Referee report on Ph.D. dissertation of Alicja Kołodziejska, M. Sc.

On random walks in a sparse random environment

The thesis of Alicja Kolodziejska deals with some properties of random walk in a sparse random environment (RWSRE). It basically concerns a single particle performing a nearest-neighbor random walk on the set of integers. Apart from a specific marked sites the movement of the particle is symmetric. In the marked sites independent random drifts are used instead. Consequently, we are dealing with a mixture of a standard random walk and a walk in an i.i.d. random environment (RWRE).

The main goal of the thesis is to examine how the mixture of two different random environments affects limiting behavior of the walk. The first important result concerns limit theorems for the position of the walk and the sequence of first passage times. More precisely, generalizing results for the RWRE there is a case in which the strong quenched central limit theorem holds true, for the position of the walk. A similar result refers to the sequence of first passage times.

The second part refers to a more complicated case of weak quenched limits. Here the sparsity plays the dominant role in governing limit behaviors of the RWSRE. There are formulated and proved weak quenched limit theorems for the sequence of first passage times as well as for the position of the walk. It has to be stressed that RWSRE can exhibit properties that are different from RWRE. There is also discussed the problem of lack of a strong quenched limit in a special setting.

The last part of the thesis is about maximal local times of the walk. The local time means here the amount of time spent by the particle in its favorite sites. There are two annealed limit theorems for the sequence of maximal local times: first for the case of dominating drift, second for the case of dominating sparsity. These results may be compared to the classic results for RWRE.

1 Brief description of the results.

In the third section there are two crucial results. The first one is Theorem 3.2.1. - this is a strong quenched central limit theorem for the position of the walk (RWSRE). The second one is Theorem 3.2.2., which refers once again to the strong quenched central limit theorem, but rather for the sequence of first passage times. The proof of Theorem 3.2.2. is based on the verification of the Lindeberg condition. The proof of Theorem 3.2.1. requires some more preparations which makes it possible to replace X by X^* - the sort of maximal process. The main point here is to use some moment bounds which can be used to show some suitable convergence.

In the fourth section some weak quenched limit theorems are proved. The concept is that under certain assumptions and under normalization a_n^2, $a_n \sim n^{1/\beta}$, $\beta \in (0,4)$ the centered sequence of first passage times weakly converges to some functional of a certain Poisson point process, more formally Theorem 4.2.1. states that

$$\mathbf{P}_\omega\left(\left(T_n - \mathbf{E}_\omega T_n\right)/a_n^2 \in \cdot\right) \Rightarrow G(N)(\cdot).$$

It is important that except of standard assumptions for RWRE model, it is required here that $\mathbf{E}\xi < \infty$. The second result deals with the case of $\mathbf{E}\xi = \infty$ and $\beta = 1$. Here it is important to understand the behavior of

$$m_n = n\mathbf{E}\xi 1_{\xi \leqslant a_n}$$

and more precisely, its asymptotic inverse c_n. Now, what can be proved is Theorem 4.2.2. i.e.

$$\mathbf{P}_\omega\left(\left(T_n - \mathbf{E}_\omega T_n\right)/a_{c_n}^2 \in \cdot\right) \Rightarrow G(N)(\cdot)$$

with the same functional of some Poisson point process. There is also a similar result -Theorem 4.2.3. which deals with $\beta \in (0,1)$ and the normalization $a_n = n$. Finally, there is also comment why strong limits may not existsat least in a special case. More formally, there is Theorem 4.2.4. which shows that a strong limit (in Prokhorov metric) does not exist. Proofs are technical, some parts follow natural schemes., whereas some require some smart ideas how to use well known result for the independent type setting to catch much more complicated case discussed in the thesis. There is an auxiliary Theorem 4.4.4. which explains weak convergence of

$$(T_{S_n} - \mathbf{E}_\omega T_{S_n} \in \cdot)$$

to $G(N_\infty)$ a functional of a Poisson point process. The approach is based on the idea of coupling and some moment type estimates. Then there is a short proof of Theorem 4.2.1. where only basic estimates suffice and quite involved proof of Theorem 4.2.2. where there are additional problems caused by J_1 topology. Finally the proof of Theorem 4.2.3. is also based on Theorem 4.4.4. accompanied by some standard bounds. The most difficult proof concerns Theorem 4.2.4. It requires some modified process $\bar{X}$ which brings a bit of independence which is very helpful to establish the result. Also some special sets have to be constructed in order to show that a specific behavior of ω_n may lead to a contradiction.

It should be noted that the model is bit modified in the fifth section. The most important parameter, here is the local time

$$L_k(n) = |\{m \leqslant T_n :\ X_m = k\}|$$

Under a certain list of assumptions, where it is wise to mention $\mathbf{E}\rho^\alpha$, $\alpha \in (0,2)$, $\mathbf{E}\xi^{\alpha+\delta} < \infty$, Theorem 5.2.1. holds true, i.e.

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\max_{k\leqslant n} L_k(n)}{n^{1/\alpha}} > x\right) = 1 - e^{-c_\alpha x^{-\alpha}},$$

where c_α is some constant. Under a second list of assumptions, where it should be recognized $\mathbf{P}(\xi > x) \sim x^{-\beta}$, $\lim_{n\to\infty} n\mathbf{P}(\xi > a_n) = 1$ Theorem 5.2.2 holds true, that is

$$\lim_{nra\infty} \mathbf{P}\left(\frac{\max_{k\leqslant n} L_k(n)}{a_n} > x\right) = 1 - e^{-c_\beta x^{-\beta}},$$

where c_β is some constant. Proofs are based on the branching theory, in particular the analysis of the process Y plays the crucial role since it translates the problem for local times into the problem for branching theory. Furthermore, in the proof of Theorem 5.2.1 the basic concept is to divide local time into independent pieces, which obviously require a lot of technical work and a lot of nifty bounds. On the other hand, Theorem 5.2.2. is related to the existence of a long block and therefore the proof strategy is different and based rather on Ray-Knight Theorem as well as deep analysis of the process Y.

2 Remarks.

The dissertation of A. Kołodziejska is very impressive. I am particularly impressed by the technical abilities of the PHD candidate in carrying out complex reasoning leading to nice new results. It happened occasionally, that it was difficult for me to read the thesis, though most of the time at least I could follow main ideas. As for critical remarks, it should be little explanation why certain lists of assumptions are of meaning. What lacks is a discussion why what is proved in the thesis is important in the sense of the general view of the problem. In the form being I can get the impression that the assumptions are chosen so that all the proofs could be covered. Sometimes, though, there is some reference to similar results for RWRE. I think also that some rough ideas for the presented proofs could be really helpful before tedious technical work. Fortunately, from time to time I could tread some comments of this type which were quite helpful.

The paper is well written, I could find only few minor mistakes e.g.

- page 34 $[\varepsilon, \infty]$ compact (I think sth. missing here)
- page 41 I am not sure what is $=^{\mathrm{st}}$
- page 61 there is one P instead of $\mathbf{P}$
- page 65 there is notation $\mathbb{Z}$ without explanation, though I know that there is a similar notation used for e.g. Y.

3 Conclusion.

In summary, I believe that the thesis of A. Kołodziejska more than meets all the requirements for a doctoral dissertation and I am pleased to recommend to the PHD committee that the degree of doctor of mathematical sciences be awarded to M. Sc. Alicja Kołodziejska.

Witold Bednorz
Warsaw, Septemeber 23-th, 2024 r

Signed by /
Podpisano przez:

Witold Bednorz

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