Report on the dissertation of Adam Malinowski entitled "Ellis groups in model theory and strongly generic sets"

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The main topic of study of the thesis are *Ellis groups* of certain topological dynamical systems in a model-theoretic context. In this setting, given a group G, a *topological dynamical system of* G (or a G*-flow*, for short) is simply a (left) action of G on a compact Hausdorff space X by homeomorphisms. In model theory, such actions arise naturally as actions of definable groups on certain type spaces.

The theory of abstract topological dynamics was initiated by Ellis in the 1960s when he associated to every such G-flow X, a semigroup (now called the *Ellis semigroup* of the flow), which is simply the closure of G in the product topology of X^X. The properties of this semigroup reflect many dynamical properties of the flow and it has been an object of intense study ever since. Ellis also observed that compactness and continuity allow to transfer the structure theory for minimal ideals in finite semigroups to this setting and in particular, one obtains that every minimal left ideal of the Ellis semigroup is a disjoint union of isomorphic groups. This group is called the *Ellis group* of the flow.

Ellis groups were first considered in a model-theoretic setting by Newelski, who was motivated by the fact that they permit to generalize ideas of the classical subject of *stable groups* in model theory to a non-stable context. In certain well-behaved situations, Ellis groups also provide invariants for first-order theories. This line of research was quite successful and has become an important direction in modern model theory. One of the main motivations of the thesis is studying how the Ellis group changes when passing to an elementary extension in more general situations than currently understood.

The main results of the thesis are in abstract dynamics but they are clearly motivated by model theory, which gives an original viewpoint to a classical subject. As type spaces in classical model theory are zero-dimensional, the author concentrates on dynamics on zero-dimensional spaces. This allows to use Stone duality to associate to every zero-dimensional G*-ambit* (a G-flow with a distinguished point with a dense orbit) a subalgebra of the Boolean algebra $\mathcal{P}(G)$ of all subsets of G invariant under the action of G. The author considers *topological* groups G and the subalgebras of $\mathcal{P}(G)$ of interest are defined using this topology, but in the corresponding flows, the action of G is *not* continuous. It is not usual in topological dynamics to consider actions of topological groups which are not continuous, but here it is essential because the group in question is often compact (or precompact) and compact groups have no interesting continuous flows.

Next, I will describe in some detail the different parts of the thesis. It consists of four chapters, the first of which is an introduction and the second establishes context and recalls some known facts. In the third chapter, there are some abstract results establishing a correspondence between almost periodic points of the Bernoulli flow 2^G and algebras of syndetic sets (called *generic* in the thesis). In Section 3.2, there is an interesting construction of minimal flows of residually finite groups G. Even though the author arrived at this construction independently, it has already been considered in the literature under the name of *Toeplitz subshifts*, by Jacobs and Keane [JK] for the group of integers and by Krieger [K] for general residually finite groups.

The bulk of the original results of the thesis are contained in Chapter 4. For a topological group G, the author defines the Boolean algebra $\mathrm{SBP}(G) \subseteq \mathcal{P}(G)$, which is generated by the open sets and the nowhere dense sets. This is by analogy with the classical algebra of sets with the *Baire property*, which is the *σ-algebra* with the same generating family. While the Baire property is only non-trivial for topological spaces which are Baire, this smaller Boolean algebra makes sense for an arbitrary topological group. This is made use of in an important way, as many of the groups considered are not complete. The algebra SBP is also intimately related to the more classical algebra of regular open sets, which is a quotient of it. Considering the algebra SBP is again motivated by model theory: for a group G definable in an o-minimal structure (equipped with the order topology), it contains all externally definable sets.

Sections 4.1, 4.2, and 4.3 are devoted to calculations of the Ellis group of systems of the form $\mathrm{S}(\mathcal{A})$, where $\mathcal{A}$ is a subalgebra of $\mathrm{SBP}(G)$ and G is respectively equipped with the profinite topology, compact, or precompact. In fact, the result in 4.3 is a common generalization of the other two. In all cases, the Ellis group is isomorphic to an appropriate quotient of (the completion of) G. In all of these cases, the Ellis semigroup has a unique minimal ideal, which, in the case where $\mathcal{A} = \mathrm{SBP}$, can be recovered as the Stone space of the Gleason cover of G. (The *Gleason cover* of a (compact) topological space is simply the Stone space of the Boolean algebra of the regular open subsets of the space; see [G]). Even though the results are formulated in a somewhat unusual setting (using actions of topological groups which are not continuous), it is possible to translate them to very natural statements about certain extensions of equicontinuous flows of discrete groups. Some model-theoretic applications in the o-minimal setting are also provided.

In the case where the group G is not precompact, the situation is (unsurprisingly) more complicated and no definitive results are obtained. However, the author has also carefully considered this situation and has identified some concrete obstacles for the application of his methods.

The thesis also contains two appendices: the first is devoted to an explicit, non-trivial calculation of the Ellis group of a system of certain wreath product groups and the second to some exploration of an open question posed in Section 3.

I have found the thesis to be very well and carefully written, with complete and detailed proofs. The author has clearly mastered the concepts and tools of abstract topological dynamics and was able to apply them successfully in a model-theoretic setting. He studies systematically the dynamical systems associated to subalgebras of $\mathrm{SBP}(G)$ for G precompact and obtains definitive results about the Ellis group in this situation. The thesis also contains a multitude of well-thought, detailed, and non-trivial examples to illustrate the results, which otherwise may appear quite abstract.

In conclusion, I recommend, without reservation, that the candidate be awarded a doctoral degree.

References

[G] A. M. Gleason, *Projective topological spaces*, Illinois J. Math. **2** (1958), 482–489.

[JK] K. Jacobs and M. Keane, $0-1$-*sequences of Toeplitz type*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **13** (1969), 123–131. MR255766

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