathcal{K}$ is fixed and the study of existentially closed models is restricted to the class. Natural examples are arbitrary extension, but also separable extensions and regular extensions. In Theorem 3.14 it shown that, if the pair $(\mathcal{B}, \varphi)$ is nice and the class $\mathcal{K}$ is definable, then the class of $\mathcal{B}_\varphi$-fields that are $\mathcal{K}$-closed is elementary (which is one of the main results of the thesis). When $\mathcal{K}$ is the class of all extensions, the model companion is denoted $\mathcal{B}_\varphi$-CF.

The rest of Chapter 3 is devoted to study the model-theoretic properties of the theory $\mathcal{B}_\varphi$-CF when $\mathcal{B}$ is local (and the unique maximal ideal is contained in the kernel of the Frobenius). For instance, in Theorem 3.26 it is shown that $\mathcal{B}_\varphi$-CF admits q.e. after adding the p-th root function. Furthermore, Theorem 3.33 shows that this theory is stable and a description of forking independence is provided. Another interesting result is Theorem 3.47 which shows that being a PAC-substructure in $\mathcal{B}_\varphi$-CF is an elementary property (a desirable property when studying PAC-substructures of stable theories).

Referee's opinion

In my opinion, the thesis is mostly well written and certainly contains strong results. The candidate demonstrates good working knowledge of the subject and the state-of-the-art results. It is also clear that they can perform independent research with potential international impact. Parts of the thesis have already been published, and I expect that the rest will produce at least one more research output suitable for a leading journal (possibly in Logic). I do have a few questions and suggestions below. In particular, I believe there is an issue with one of the cases covered by nice pairs. Regardless of the answers to my questions (including my issue with nice pairs), I have a very positive impression of the research work and I recommend the thesis to be accepted.

Yours sincerely,

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Corrections and suggestions

Issue with the commuting case of nice pairs:

(*) The issue is in the proof of Proposition 2.68 and is related to Part (3) of Definition 2.67. The notion of nice pair includes the case of several *commuting* derivations in positive characteristic. Then, in Proposition 2.68, the third part of the proof is arguing that in such contexts you can get a finitely generated field extension that contains a and its derivatives (of order one). But the argument is setting derivations to be zero and claims that because of this choice the derivations will commute. However, this is not generally true. Simply take $\mathbb{F}_p(t,s)$ where $\delta_1(t)=t$ and $\delta_2(t)=s$ (and also $\delta_1(s)=s$ and $\delta_2(s)=t$). This gives a field with commuting derivations. But if I repeat the argument on the indeterminate s, I would be extending the derivations as $\delta_1(s)=0$ and $\delta_2(s)=0$; however, this would yield

$$0=\delta_1(s)=\delta_1\delta_2(t)=\delta_2\delta_1(t)=\delta_2(t)=s.$$

Here one uses that δ_1 and δ_2 must commute.

Generally speaking I don't expect Proposition 2.68 to hold for several commuting derivations in arbitrary characteristic (we know it does not hold in characteristic zero). While I don't have a counterexample in positive characteristic, the problem of extending 'commuting' derivations in this manner is quite subtle in any characteristic. (it might be that the characteristic zero example could be adapted to positive characteristic, but I am not sure about this).

If this cannot be fixed, I guess the case of commuting operators should be removed from the definition of nice pair. Also, see Remark 3.17(5) and Remark 3.55 (as these are affected).

Minor corrections/suggestions: X^y means page X line y from the top; and X_y means line y from the bottom.

(1) I'd suggest adding an Abstract.
(2) In 6_5, there is an extra 'algebra'.
(3) I'd advise expanding the introduction. Discuss more motivation and more details on what is known. Also, be more precise on the main results of the thesis (some explicit statements would be good).
(4) In 9_7, the definition of regular extension should be: L and K^{alg} are **linearly** disjoint over K.
(5) In 10^2, Fact 1.2(2), should say the Jacobian of $(f_1, \dots, f_n)$ at a.
(6) In 11^{10}, why fix m? There is another use of m in $V_1 \cup \cdots \cup V_m$; but this is not the m you fixed.
(7) In p.11, in the definition of $I_K(V)$ it should be $a \in V$ (not $a \in V(K)$).
(8) In 12^{11}, there is a K missing in $K[V]$
(9) 12^{13}, there is a typo in 'sat that'
(10) In 1.5, are you assuming k is a field or just a ring? (in 1.3 it was just a ring)
(11) In the definition of algebraic group, why are you not including "reduced scheme"?
(12) In Fact 1.3 (2), are you assuming $char(k) = p > 0$? Also, is this result only for algebraic groups (not for group schemes)?
(13) In 14^5 one T should be T^*
(14) p.14, (p5) local character, you have $|B_0| \leq \omega$ but it should be $|B_0| \leq |T|$
(15) 15^6, p should be q
(16) middle of p.17, you say 'Fortunately, not every derivation comes from....', why 'fortunately', perhaps simply use "Note that not every..."
(17) In Remark 2.8(1), explain what is the 'basic data' in Moosa-Scanlon. It'd be good if somewhere (maybe in Preliminaries) you add more details of the setup and basic data of Moosa-Scanlon and how it differs from your presentation.
(18) In Remark 2.8(2), I'd suggest 'In general one cannot...' (refer to Ex.2.10)
(19) In Ex.2.10, please remind us how $\mathcal{F}$ is defined. Also, at the end of the example you say that when R is of infinite imperfection degree then $\dim_R(\mathcal{F})$ is infinite as well. Please give an argument for this.
(20) In Ex.2.10, you should point out that as long as K is not perfect we have $\mathcal{F}(K) \neq K[X]/(X^2)$.
(21) Please provide more examples in the spirit of Ex. 2.10. At least, mention the case of p^n-derivations, i.e., $\delta(xy) = \delta(x)y^{p^n} + x^{p^n}\delta(y)$.
(22) In Lemma 2.12, recall what monic 'additive' polynomials are.

(23) 23^4, you say 'there exists an algorithm', where is this algorithm? Is it in the literature or you mean that it is contained in one of your arguments? Please explain.
(24) In Lemma 2.13, is K perfect?
(25) In Notation 2.16, explain how to compute the map Θ
(26) In the proof of 2.17, a couple of β's should be $\mathcal{B}$
(27) 28^5, when the field k is perfect
(28) In Ex.2.23, what exactly is $B_{\otimes}^{Fr}$, i.e,, do you require $n_1 = 0$ or not? In this example you need $n_1 = 0$, but I don't think this is part of the definition of $B_{\otimes}^{Fr}$, is it?
(29) In Remark 2.24, you mention how to translate the formalism of $\mathcal{B}$-operators to the first-order setup. You mention that the coordinate-operators ∂_i's should be additive and satisfy some form of Leibniz rule. This yields that the operator $\partial : R \to B(R)$ is a ring homomorphism. But what about conditions on the ∂_i's that are equivalent to ∂ being a 'k-algebra' homomorphism. In the case where $\mathcal{B}$ is of the form $- \otimes \mathcal{B}(k)$ (i.e., the scheme group isomorphism also preserves the module structure), you can say this by asking the ∂_i's to be k-linear, but what about in your general set up (where the isomorphism is only at the group level, not of modules)? Note that at this point you are not making any additional assumptions (for instance, you are not assuming that $\mathcal{B}$ is local nor that k is perfect). This corresponds to some equations of the ∂_i's on elements of k, but these equations have to be worked out.
(30) In Cor.2.30, assume that $\mathcal{B}$ is local
(31) Def.2.39, the composition is backwards
(32) Def 2.65, state that φ is an iterativity condition, or before the definition say that from now on φ is an iteratitvity condition
(33) Def. 2.67, say that $\mathcal{B}$ is a coordinate k-algebra scheme and that φ is an iterativity condition
(34) p.56, in the axioms of $\mathcal{K}\mathcal{B}_{\varphi}$-CF, you should say 'every $\mathcal{B}_{\varphi}$-variety over k of type $\mathcal{K}$'
(35) In Theorem 3.16, you should add Assumption 3.10
(36) In Lemma 3.20, doesn't $Fr(\ker \pi) = 0$ imply $L^p \subseteq L^{\partial}$? So, why do you have $L^p \subseteq L^{\partial}$ as an extra assumption?
(37) Lemma 3.23, K should be K_0
(38) in proof of 3.26, you refer to Prop.2.70, but this proposition is in the language $\mathcal{L}_{\mathcal{B}}^{\lambda}$ rather than $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$, so a comment has to be made on how to apply the proposition when the countably-many λ-functions are replaced by the single λ_0.
(39) mid p.64, you say $\mathcal{B} = B_{(n_1,\ldots,n_e)}$, are you assuming k is prefect?
(40) In Proposition 3.30 you seem you be assuming that forking independence in $\mathrm{SCF}_{p,e}$ implies algebraic disjointness. But this is not generally the case in the case where the imperfection degree 'e' is finite. For this to hold you should work in the language where you name a p-basis (not just λ-functions). The same issue appears in 3.31 and 3.32.
(41) In Lemma 3.32, it seems to claim that linear disjointness implies that the field extension $K_a M < C$ is separable. This is not generally the case (when your ambient model is a monster model of SCF), for this you need

to know that K_a and M are p-independent over the base field. Again, in finite imperfection degree this is immediate if you work in the language after naming a p-basis. In the case of infinite imperfection degree p-independence is implied by forking independence (you can use full existence to get a forking independent copy and then the rest of the argument goes through).

(42) In the proof 3.37, there are two definitions of $\partial(X_n)$, which one is it?
(43) Prop 3.38, what are the assumptions on $\mathcal{B}$?
(44) Remark 3.48, what is m? State what $\mathrm{DCF}_{p,m}$ stands for.